Outline for Midterm #1
Math 4318, Spring 2011

I. Single Variable Functions: Differentiation

A. Definitions and first properties

1. We recalled a few definitions concerning limits form Analysis I.

2. A function $f : (a,b) \to \mathbb{R}$ is **differentiable** at a point $p \in (a,b)$ if the limit

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$$

exists. If the limit exists we denote it by $f'(p)$ and call it the **derivative of** f **at** p.

3. Interpreting the derivative as a slope we could also define

$$f'(p) = \lim_{t\to p} \frac{f(t)-f(p)}{t-p}.$$

4. Theorem: If f is differentiable at p then f is continuous at p.

5. Theorem: Suppose f and g are differentiable at p. Then

- **i.** The function $f \pm g$ is differentiable at p with

$$(f\pm g)'(p) = f'(p) \pm g'(p).$$

- **ii.** The function fg is differentiable at p with

$$(fg)'(p) = f'(p)g(p) + f(p)g'(p).$$

- **iii.** If $g(p) \neq 0$ then the function f/g is differentiable at p and

$$(f/g)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{g^2(p)}.$$

- **iv.** If $h(x) = c$ for some constant $c \in \mathbb{R}$ then $h'(x) = 0$ for all x.
- **v.** If $h(x) = x$ then $h'(x) = 1$ for all x.
- **vi.** If f is differentiable at p and g is differentiable at $f(p)$ then $g\circ f$ is differentiable at p and

$$(g\circ f)'(p) = g'(f(p))f'(p).$$

6. Discussed the example

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

It is differentiable on $\mathbb{R}$ (including 0) but its derivative is not continuous at 0.

7. The n**th order derivative of** f **at** p is defined to be the derivative of the $(n-1)$st derivative of f at p, if it exists. Denote the nth derivative by $f^{(n)}(p)$. With this notation we take $f^{(0)}$ to denote f.

8. A function is **continuously differentiable of order** r **on the interval** I if $f^{(r)}$ exists and is continuous on I. (Note this implies $f^{(k)}$ exists and is continuous on I for all $k \leq r$.) We denote the set of continuously rth order differentiable functions on I by $C^r(I)$. The set $C^\infty(I)$ denotes functions whose derivative of all orders exist on I.

9. Clearly $C^r(I) \supset C^{r+1}(I)$. This inclusion is strict (that is for any r there are functions that are rth order continuously differentiable that are not continuously differentiable to order $r+1$). The first few examples are $f(x) = |x|, g(x) = x|x|, h(x) = |x|^3$...

B. The Mean Value Theorem

1. **Theorem (Mean value theorem):** If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b) then there is a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

2. **Theorem:** If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ and f has a local maximum of minimum at c then $f'(c) = 0$.
3. **Corollary:** If f is differentiable on (a, b) and there is some K such that $|f'(x)| leqK$ for all $x \in (a, b)$ then

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a, b]$. (In particular f is Lipschitz.)
4. Suppose that f is differentiable on (a, b)
   i. If $f'(x) \geq 0$ for all x then f is increasing.
   ii. If $f'(x) \leq 0$ for all x then f is decreasing.
   iii. If $f'(x) > 0$ for all x then f is strictly increasing.
   iv. If $f'(x) < 0$ for all x then f is strictly decreasing.
5. **Corollary (Intermediate value theorem for derivatives):** Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable. Given any λ between $f'(a)$ and $f'(b)$ there is some point $c \in (a, b)$ such that $f'(c) = \lambda$.
6. **Corollary (Inverse function theorem):** Suppose that $f : (a, b) \to \mathbb{R}$ is a differentiable function and $f'(x) \neq 0$ for all $x \in (a, b)$. Then f is a bijection onto its image. The inverse of f is continuous and

$$(f^{-1})'(y) = \frac{1}{f(x)}$$

where $y = f(x)$.
7. **Theorem:** Let f and g be two continuous functions on $[a, b]$ that are differentiable on (a, b). There is a point $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

8. **Corollary (L'Hopital's Rule):** Let f, g be differentiable functions on (a, b). Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to b$ and $g(x)$ and $g'(x)$ are not zero near b. If

$$\lim_{x \to b} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to b} \frac{f(x)}{g(x)} = L.$$

There are of course many other cases of L'Hopital's rule.

C. Taylor Polynomials

1. Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. If $f(c), f'(c), \ldots, f^{(n)}(c)$ exist then the **nth order Taylor polynomial of f at c** is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

2. **Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a function such that $f', f'', \ldots, f^{(n+1)}$ all exists on (a, b). Let $P_n(x)$ be the nth order Taylor polynomial of f at c. Then for each $x \in [a, b]$ there is some point t between c and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(t)}{(n + 1)!}(x - c)^{n+1}.$$

II. Single Variable Functions: Integration

A. Riemann integrability

1. A **partition** of an interval $[a, b]$ is a finite collection of points $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ such that $x_i < x_{i+1}$, $x_0 = a$ and $x_n = b$. The **intervals of a partition** are $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots n$. The **lengths of a partition** are $\Delta x_i = x_i - x_{i-1}$. The **size of the partition** $\mathcal{P}$ is $\|\mathcal{P}\| = \max\{\Delta x_1, \ldots \Delta x_n\}$.

2. A **tagged partition** $\mathcal{P}^t$ is a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ together a choice of point t_i in each interval $[x_{i-1}, x_i]$.

