Outline for Midterm #2
Math 4318, Spring 2011

I. Single Variable Functions: Differentiation

A. Definitions and first properties

1. We recalled a few definitions concerning limits form Analysis I.

2. A function $f : (a,b) \to \mathbb{R}$ is **differentiable** at a point $p \in (a,b)$ if the limit

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

exists. If the limit exists we denote it by $f'(p)$ and call it the **derivative of f at p**.

3. Interpreting the derivative as a slope we could also define

$$f'(p) = \lim_{t \to p} \frac{f(t) - f(p)}{t - p}.$$

4. Theorem: If f is differentiable at p then f is continuous at p.

5. Theorem: Suppose f and g are differentiable at p. Then

i. The function $f \pm g$ is differentiable at p with

$$(f \pm g)'(p) = f'(p) \pm g'(p).$$

ii. The function fg is differentiable at p with

$$(fg)'(p) = f'(p)g(p) + f(p)g'(p).$$

iii. If $g(p) \neq 0$ then the function f/g is differentiable at p and

$$(f/g)'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{g^2(p)}.$$

iv. If $h(x) = c$ for some constant $c \in \mathbb{R}$ then $h'(x) = 0$ for all x.

v. If $h(x) = x$ then $h'(x) = 1$ for all x.

vi. If f is differentiable at p and g is differentiable at $f(p)$ then $g \circ f$ is differentiable at p and

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

6. Discussed the example

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

It is differentiable on $\mathbb{R}$ (including 0) but its derivative is not continuous at 0.

7. The **nth order derivative of f at p** is defined to be the derivative of the $(n-1)$st derivative of f at p, if it exists. Denote the nth derivative by $f^{(n)}(p)$. With this notation we take $f^{(0)}$ to denote f.

8. A function is **continuously differentiable of order r on the interval I** if $f^{(r)}$ exists and is continuous on I. (Note this implies $f^{(k)}$ exists and is continuous on I for all $k \leq r$.) We denote the set of continuously rth order differentiable functions on I by $C^r(I)$. The set $C^\infty(I)$ denotes functions whose derivative of all orders exist on I.

9. Clearly $C^r(I) \supset C^{r+1}(I)$. This inclusion is strict (that is for any r there are functions that are rth order continuously differentiable that are not continuously differentiable to order $r+1$). The first few examples are $f(x) = |x|, g(x) = x|x|, h(x) = |x|^3$...

B. The Mean Value Theorem

1. **Theorem (Mean value theorem):** If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b) then there is a point $c \in (a, b)$ such that
$$f(b) - f(a) = f'(c)(b - a).$$
2. **Theorem:** If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ and f has a local maximum of minimum at c then $f'(c) = 0$.
3. **Corollary:** If f is differentiable on (a, b) and there is some K such that $|f'(x)| leqK$ for all $x \in (a, b)$ then
$$|f(x) - f(y)| \leq K|x - y|$$
for all $x, y \in [a, b]$. (In particular f is Lipschitz.)
4. Suppose that f is differentiable on (a, b)
   i. If $f'(x) \geq 0$ for all x then f is increasing.
   ii. If $f'(x) \leq 0$ for all x then f is decreasing.
   iii. If $f'(x) > 0$ for all x then f is strictly increasing.
   iv. If $f'(x) < 0$ for all x then f is strictly decreasing.
5. **Corollary (Intermediate value theorem for derivatives):** Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable. Given any λ between $f'(a)$ and $f'(b)$ there is some point $c \in (a, b)$ such that $f'(c) = \lambda$.
6. **Corollary (Inverse function theorem):** Suppose that $f : (a, b) \to \mathbb{R}$ is a differentiable function and $f'(x) \neq 0$ for all $x \in (a, b)$. Then f is a bijection onto its image. The inverse of f is continuous and
$$(f^{-1})'(y) = \frac{1}{f(x)}$$
where $y = f(x)$.
7. **Theorem:** Let f and g be two continuous functions on $[a, b]$ that are differentiable on (a, b). There is a point $c \in (a, b)$ such that
$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$
8. **Corollary (L'Hopital's Rule):** Let f, g be differentiable functions on (a, b). Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to b$ and $g(x)$ and $g'(x)$ are not zero near b. If
$$\lim_{x \to b} \frac{f'(x)}{g'(x)} = L$$
then
$$\lim_{x \to b} \frac{f(x)}{g(x)} = L.$$
There are of course many other cases of L'Hopital's rule.

