Math 4318 Midterm Exam 1 Spring 2011
Solutions

1) For a fixed $c \in [a,b]$ define the function

$$\delta_c : C^0([a,b]) \to \mathbb{R} : f \mapsto f(c).$$

If we give $C^0([a,b])$ the usual sup-norm $\|\cdot\|_\infty$, then show that δ_c is continuous.
Solution: Notice that

$$|\delta_c(f) - \delta_c(g)| = |f(c) - g(c)| \leq \|f - g\|_\infty.$$

So given $\epsilon > 0$ let $\delta = \epsilon$ and we see that if $\|f - g\|_\infty < \delta$ then

$$|\delta_c(f) - \delta_c(g)| < \delta = \epsilon.$$

So δ_c is continuous.

2) Show that for any function $f \in C^1([a,b])$ there is a sequence of polynomials p_n that converge to f in the $\|\cdot\|_{C^1}$ norm (that is the p_n converge uniformly to f and p_n' converge uniformly to f'). Hint: use the Weierstrass Theorem to approximate f' first.
Solution: Since f is C^1 we know that $f' \in C^0([a,b])$. Thus the Weierstrass Theorem tells us that for each n there is a polynomial q_n such that

$$\|f' - q_n\|_\infty < 1/n.$$

Thus we see that $\{q_n\}$ converges uniformly to f' on $[a,b]$. Let

$$p_n(x) = f(a) + \int_a^x q_n(y)\, dy.$$

Notice that

$$\begin{aligned} |f(x) - p_n(x)| &= \left| \int_a^x f'(y)\, dy + f(a) - \left(f(a) + \int_a^x q_n(y)\, dy \right) \right| \\ &= \left| \int_a^x f'(y) - q_n(x)\, dy \right| \leq \int_a^x |f'(x) - q_n(x)|\, dy \leq \|f' - q_n\|_\infty (b-a). \end{aligned}$$

Thus $\|f - p_n\|_\infty \leq \|f' - q_n\|_\infty (b-a)$ and $\|f - p_n\|_{C^1} = \|f - p_n\|_\infty + \|f' - p_n'\|_\infty \leq (1+b-a)\|f' - q_n\|_\infty \leq (1+b-a)\frac{1}{n}$. Thus we see that $\{p_n\}$ converges to f in the C^1-norm.

3) a) Give and example of a sequence of functions $\{f_n\}$ on the interval $[a,b]$ that are integrable and converge point-wise to a function f that is also integrable but where

$$\int_a^b f(x)\, dx \neq lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

Solution: Let

$$f(x) = \begin{cases} n^2 x & x \in [0, 1/n] \\ -n^2 x + 2n & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

Notice that $\{f_n\}$ converges point-wise to $f(x) = 0$. Clearly

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n} (-n^2x + 2n)\,dx \\ &= \frac{1}{2}n^2x^2|_0^{1/n} + (-\frac{1}{2}n^2x^2 + 2nx)|_{1/n}^{2/n} \\ &= \frac{1}{2}(1-0) + (-\frac{1}{2}(4-1) + 2n(2/n - 1/n) = 1\end{aligned}$$

and

$$\int_0^1 f(x)\,dx = 0.$$

So $\int_a^b f(x)\,dx \neq lim_{n\to\infty} \int_a^b f_n(x)\,dx$.

b) Show that if $f_n \to f$ uniformly on $[a, b]$ then f must be integrable on $[a, b]$. (You only need to show f is integrable, but not anything about the integral.)

Solution: Recall a function is integrable if and only if it is bounded and continuous almost everywhere. Since $f_n \to f$ uniformly we know that given $\epsilon = 1$ there is an N such that $|f_n(x) - f(x)| \leq 1$ for all $n \geq N$. Now since f_N is integrable it is bounded. Say M is the bound on f_N. Then $|f(x)| \leq M + 1$. So f is bounded too.

Now let C_n be the set of points at which f_n is discontinuous and C the set of points at which f is discontinuous. We know that if $x \notin \cap C_n$ then f is continuous so $x \notin C$. In other works $C \subset \cup C_n$. We know that since f_n is integrable the sets C_n have measure zero. The countable union of sets of measure zero have measure zero so $\cup C_n$ has measure zero. Finally C being a subset of a set of measure zero has measure zero. Thus f in integrable (by the Riemann-Lebesgue Theorem).

Another solution: As above we know f is bounded (since the f_n are). Now given any $\epsilon > 0$ there is some N such that $|f(x) - f_n(x)| \leq \epsilon/3$ for all $n \geq N$ and $x \in [a, b]$. Thus $f_n(x) - \epsilon/3 \leq f(x) \leq f_n(x) + \epsilon/3$ for all $x \in [a, b]$ and $n \geq N$. Now since f_N is integrable there is a partition $\mathcal{P}$ such that $U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) < \epsilon/3$. So

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq U(f_N + \epsilon/3, \mathcal{P}) - L(f_N - \epsilon/3, \mathcal{P}) = U(f_N, \mathcal{P}) - L(f_N, \mathcal{P}) + 2\epsilon/3 < \epsilon.$$

Thus f is integrable.

4) a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(x) = g(x) + c$ where $c \in \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\|g(x)\| \leq M\|x\|^2$$

for some constant M. Use the definition of derivative to prove that $Df(0) = 0$.

Solution: Notice that $\|f(0) - c\|] = \|g(0)\| \leq M\|0\| = 0$. So $f(0) = c$. Consider

$$\begin{aligned}\lim_{x\to 0} \frac{\|f(x) - f(0) - 0(x-0)\|}{\|x - 0\|} &= \lim_{x\to 0} \frac{\|c + g(x) - c\|}{\|x\|} \\ &= \lim_{x\to 0} \frac{\|g(x)\|}{\|x\|} \leq \lim_{x\to c} \frac{M\|x\|^2}{\|x\|} \\ &= \lim_{x\to 0} M\|x\| = 0.\end{aligned}$$

So $Df(0) = 0$.

b) Compute the derivative of $f(x, y, z) = (x^2 z, 3x + 4z)$.
Solution:

$$Df = \begin{bmatrix} 2xz & 0 & x^2 \\ 3 & 0 & 4 \end{bmatrix}$$

5) Answer the following questions **T**rue or **F**alse. Circle either **T** or **F** to indicate your answer. You do not need to justify your answer.

1. Smooth functions are analytic.

   F We had a counter example in class.

2. If a sequence of function $\{f_n\}$ that are integrable on $[a, b]$ and converge to f points-wise then f is integrable.

   F We had a counter example in class.

3. If the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ exists then all the partial derivatives of the coordinate functions exist and are continuous.

   F We had a counter example in class.

4. If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c$ then it defines an differentiable function on the interval $(-|c|, |c|)$.

   T The hypothesis implies that the radius of convergence is at least $|c|$.

5. Given any continuous function $f : [5, 10] \to \mathbb{R}$ there is a sequence of polynomials that converge to f uniformly on $[5, 10]$.

   T This is the Weierstrass theorem.

6. If all the directional derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ exist and are continuous then the derivative of f exists.

   T If all the directional derivatives are continuous then all the partial derivatives are continuous so this follows from a theorem in class.

7. A continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable somewhere.

   F We had a counter example in class.

8. A Lipschitz map must be differentiable everywhere.

   F Consider $f(x) = |x|$.

9. A Lipschitz map on a compact interval must have a fixed point.

F Only if the LIpschitz constant is less than 1 (so the mapping is a contraction).

10. Cauchy sequences must converge in a Banach space.

T This is part of the definition of a Banach space.