3. If $f : [a, b] \to \mathbb{R}$ is a function and $\mathcal{P}$ is a tagged partition of $[a, b]$ then the **Riemann sum of** f **associated to** $\mathcal{P}^t$ is

$$S(f, \mathcal{P}^t) = \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

We say f is **Riemann integrable** if there is come number $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there is some $\delta > 0$ such that for any tagged partition $\mathcal{P}^t$ of $[a, b]$ with $\|\mathcal{P}^t\| < \delta$ we have

$$|S(f, \mathcal{P}^t) - I| < \epsilon.$$

Let $\mathcal{R}([a, b])$ be the set of Riemann integrable functions on $[a, b]$.

4. Lemma: If $f \in \mathcal{R}([a, b])$ then the I in the definition above is uniquely determined.

5. If $f \in \mathcal{R}([a, b])$ then the **Riemann integral of** f **over** $[a, b]$ is the number I in the definition above. We denoted this number by

$$\int_a^b f(x)\, dx.$$

6. Proposition:

i. $\mathcal{R}([a, b])$ is a vector space.

ii. The map

$$\mathcal{R}([a, b]) \to \mathbb{R} : f \mapsto \int_a^b f(x)\, dx$$

is a linear map.

iii. The constant function $f(x) = k$ is in $\mathcal{R}([a, b])$ for any $[a, b]$ and

$$\int_a^b k\, dx = k(b - a)$$

iv. If $f(x) \leq g(x0$ and $f, g \in \mathcal{R}([a, b])$ then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

7. Theorem: If $f \in \mathcal{R}([a, b])$ then f is bounded on $[a, b]$.

B. Darboux integrability

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P} = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$. For each $i = 1, \ldots, n$ set

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

The **lower sum of** f **associated to the partition** $\mathcal{P}$ is

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i$$

and the **upper sum** is

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i.$$

The **lower integral of** f **over** $[a, b]$ is

$$\underline{\int_a^b} f(x)\, dx = \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\}$$

and the **upper integral of** f **over** $[a, b]$ is

$$\overline{\int_a^b} f(x)\, dx = \int \{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\}.$$

We say f is **Darboux integrable on** $[a, b]$ if

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

2. **Theorem:** A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. And if it is integrable then

$$\int_a^b f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

3. We say a partition $\mathcal{P}'$ **refines** a partition $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}'$.
4. **Lemma:** If $\mathcal{P}'$ refines $\mathcal{P}$ then

$$L(f, \mathcal{P}) \le L(f, \mathcal{P}') \le U(f, \mathcal{P}') \le U(f, \mathcal{P}).$$

5. **Lemma:** A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\epsilon > 0$ there is some partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

6. **Corollary:** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there is some partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

7. **Corollary:** A continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

C. Sets of measure zero and the Riemann-Lebesgue theorem

1. A set $S \subset \mathbb{R}$ has **measure zero** or is **a set of measure zero** if for every $\epsilon > 0$ there is a countable collection of intervals (a_i, b_i) such that $S \subset \cup(a_i, b_i)$ (that is the intervals are a cover of S) and

$$\sum (b_i - a_i) \le \epsilon.$$

2. **Lemma:**
   - **i.** A finite set has measure zero.
   - **ii.** A subset of a set of measure zero has measure zero.
   - **iii.** A countable union of sets of measure zero has measure zero.
   - **iv.** A countable set has measure zero.
   - **v.** The middle thirds Cantor set has measure zero.
3. A function $f : [a, b] \to \mathbb{R}$ is **continuous almost everywhere** if the set of points at which f is discontinuous has measure zero.
4. **Theorem (the Riemann-Lebesgue Theorem):** Function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and continuous almost everywhere.

5. **Corollary:** Every continuous and bounded piecewise continuous function on $[a, b]$ is Riemann integrable.
6. **Corollary:** If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $g : [c, d] \to \mathbb{R}$ is continuous with $f([a, b]) \subset [c, d]$ then $g \circ f$ is Riemann integrable.
7. **Corollary:** If $f \in \mathcal{R}([a, b])$ then $|f| \in \mathcal{R}([a, b])$ and
$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$
8. **Corollary:** The product of Riemann integrable functions is Riemann integrable.
9. **Corollary:** If $a < c < b$ then $f \in \mathcal{R}([a, b])$ if and only if $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$. Moreover,
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$
10. **Corollary:** Suppose $f : [a.b] \to \mathbb{R}$ is non-negative (that is $f(x) \geq 0$) and integrable. Then $\int_a^b f(x)\, dx = 0$ implies that $f(x) = 0$ at every point x where f is continuous.

D. Fundamental theorem of calculus

1. **Theorem (Fundamental theorem of calculus I):** Let $f \in \mathcal{R}([a, b])$ and set
$$F(x) = \int_a^x f(t)\, dt$$
for all $x \in [a, b]$. Then the function F is continuous on $[a, b]$ and if f is continuous at c then F is differentiable at c and $F'(c) = f(c)$.
2. **Corollary:** Any continuous function $f : [a, b] \to \mathbb{R}$ has han anti-derivative (*i.e.* a function F such that $F'(x) = f(x)$ for all $x \in [a, b]$).
3. **Theorem (Fundamental theorem of calculus II):** If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$ then
$$\int_a^b f'(x)\, dx = f(b) - f(a).$$
4. **Theorem (Integration by parts):** Let f and g be functions on $[a, b]$ for which f' and g' are both integrable. Then
$$\int_a^b f'(x)g(x)\, dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\, dx.$$
5. **Theorem (Change of variables):** Let $\phi : [c, d] \to \mathbb{R}$ have continuous derivative and $f : [a, b] \to \mathbb{R}$ be continuous with $[a, b] \subset \phi([c, d])$. Then
$$\int_c^d f(\phi(t))\phi'(t)\, dt = \int_{\phi(c)}^{\phi(d)} f(x)\, dx.$$