C. Taylor Polynomials

1. Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. If $f(c), f'(c), \ldots, f^{(n)}(c)$ exist then the **nth order Taylor polynomial of f at c** is
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$
2. **Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a function such that $f', f'', \ldots, f^{(n+1)}$ all exists on (a, b). Let $P_n(x)$ be the nth order Taylor polynomial of f at c. Then for each $x \in [a, b]$ there is some point t between c and x such that
$$f(x) = P_n(x) + \frac{f^{(n+1)}(t)}{(n + 1)!}(x - c)^{n+1}.$$

II. Single Variable Functions: Integration

A. Riemann integrability

1. A **partition** of an interval $[a, b]$ is a finite collection of points $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ such that $x_i < x_{i+1}$, $x_0 = a$ and $x_n = b$. The **intervals of a partition** are $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots n$. The **lengths of a partition** are $\Delta x_i = x_i - x_{i-1}$. The **size of the partition** $\mathcal{P}$ is $\|\mathcal{P}\| = \max\{\Delta x_1, \ldots \Delta x_n\}$.

2. A **tagged partition** $\mathcal{P}^t$ is a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ together a choice of point t_i in each interval $[x_{i-1}, x_i]$.

3. If $f : [a, b] \to \mathbb{R}$ is a function and $\mathcal{P}$ is a tagged partition of $[a, b]$ then the **Riemann sum of f associated to $\mathcal{P}^t$** is

$$S(f, \mathcal{P}^t) = \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

We say f is **Riemann integrable** if there is come number $I \in \mathbb{R}$ such that for every $\epsilon > 0$ there is some $\delta > 0$ such that for any tagged partition $\mathcal{P}^t$ of $[a, b]$ with $\|\mathcal{P}^t\| < \delta$ we have

$$|S(f, \mathcal{P}^t) - I| < \epsilon.$$

Let $\mathcal{R}([a, b])$ be the set of Riemann integrable functions on $[a, b]$.

4. Lemma: If $f \in \mathcal{R}([a, b])$ then the I in the definition above is uniquely determined.

5. If $f \in \mathcal{R}([a, b])$ then the **Riemann integral of f over** $[a, b]$ is the number I in the definition above. We denoted this number by

$$\int_a^b f(x)\, dx.$$

6. Proposition:

i. $\mathcal{R}([a, b])$ is a vector space.

ii. The map

$$\mathcal{R}([a, b]) \to \mathbb{R} : f \mapsto \int_a^b f(x)\, dx$$

is a linear map.

iii. The constant function $f(x) = k$ is in $\mathcal{R}([a, b])$ for any $[a, b]$ and

$$\int_a^b k\, dx = k(b - a)$$

iv. If $f(x) \leq g(x)$ and $f, g \in \mathcal{R}([a, b])$ then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

7. Theorem: If $f \in \mathcal{R}([a, b])$ then f is bounded on $[a, b]$.

B. Darboux integrability

1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P} = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$. For each $i = 1, \ldots, n$ set

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

The **lower sum of f associated to the partition $\mathcal{P}$** is

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i$$

and the **upper sum** is

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i.$$

The **lower integral of f over** $[a, b]$ is

$$\underline{\int_a^b} f(x)\, dx = \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\}$$

and the **upper integral of f over** $[a, b]$ is

$$\overline{\int_a^b} f(x)\, dx = \int \{U(f, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a, b]\}.$$

We say f is **Darboux integrable on** $[a, b]$ if

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

2. **Theorem:** A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable. And if it is integrable then

$$\int_a^b f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx.$$

3. We say a partition $\mathcal{P}'$ **refines** a partition $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}'$.
4. **Lemma:** If $\mathcal{P}'$ refines $\mathcal{P}$ then

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P}).$$

5. **Lemma:** A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\epsilon > 0$ there is some partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

6. **Corollary:** A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there is some partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

