Math 4318 - Spring 2011
Homework 1 — Solutions

3. Let
$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational.} \end{cases}$$
Show that f is differentiable at 0 and compute $f'(0)$.
Solution: Notice that if the derivative of x^2 at $x = 0$ is 0 and the derivative of the zero function at 0 is 0. So it would be reasonable to try to prove that $f'(0) = 0$. To this end let $\epsilon > 0$ we need to find some $\delta > 0$ such that if $0 < |h| < \delta$ then
$$\left| \frac{f(0+h) - f(0)}{h} - 0 \right| < \epsilon.$$
Rewriting this and recalling that $f(0) = 0$ we are trying to find $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(h)/h| < \epsilon$. If we simply take $\delta = \epsilon$ then notice that if $0 < |h| < \delta$ we either have h rational, in which case
$$|f(h)/x| = |h^2/h| = |h| < \delta = \epsilon,$$
or we have h irrational, in which case
$$|f(h)/h| = |0/h| = 0 < \epsilon.$$
In all cases we have the desired $|f(h)/h| < \epsilon$. Thus we have shown that
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and that $f(a) = 0$. If $g(x) = |f(x)|$ show that g is differentiable at a if and only if $f'(a) = 0$.
Solution: We begin by assuming that $f'(a) \neq 0$. With out loss of generality we assume that $f'(a) > 0$. Thus we have that
$$\lim_{h \to 0} \frac{f(a+h)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) > 0.$$
So if $\{h_n\}$ is a sequence of positive numbers such that $h_n \to 0$ then we have that $f(a+h_n)/h \to f'(a)$. Thus there is some N such that $|f(a+h_n)/h_n - f'(a)| < f'(a)/2$ for all $n \geq N$. In particular $f(a+h_n)/h_n$ is positive and hence $f(a+h_n)$ is positive for all $n \geq N$. Now for $n \geq N$ we have
$$\frac{g(a+h_n) - g(a)}{h_n} = \frac{|f(a+h_n)|}{h_n} = \frac{f(a+h_n)}{h_n} \to f'(a).$$
So if $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$ exists then it must equal $f'(a)$.
Now consider a sequence $\{k_n\}$ of negative numbers that converge to 0. As above we see that there is some N' such that $n \geq N'$ implies that $f(a+k_n)/k_n$ is positive and hence $f(a+k_n)$ is negative. So for $n \geq N'$ we see that
$$\frac{g(a+k_n) - g(a)}{k_n} = \frac{|f(a+k_n)|}{k_n} = \frac{-f(a+k_n)}{k_n} \to -f'(a).$$

Thus if $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exists then it must equal $-f'(a)$. But since $f'(a) \neq -f'(a)$ we see that $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ does not exist. That is g is not differentiable at a. So g differentiable at a implies that $f'(a) = 0$.
We now assume that $f'(a) = 0$. Notice that

$$\left|\frac{g(a+h)-g(a)}{h}\right| = \left|\frac{|f(a+h)|-|f(a)|}{h}\right| \leq \left|\frac{f(a+h)-f(a)}{h}\right|.$$

(The inequality follows from $||a|-|b|| \leq |a-b|$.) Since $\frac{f(a+h)-f(a)}{h} \to 0$ as $h \to 0$ we see that $|g'(a)| = 0$. So $g'(a) = 0$ and g is differentiable.

6. Recall that a function f is **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. If f is an even function then show that f' is an odd function.
   Solution: If f is an even function the we compute

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{k\to 0} \frac{f(-x-k)-f(-x)}{-k} \\ &\qquad \text{(substitute } h=-k\text{)} \\ &= -\lim_{k\to 0} \frac{f(x+k)-f(x)}{k} = -f'(x). \end{aligned}$$

   Thus we see that $f'(x)$ is an odd function.

11. Suppose that f is defined and twice differentiable in some interval containing c. Show that

$$f''(c) = \lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}.$$

   Given and example that shows the limit on the right hand side might exist even if the second derivative of f at c does not exits. **Hint:** L'Hopital and the previous problem (which you should do but don't have to write up).
   Solution: Since the limit of the numerator and the limit of the denominator is zero, L'Hopital's rule give

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = \lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h}.$$

   But now Problem 10 says that the limit on the right is computes the derivative of f' at c, that is $f''(c)$.
   One of the standard examples of a function whose second derivative does not exist is $f(x) = x|x|$ at $x = 0$. Indeed you can easily check that $f'(x) = 2|x|$ and that

$$f''(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$

   and is not defined at $x = 0$. But we see that for $c = 0$ we have

$$\lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} = \lim_{h\to 0} \frac{h|h|-h|h|}{h^2} = 0.$$

   So this limit can exist even when the second derivative does not.

