Math 4317 Practice Final Exam Fall 2010

1) Show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$ and that equality holds if and only if $x = 0$.

2) Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Suppose that for every $0 \leq a < b \leq 1$ there is a point $c \in [a,b]$ such that $f(c) = 0$. Show that $\int_0^1 f(x)\,dx = 0$. Does f have to be the zero function? What if f is continuous?

3) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume there is no $x \in \mathbb{R}$ such that $f(x)$ and $f'(x)$ are both zero. Show that the set $\{x \in [0,1] : f(x) = 0\}$ is finite.

4) Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$. Set $M_n = \left(\int_a^b f^n(x)\,dx\right)^{1/n}$. Show that $\lim_{n\to\infty} M_n = \sup\{f(x) : x \in [a,b]\}$.

5) Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is differentiable on $(0,1)$. If $f(0) = 0, f(1) = 1$ and $\int_0^1 f(x)\,dx = 0$, then show that there is some point $c \in [0,1]$ such that $f'(c) = 0$.

6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $f_n(x) = f(nx)$ for $n = 1, 2, \ldots$. If the sequence $\{f_n\}$ is equicontinuous on $[0,1]$ then show that f is constant on $[0,\infty)$. Hint: Show that $f(0) = f(p/q)$ for all rational numbers p/q.

7) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = e^x \cos y$. Compute Df, $D^2 f$ and the second order Taylor polynomial at $(0,0)$.

8) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\|f(x_0) - f(x)\| \leq K\|x_0 - x\|^\alpha$ for some fixed real number $\alpha > 1$ and fixed point $x_0 \in \mathbb{R}^n$. Compute $Df(x_0)$.

9) Suppose that $B \subset \mathbb{R}^n$ is a bounded set and that $f : B \to \mathbb{R}$ is integrable over B and $f(x) \geq 0$ for all $x \in B$. If $A \subset B$ and f is integrable over A, then show that $\int_A f \leq \int_B f$. Is this true if we do not assume that $f(x) \geq 0$?