7. **Corollary:** A continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

C. Sets of measure zero and the Riemann-Lebesgue theorem

1. A set $S \subset \mathbb{R}$ has **measure zero** or is **a set of measure zero** if for every $\epsilon > 0$ there is a countable collection of intervals (a_i, b_i) such that $S \subset \cup(a_i, b_i)$ (that is the intervals are a cover of S) and

$$\sum (b_i - a_i) \leq \epsilon.$$

2. **Lemma:**
   - **i.** A finite set has measure zero.
   - **ii.** A subset of a set of measure zero has measure zero.
   - **iii.** A countable union of sets of measure zero has measure zero.
   - **iv.** A countable set has measure zero.
   - **v.** The middle thirds Cantor set has measure zero.
3. A function $f : [a, b] \to \mathbb{R}$ is **continuous almost everywhere** if the set of points at which f is discontinuous has measure zero.
4. **Theorem (the Riemann-Lebesgue Theorem):** Function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and continuous almost everywhere.

5. **Corollary:** Every continuous and bounded piecewise continuous function on $[a, b]$ is Riemann integrable.
6. **Corollary:** If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $g : [c, d] \to \mathbb{R}$ is continuous with $f([a, b]) \subset [c, d]$ then $g \circ f$ is Riemann integrable.
7. **Corollary:** If $f \in \mathcal{R}([a, b])$ then $|f| \in \mathcal{R}([a, b])$ and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

8. **Corollary:** The product of Riemann integrable functions is Riemann integrable.
9. **Corollary:** If $a < c < b$ then $f \in \mathcal{R}([a, b])$ if and only if $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$. Moreover,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

10. **Corollary:** Suppose $f : [a.b] \to \mathbb{R}$ is non-negative (that is $f(x) \geq 0$) and integrable. Then $\int_a^b f(x)\, dx = 0$ implies that $f(x) = 0$ at every point x where f is continuous.

D. Fundamental theorem of calculus

1. **Theorem (Fundamental theorem of calculus I):** Let $f \in \mathcal{R}([a, b])$ and set

$$F(x) = \int_a^x f(t)\, dt$$

for all $x \in [a, b]$. Then the function F is continuous on $[a, b]$ and if f is continuous at c then F is differentiable at c and $F'(c) = f(c)$.

2. **Corollary:** Any continuous function $f : [a, b] \to \mathbb{R}$ has han anti-derivative (*i.e.* a function F such that $F'(x) = f(x)$ for all $x \in [a, b]$).
3. **Theorem (Fundamental theorem of calculus II):** If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$ then

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

4. **Theorem (Integration by parts):** Let f and g be functions on $[a, b]$ for which f' and g' are both integrable. Then

$$\int_a^b f'(x) g(x)\, dx = f(x)g(x)|_a^b - \int_a^b f(x) g'(x)\, dx.$$

5. **Theorem (Change of variables):** Let $\phi : [c, d] \to \mathbb{R}$ have continuous derivative and $f : [a, b] \to \mathbb{R}$ be continuous with $[a, b] \subset \phi([c, d])$. Then

$$\int_c^d f(\phi(t))\phi'(t)\, dt = \int_{\phi(c)}^{\phi(d)} f(x)\, dx.$$

E. Improper integrals

1. If $f : (a, b] \to \mathbb{R}$ is a function and $c \in (a, b]$ then if f is integrable on $[c, b]$ set

$$I_c = \int_c^b f(x)\, dx.$$

Define the **improper integral of** f **on** $[a, b]$ to be

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} I_c$$

if the limit exists. (One can similarly define the improper integral of $f : [a, b) \to \mathbb{R}$.)

2. If $f : [a,b] \to \mathbb{R}$ is integrable then this definition agrees with the Riemann integral of f.

III. Sequences of functions and function spaces

A. Sequences of functions

1. If $S \subset \mathbb{R}$ and $\{f_n : S \to \mathbb{R}\}$ is a sequence of functions on f (that is if $\mathcal{F}(S, \mathbb{R})$ is the set of all functions from S to $\mathbb{R}$, then $\{f_n\}$ is a sequence in the set $\mathcal{F}(S, \mathbb{R})$), the we say the sequence **converges point wise to** f if for each $x \in S$ we have the sequence of numbers $\{f_n(x)\}$ converges to $f(x)$. That is for all $x \in S$ and $\epsilon > 0$ there is some N such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$.

2. We say that a sequence of functions $\{f_n : S \to \mathbb{R}\}$ **converges uniformly to** f **on** S if for all $\epsilon > 0$ there is a N such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and $x \in S$.