12. Suppose that $f : (a, b) \to \mathbb{R}$ and that f'' exists everywhere. Then show that $f''(x) \geq 0$ for all $x \in (a, b)$ if and only if f is **convex** by which we mean

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in (a, b)$ and $t \in [0, 1]$. **Hint**: Problem 11.
Solution: Assuming f is convex notice that we can take $x = a + h$ and $y = a - h$ and $t = \frac{1}{2}$ to see that

$$f(\frac{1}{2}(a+h) + \frac{1}{2}(a-h)) \leq \frac{1}{2}f(a+h) + \frac{1}{2}f(a-h).$$

So we have

$$f(a+h) + f(a-h) - 2f(a) \geq 0.$$

If $h \neq 0$ then $h^2 > 0$ so for all $h \neq 0$ we have

$$\frac{f(a+h) + f(a-h) - 2f(a)}{h^2} \geq 0.$$

Thus as we take the limit as $h \to 0$ we see that $f''(a) \geq 0$ for all a.
We now assume that $f''(x) \geq 0$ for all x. For a fixed x and y with $x < y$ and $t \in [0, 1]$ let $z = tx + (1-t)y$. Now consider the order one Taylor polynomial for f at z. If we evaluate this polynomial at x then there is some $c \in [x, z]$ such that

$$f(x) = f(z) + f'(z)(x-z) + \frac{f''(c)}{2}(x-z)^2$$

and if we evaluate at y then there is some $d \in [z, y]$ such that

$$f(y) = f(z) + f'(z)(y-z) + \frac{f''(d)}{2}(y-z)^2.$$

Since $f''(c)/2(x-z)^2$ and $f''(d)/2(y-z)$ are both non-negative so is

$$t\frac{f''(c)}{2}(x-z)^2 + (1-t)\frac{f''(d)}{2}(y-z).$$

Thus

$$\begin{aligned} tf(x) + (1-t)f(y) &= [tf(z) + (1-t)f(z)] + [tf'(z)(x-z) + (1-t)f'(z)(y-z)] \\ &\qquad + tf''(c)/2(x-z)^2 + (1-t)'(d)/2(y-z) \\ &= f(z) + f'(z)(tx + (1-t)y - z) + t\frac{f''(c)}{2}(x-z)^2 + (1-t)\frac{f''(d)}{2}(y-z) \\ &\geq f(z) = f(tx + (1-t)y). \end{aligned}$$

So f is convex.

16. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If there is a constant $C \in (0, 1)$ such that $|f'(x)| < C$ for all x then show that f does have a fixed point. Show that it is not sufficient to assume that $|f'(x)| < 1$ to guarantee a fixed point by considering the function

$$f(x) = x + \frac{1}{1 + e^x}.$$

Hint: When C exists let x_1 be any point and set $x_n = f(x_{n-1})$. What can you say about $\lim x_n$? Does it exits?
Solution: We know form a lemma in class that

$$|f(x) - f(y)| \leq C|x - y|.$$

So if we let x_0 be any point in $\mathbb{R}$ and set $x_n = f(x_{n-1})$ then notice that

$$|x_n - x_{n+1}| \leq C|x_{n-1} - x_n| \leq \ldots \leq C^n|x_0 - x_1|.$$

Set $d = |x_0 - x_1|$ and notice that for $n > m$ we have

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{x-1}| + |x_{n-1} - x_{n_2}| + \ldots |x_{m+1} + x_m| \\ &\leq C^{n-1}|x_0 - x_1| + C^{n-2}|x_0 - x_1| + \ldots + C^m|x_0 - x_1| \\ &\leq dC^m(C^{n-m-1} + C^{n-m-2} + \ldots + C + 1) \\ &= dC^m \frac{1 - C^{n-m}}{1 - C} \leq d\frac{C^m}{1 - C}. \end{aligned}$$

