Math 4318 Midterm Exam 1 Solutions Spring 2011

1) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and set

$$F(x) = \int_a^x f(t)\, dt.$$

Prove that F is Lipschitz.

Solution: Since f is Riemann integrable it is bounded, that is there is some M such that $|f(x)| \leq M$ for all $x \in [a, b]$. Now if $y \leq x$ we have

$$|F(x) - F(y)| = |\int_y^x f(t)\, dt| \leq \int_y^x |f(t)|\, dt \leq \int_y^x M\, dt = M(x - y) = M|x - y|.$$

Similarly $|F(x) - F(y)| \leq M(y - x) = M|x - y|$ if $y \geq x$. So we see that for any $x, y \in [a, b]$ we have

$$|F(x) - F(y)| \leq M|x - y|.$$

That is F is Lipschitz.

2) a) Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at all $x \in \mathbb{R}$ by computing the derivative. (Thought we did not prove it in class you may use, without proof, the fact that the derivative of $\sin x$ is $\cos x$ as well as other derivative rules you know.)

Solution: For $x \neq 0$ the function is a composition and product of differentiable functions so we can use the product and chain rule to get

$$f'(x) = 2x \sin(1/x^2) + x^2 \cos(1/x^2)(-2x^{-3}) = 2x \sin(1/x^2) - 2(1/x) \cos(1/x^2).$$

Now for $x = 0$ we compute

$$\left|\frac{f(0 + h) - f(0)}{h}\right| = \left|\frac{h^2 \sin(1/h^2)}{h}\right| = |h \sin 1/h^2| \leq |h|.$$

So as h goes to zero the difference quotient goes to zero thus $f'(0) = 0$.

b) Show that f' is unbounded on $[0, 1]$.

Solution: Let

$$x_n = \sqrt{\frac{1}{2n\pi}}.$$

Notice that $x_n \to 0$ and $n \to \infty$ but that $f'(x_n) = -2(\sqrt{2n\pi})$ which is unbounded as $n \to \infty$. So f' is unbounded near 0.

3 a) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies $f'(x) \geq 0$ for all $x \in [a, b]$. Show that f is increasing on $[a, b]$. (That is $f(x) \geq f(y)$ if $x > y$.)

Solution: Let $x > y$ be two points in $[a, b]$ we know by the mean value theorem that there is a c between x and y such that

$$f(x) - f(y) = f'(c)(x - y) \geq 0.$$

So $f(x) \geq f(y)$. Thus f is increasing on $[a, b]$.
b) If in addition $f'(x)$ is not identically zero on any sub-interval of $[a, b]$ then f is strictly increasing. (That is $f(x) > f(y)$ if $x > y$.)

Solution: Suppose that f is not strictly increasing. Then there is some x and y with $x > y$ such that $f(x) = f(y)$. But then $f(z) = f(x)$ for all $z \in [y, x]$. That is f is constant on $[y, x]$. But this implies that $f'(z) = 0$ on (y, x) contradicting the assumption. Thus f is strictly increasing.

4) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. For $c \in (a, b)$ we know that f restricted to $[a, c]$ and to $[c, b]$ gives an integrable function too (you do not have to prove this). Using the definition of the integral (either Riemann or Darboux) show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Hint: Given $\epsilon > 0$ choose a good partition that shows that both the right and left sides of the above equation are with ϵ of each other.

Solution: Since f is integrable we know that for any $\epsilon > 0$ there is some partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Now let $\mathcal{P}' = \mathcal{P}$ with the point c added. Since $\mathcal{P}'$ is a refinement of $\mathcal{P}$ we know the above inequality holds for $\mathcal{P}'$ too (since the upper sum cannot be larger and the lower sum cannot be smaller). We know that $\int_a^b f(x)\,dx$ must be between $U(f, \mathcal{P}')$ and $L(f, \mathcal{P}')$ since it is equal to the upper and lower Darboux integral. Moreover notice that if $\mathcal{P}_1$ consists of the points of $\mathcal{P}'$ that are in $[a, c]$ and $\mathcal{P}_2$ are the points of $\mathcal{P}'$ that are in $[c, b]$, then

$$U(f, \mathcal{P}') = U(f|_{[a,c]}, \mathcal{P}_1) + U(f_{[c.b]}, \mathcal{P}_2).$$

So

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx \leq U(f, \mathcal{P}').$$

Similarly

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \underline{\int_a^c} f(x)\,dx + \underline{\int_c^b} f(x)\,dx \geq L(f, \mathcal{P}').$$

That is $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ is between $L(f, \mathcal{P}')$ and $U(f, \mathcal{P}')$. Thus we know that

$$\left| \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \right) - \int_a^b f(x)\,dx \right| < \epsilon$$

since both the quantities on the left are in an interval of length less than ϵ, namely $[L(f, \mathcal{P}'), U(f, \mathcal{P}')]$. Since the above inequality is true for all $\epsilon > 0$ we know the left hand side is zero and this establishes our desired equality.

5) Answer the following questions **T**rue or **F**alse. Circle either **T** or **F** to indicate your answer. You do not need to justify your answer.

1. Any union of sets of measure zero have measure zero.

F

2. Every continuous function has an anti-derivative.

T

3. Given any two partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$ and any function $f : [a, b] \to \mathbb{R}$ we must have $L(f, \mathcal{P}) \le U(f, \mathcal{Q})$.

T

4. For any f and g with $g(x) \neq 0$ near c we have $\lim\limits_{x = \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$.

F

5. A Lipschitz function is differentiable.

F

6. If $f : [a, b] \to \mathbb{R}$ is integrable and $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.

F

7. A function $f : [a, b] \to \mathbb{R}$ is continuous at $c \in (a, b)$ if and only if the oscillation of f at c is zero. Recall the oscillation of f at c is

$$\mathrm{osc}_c(f) = \lim_{t \to 0} (\sup\{f(x) : x \in [c - t, c + t]\} - \inf\{f(x) : x \in [c - t, c + t]\}).$$

T

8. A continuously differentiable function on a compact interval is Lipschitz on that interval.

T

9. If a function $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ then it is integrable on $[a, b]$.

T

10. A function that is Riemannian integrable on $[a, b]$ must be bounded on $[a, b]$.

T