3. Lemma: The sequence $\{f_n : S \to \mathbb{R}\}$ converges uniformly to some function on S if and only if for all $\epsilon > 0$ there is an N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ and $x \in S$.

4. Theorem: If $f_n \to f$ uniformly on S and the f_n are continuous at $c \in S$ then f is continuous at c.

5. Theorem: If $\{f_n|\}$ is a sequence in $\mathcal{R}([a,b])$ and $f_n \to f$ uniformly on $[a,b]$ then $f \in \mathcal{R}(a,b])$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

6. Theorem: Let $\{f_n : [a,b] \to \mathbb{R}\}$ be a sequence of functions. Suppose that

i. there is some $c \in [a,b]$ such that $\{f_n(c)\}$ converges,

ii. the functions f_n are all differentiable on $[a,b]$, and

iii. $f_n' \to g$ uniformly on $[a,b]$ for some function g.

Then there is some function f such that $f_n \to f$ uniformly on $[a,b]$ and $f' = g$.

B. Metric space topology

1. Let X be a set. A **metric** on X is a function

$$d : X \times X \to \mathbb{R}$$

such that

i. $d(p,q) \geq 0$ for all $p, q \in X$,

ii. $d(p,q) = 0$ if and only if $p = q$

iii. $d(p,q) = d(q,p)$, and

iv. $d(p,q) \leq d(p,r) + d(r,q)$ for all $p,q,r \in X$. We think of $d(p,q)$ as being the distance between p and q. The pair (X,d) is called a **metric space**.

2. Example: Given a norm $\|\cdot\|$ on a vector space V one has a metric associated to the norm by setting

$$d(v,w) = \|v - w\|$$

for all $v, w \in V$. Thus if $\langle\cdot,\cdot\rangle$ is an inner product on V then $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm and hence we get a metric associated to the inner product too.

3. Given a metric space (X,d) most all the concepts form Analysis I can be defined for (X,d) since they only depended on some notion of distance (where there we used the standard norm on $\mathbb{R}^n$). For example

i. A set $U \subset X$ is **open** if for all $p \in U$ there is some $r > 0$ such that $B_r(p) \subset U$ where $B_r(p) = \{x \in X : d(x,p) < r\}$.

ii. A **neighborhood** of a point $p \in X$ is an open set N containing p.

iii. A subset $S \subset X$ has $p \in X$ as a **cluster point** if for all neighborhoods N of p we have $(N - \{p\}) \cap S \neq \emptyset$. (This is equivalent to saying for all $\epsilon > 0$ there is some $q \in S$ with $q \neq p$ and $d(q,p) < \epsilon$.)

iv. A set S in X is **closed** if it contains all of its cluster points.
v. A set S is bounded if there is some $p \in X$ and R such that $S \subset B_r(p)$.
vi. A sequence $\{x_n\}$ in X is **Cauchy** if for all $\epsilon > 0$ there is some N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Most all the other concepts can be rephrased in terms of d too.

4. A metric space is complete if every Cauchy sequence converges.

5. Theorem (from Analysis I): The space $(\mathbb{R}^n, d)$ is complete. (Here $d(x, y) = \sqrt{\sum (x_i - y_i)^2}$.)

6. Suppose V is a vector space and d is a metric on V that is complete. If d comes from a norm then (V, d) is called a **Banach space**. If d comes from an inner product then (V, d) is called a **Hilbert space**.

C. Function spaces

1. Let $D \subset \mathbb{R}$. We set

$$\mathcal{B}(D, \mathbb{R}) = \{\text{bounded functions } f : D \to \mathbb{R}\}$$

and for $f \in \mathcal{B}(D, \mathbb{R})$ let

$$\|f\|_\infty = \sup\{|f(x)| : x \in D\}.$$

We call this the **sup-norm** or **uniform norm** on $\mathcal{B}(D, \mathbb{R})$ and one easily checks that this is a norm.

2. Lemma: Let $\{f_n\}$ be a sequence in $\mathcal{B}(D, \mathbb{R})$. Then the following are equivalent
i. $f_n \to f$ uniformly on D.
ii. f_n converges to f in the sup-norm.
iii. $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

3. Theorem: The sequence $\{f_n\}$ in $\mathcal{B}(D, \mathbb{R})$ is Cauchy in the sup-norm if and only if it converges in the sup-norm. That is $(\mathcal{B}(D, \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

4. Let

$$C^0_b\{D\} = \{f \in \mathcal{B}(D, \mathbb{R}) : f \text{ is continuous}\}.$$

(Notice that if D is compact then $C^0_b(D) = C^0(D)$.)