(For the last equality we used the formula $1 + r + \ldots + r^k = \frac{1-r^{k+1}}{1-r}$.) Since $d\frac{C^m}{1-C}$ goes to 0 as $m \to \infty$ we see that $\{x_n\}$ is a Cauchy sequence. Indeed, if $\epsilon > 0$ is given there is some N such that $d\frac{C^n}{1-C} < \epsilon$ for $n \geq N$. Thus if $n \geq m \geq N$ we that $|x_n - x_m| \leq \epsilon$. Since Cauchy sequences converge we know there is some x such that $x_n \to x$. Notice that since f is continuous we have that

$$f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x.$$

Now consider the function $f(x) = x + \frac{1}{1+e^x}$. Suppose x is a fixed point, then we have

$$x = x + \frac{1}{1 + e^x}.$$

That is $\frac{1}{1+e^x} = 0$. But since $1 + e^x \neq 0$ we see that implies $1 = 0$. This contradiction implies that f does not have a fixed point. But now notice that $f'(x) = 1 - \frac{e^x}{(1+e^x)^2}$. The second term is always positive so $f'(x) < 1$ for all x but as $x \to -\infty$ the second term goes to 0 so $f'(x)$ is not bounded above by any $C < 1$.

19. Let f be a twice differentiable function on the interval (a, ∞). Show that

$$\left(\sup_{x\in(a,\infty)} \{|f'(x)|\} \right)^2 \leq 4 \left(\sup_{x\in(a,\infty)} \{|f(x)|\} \right) \left(\sup_{x\in(a,\infty)} \{|f''(x)|\} \right).$$

Notice that this says we can bound the first derivative of f in terms of f and the second derivative. **Hint:** Consider Taylor polynomial expanded about x evaluated at $x + h$ to get a quadratic equation in h.
Solution: Taylor's theorem gives

$$f(y) = f(x) + f'(x)(y - x) + \frac{f''(c)}{2}(y - x)^2$$

for some c between x and y. So plugging in $y = x + h$ we see

$$f(x + h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2.$$

Rearranging we have

$$-f'(x)h = f(x)h - f(x+h) + \frac{f''(c)}{2}h^2.$$

Taking absolute values we get

$$|f'(x)h| \le |f(x)| + |f(x+h)| + |\frac{f''(c)}{2}h^2|.$$

If we let

$$M_k = \sup\{|f^{(k)}(x)| : x \in (a,\infty)\}$$

then we see that

$$|f'(x)h| \le 2M_0 + \frac{M_2}{2}|h|^2$$

or

$$0 \le 4M_0 - 2|f'(x)||h| + M_2|h|^2.$$

The right hand side is a quadratic expression in $|h|$. Recall the roots of a quadratic ax^2+bx+c occur at $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Thus if $b<0$ for the quadratic to be non-negative for all positive x there is either a double root at $\frac{-b}{2a}$ or the roots are complex. So we see that we must have $b^2-4ac \le 0$. Thus we see that

$$4|f'(x)|^2 - 16M_0M_2 \le 0.$$

That is, we have

$$|f'(x)|^2 \le 4M_0M_2.$$

Since this is true for any $x \in (a,\infty)$ we can take the supremum over x on the left and get

$$M_1^2 \le 4M_0M_2,$$

which is equivalent to the formula we were trying to establish.

20. Suppose f is n times continuously differentiable on some interval (a,b) that contains c. If $f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \ne 0$, then show that

   (a) If n is even and $f^{(n)}(c) > 0$, then f has a relative minimum at c.
   (b) If n is even and $f^{(n)}(c) < 0$, then f has a relative maximum at c.
   (c) If n is odd, then f has neither a relative minimum or a relative maximum at c.

   This, of course, is a large generalizations of the "second derivative test" you learned in calculus for determining if a critical point is a max or a min.

   Solution: Let's suppose that $f^{(n)}(c) > 0$. Since $f^{(n)}$ is continuous we know there is some interval (a,b) that contains c on which $f^{(n)}$ is positive. Now using the $(n-1)$-order Taylor polynomial about c we have for $x \in (a,b)$ that

$$f(x) = f(c) + \frac{f^{(n)}(d)}{n!}(x-c)^n,$$

   for some d between c and x. (Notice that the other terms in the Taylor polynomial are zero since $f^{(k)}(c) = 0$ for $k = 1,\ldots,n-1$.) Now since n is even and $f^{(n)}(c) > 0$ we see that the last term is positive. Therefor

$$f(x) \ge f(c).$$

That is c is a relative minimum of f.
We have a similar argument for $f^{(n)}(c) < 0$. If n is odd then we see that the second term on the right in the equation above can be both positive and negative for x near c (it will be positive for x on one side of c and negative on the other side of c), thus c will be neither a relative minimum or maximum.