5. Theorem: $C^0_b(D)$ is a Banach space (in the sup-norm).

6. Theorem: The set $\mathcal{R}([a, b])$ (which is a subset of $\mathcal{B}([a, b], \mathbb{R})$) is a Banach space in the sup-norm and the function

$$I : \mathcal{R}([a, b]) \to \mathbb{R} : f \to \int_a^b f(x)\, dx$$

is continuous.

7. For a function in $f \in C^n([a, b])$ and any $k \leq n$ define

$$\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + \ldots + \|f^{(k)}\|_\infty.$$

It is easy to check this is a norm on $C^n([a, b])$.

8. Theorem: $(C^n([a, b]), \|\cdot\|_{C^n})$ is a Banach space.

D. Approximation of functions

1. Theorem (Weierstrass): Polynomials are dense in $C^0([a, b])$. (That is for every $f \in C^0([a, b])$ and $\epsilon > 0$ there is a polynomial p such that $\|f - p\| < \epsilon$.)

2. Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ we define the **convolution** of f and g to be

$$f * g(x) = \int_{\mathbb{R}} f(x - t) g(t)\ , dt,$$

if the integral is well defined (and it will be if, for example, one of the functions has compact support and the functions are integrable on compact intervals).

3. A sequence of functions $K_n(x)$ is called an **approximation to the identity** if

i. $K_n(x) \geq 0$ for all x and n,
ii. $\int K_n(x)\,dx = 1$ for all n, and
iii. $\int_{|x|\geq\epsilon} K_n(x)\,dx \to 0$ as $n \to \infty$ for any $\epsilon > 0$.

4. **Lemma:** If $\{K_n\}$ is an approximation to the identity and f is a compactly supported continuous function then $(f * K_n) \to f$ uniformly.
5. **Lemma:** If p is a polynomial and $f \in \mathcal{R}([a,b])$ has compact support then $p * f$ is a polynomial.

E. Fixed point theorems and differential equations

1. **Theorem (Contraction mapping theorem):** Let (M,d) be a complete metric space and $f : M \to M$ a contraction mapping (that is there is some $0 \leq k < 1$ such that $d(f(x), f(y)) \leq kd(x,y)$ for all $x, y \in M$). Then there is a unique fixed point $p \in M$ for f. (That is $f(p) = p$ and p is the only point with this property.)
2. **Theorem (Existence ad uniqueness of solutions to ODEs):** Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function on a neighborhood D of (t_0, x_0) in $\mathbb{R}^2$. Assume that
   i. f is Lipschitz in the x variable (that is there is some K such that $|f(t,x) - f(t,x')| \leq K|x - x'|$) and
   ii. f is continuous on D.

   Then there is some $\delta > 0$ and a unique C^1 solution to the initial value problem
   $$\frac{dx}{dt} = f(t,x) \qquad x(t_0) = x_0$$
   on $(t_0 - \delta, t_0 + \delta)$. (That is there is some continuously differentiable function $\gamma : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ such that $\gamma'(t) = f(t, \gamma(t))$ and $\gamma(t_0) = x_0$.)
3. A function γ solves the initial value problem if and only if it is a fixed point of
   $$\Phi(\gamma)(t) = x_0 + \int_{t_0}^{t} f(s, \gamma(s))\,ds.$$

F. Compactness in function spaces

1. A subset $S \subset \mathcal{F}(D, \mathbb{R}) = \{\text{functions from } D \text{ to } \mathbb{R}\}$ is called **equicontinuous** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in D$ with $|x - y| < \delta$ and $f \in S$. (Of course sequences are subsets so we can talk about equicontinuous sequences.)
2. **Theorem:** A subset $S \subset \mathcal{F}(D, \mathbb{R})$ is compact (in the sup-norm) if and only if it is closed, bounded and equicontinuous.
3. Recall that a set in a metric space is called compact if every open cover of the set has a finite sub-cover. This is equivalent to saying that every sequence in the set has a convergent sub-sequence.
4. **Theorem (Arzelá-Ascoli Theorem):** Let $\{f_n\}$ be a sequence of functions $f_n : D \to \mathbb{R}$. If
   i. D is compact,
   ii. $\{f_n\}$ is bounded in the sup-norm and
   iii. $\{f_n\}$ is equicontinuous (not this implies that each f_n is continuous),

   then there is a sub-seqence of $\{f_n\}$ that converges uniformly on D.
5. **Theorem (Peano's Theorem):** Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function on a neighborhood D of (t_0, x_0) in $\mathbb{R}^2$. If f is continuous on D then there is some $\delta > 0$ and a unique C^1 solution to the initial value problem
   $$\frac{dx}{dt} = f(t,x) \qquad x(t_0) = x_0$$
   on $[t_0, t_0 + \delta]$.

6. Notice that when you drop the hypothesis of f being Lipschitz in the x variable you loose uniqueness of solutions. For example you can see that $x'(t) = \sqrt{x(t)}$ with the initial condition $x(0) = 0$ has infinitely many solutions. Indeed for $c \geq 0$ let $\gamma_c(t) = 0$ for $t \leq c$ and $\frac{1}{4}(t-c)^2$ for $t \geq c$ and we see that γ_c solves the initial value problem.

G. Series of functions

1. Given a sequence $\{g_k\}$ of functions on a set $D \subset \mathbb{R}$ we can associated the partial sums

$$s_n(x) = \sum_{k=0}^{n} g_k(x).$$

We say the **series** $\sum_{k=0}^{\infty} g_k$ **converges uniformly, respectively point-wise** to g if the sequence of partial sums $\{s_n\}$ converges uniformly, respectively point-wise, to g. We say the series **converges absolutely** if for each $x \in D$ the series of real numbers $\sum_{k=0}^{\infty} |g_k(x)|$ converges.

2. Theorem (Weierstrass M-test): Let $\{g_k\}$ be a sequence of functions on D and M_k be constants such that $|g_k(x)| \leq M_k$ for all $x \in D$. If $\sum_{k=0}^{\infty} M_k$ converges then $\sum_{k=0}^{\infty} g_k$ converges absolutely and uniformly on D.

3. Theorem: If $\sum_{k=0}^{\infty} g_k$ converges uniformly to g on $[a,b]$ and each $g_k \in \mathcal{R}([a,b])$ then $g \in \mathcal{R}([a,b])$ and

$$\int_a^b g(x)\,dx = \int_a^b \sum_{k=0}^{\infty} g_k(x)\,dx = \sum_{k=0}^{\infty} \int_a^b g_k(x)\,dx.$$

4. Suppose that $\sum_{k=0}^{\infty} g_k$ converges point-wise on $[a,b]$ and each of the g_k is differentiable. If $\sum_{k=0}^{\infty} g_k'$ converges uniformly on $[a,b]$ then

$$\left(\sum_{k=0}^{\infty} g_k(x)\right)' = \sum_{k=0}^{\infty} g_k'(x).$$

5. Theorem: There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable!

6. Facts we did not prove: Lipschitz functions are differentiable almost everywhere. The set of nowhere differentiable functions is dense in the set of continuous functions.

H. Power series

1. A **power series** about the point c is a series of the form

$$\sum_{k=0}^{\infty} a_n (x-c)^n.$$

2. Recall from Analysis I:

i. The **radius of convergence** is defined to be $R = \frac{1}{\rho}$ where

$$\rho = \limsup_k |a_k|^{\frac{1}{k}}.$$

ii. If $\lim_{n\to infty} \frac{|a_n|}{|a_{n+1}|}$ exists then it is equal to the radius of convergence.

iii. For any $|x-c| < R$ the series $\sum_{k=0}^{\infty} a_n(x-c)^n$ converges absolutely and for $|x-c| > R$ the series diverges.

iv. A power series converges uniformly on any compact subset of $(c-R, c+R)$.

v. A power series defines a continuous function on $(c-R, c+R)$.

3. Theorem: If R is the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Then f is differentiable of all orders on $(c-R, c+R)$ and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(x-c)^{n-k}.$$

4. **Corollary:** Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-c)^n$ converge on some interval (a, b) containing c. If $f(x) = g(x)$ for all $x \in (a, b)$ then $a_n = b_n$ for all n.
5. A function f on an open interval is called **analytic** if it can be represented as a power series with non-zero radius of convergence, about each point in its domain.
6. **Theorem:** if $f : (a, b) \to \mathbb{R}$ is C^∞ and there is some constant M such that

$$|f^{(k)}(x)| \leq M$$

for all $x \in (a, b)$ then f is analytic.
7. Given a function $f : (a, b) \to \mathbb{R}$ that is infinitely differentiable, $c \in (a, b)$ and $\delta > 0$ such that $I_{c,\delta} = [c-\delta, c+\delta] \subset (a, b)$, we can set $M^k_{c,\delta} = \sup\{|f^{(k)}(x)| : x \in I_{c,\delta}\}$. We call

$$g_{c,\delta} = \limsup_{k\to\infty} (M^k_{c,\delta}/k!)^{1/k}$$

the **derivative growth rate of** f **δ-near** c. We say f has **locally bounded derivative growth rate in** (a, b) if for all $c \in (a, b)$ there is some δ so that $g_{c,\delta}$ is finite.
8. **Theorem:** An infinitely differentiable function $f : (a, b) \to \mathbb{R}$ is analytic if and only if it has locally bounded derivative growth rate.

IV. Derivatives in higher dimensions

A. Definitions and first properties

1. A map $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable at** $c \in A$ if there is a linear map

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{x\to c} \frac{\|f(x) - (f(c) + L(x-c)\|}{\|x-c\|} = 0.$$

If such an L exists then we call it the **derivative of f at** c and denote it by $Df(c)$.
2. Alternate definition: L is the derivative of f at c if and only if for all $\epsilon > 0$ there is some $\delta > 0$ such that $\|x-c\| < \delta$ implies that $\|f(x) - (f(c) + L(x-c))\| \leq \epsilon\|x-c\|$.
3. **Lemma:** If there is a linear map L as in the definition above then it is uniquely determined by f and c.
4. **Theorem:** If $f : A \to \mathbb{R}^m$ is differentiable at $c \in A$ with A and open set in $\mathbb{R}^n$, then f is continuous at c.
5. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function on the open set A. We can write

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)).$$

The i^{th} **partial derivative of** f_j at $(x_1, \dots, x_n)$ is

$$\frac{\partial f_j}{\partial x_i} = \lim_{h\to 0} \frac{f_j(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

(if the limit exists.
6. **Theorem:** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function on the open set A. If f is differentiable on A then the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist for all i and j and in the standard basis for $\mathbb{R}^n$

and $\mathbb{R}^m$ the linear map $Df(x_1, \ldots, x_n)$ is given by the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

This matrix is called the **Jacobian matrix of f at** $(x_1, \ldots, x_n)$.

7. If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function then $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \end{bmatrix}$. That is it is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$. This is very similar to a familiar concept: the **gradient of** f. The gradient is a vector containing the partial derivatives of f

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_n} \end{bmatrix}.$$

While it is common to write this vector as a row vector, we will always write it as a column vector (actually all our vectors are column vectors, so when we think of a point $x \in \mathbb{R}^n$ we will think of it as a column vector so our matrix can act on it by matrix multiplication).

8. Theorem: Let $A \subset \mathbb{R}^m$ be an open set and $f : A \to \mathbb{R}^n$. Suppose that $f = (f_1, \ldots, f_m)$ and each partial derivative $\frac{\partial f_j}{\partial x_i}$ exists and is continuous near on A. Then f is differentiable on A.

9. Let $A \subset \mathbb{R}^m$ be an open set and $f : A \to \mathbb{R}^n$. For e a unit vector in $\mathbb{R}^n$ and $c \in A$ we define the **directional derivative of f at c in the direction of e** to be

$$f'(c, e) = \lim_{h \to 0} \frac{f(c + he) - f(c)}{h}$$

if the limit exists.

10. It is simple to check that $(Df(c))e = f'(c, e)$.

B. Chain rule and product rule

1. Theorem (Chain Rule): Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be open sets. Suppose that $f : A \to \mathbb{R}^m$, $g : B \to \mathbb{R}^p$ and $f(A) \subset B$. If f is differentiable at $x_0 \in A$ and g is differentiable at $f(x_0) \in B$ then $g \circ f$ is differentiable at x_0 and

$$D(g \circ f)(x_0) = (Dg(f(x_0)))\,(Df(x_0))\,.$$