Math 4318 Solutions for Practice Midterm Exam 1 Spring 2011

1) Let $f : [a, b] \to \mathbb{R}$ and suppose that there is some M such that $|f'(x)| \leq M$. Prove using the definitions that f if Lipschitz and continuous on $[a, b]$.

Solution: For any x, y in $[a, b]$ the Mean Value Theorem says there is some c between x and y such that

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

So f is Lipschitz with Lipschitz constant M. We know Lipschitz functions are continuous, but since we have to establish everything from definitions here is the proof. Given $x \in [a, b]$ and $\epsilon > 0$ let $\delta = \epsilon/M$, then if $y \in [a, b]$ and $|x - y| < \delta$ we see that

$$|f(x) - f(y)| \leq M|x - y| < M\delta = M(\epsilon/M) = \epsilon.$$

So f is continuous.

2) Assuming that f' exists on $[a, b]$ and $\lim_{x\to c} f'(x) = L$ for some $c \in (a, b)$, prove that f' is continuous at c.

Solution: Since c is a cluster point of $[a, b]$, $f'(x)$ being continuous at c means that $\lim_{x\to c} f'(x) = f'(c)$. Thus we must show that $L = f'(c)$. If $L \neq f'(c)$ then let $\epsilon = \frac{|f'(c)-L|}{2}$. Since $\lim_{x\to c} f'(x) = L$ there is some $\delta > 0$ such that $|f'(x)-L| < \epsilon$ whenever $0 < |x-c| < \delta$ and $x \in [a, b]$. By making δ smaller if necessary we can assume that $c-\delta$ or $c+\delta$ is in $[a, b]$. Assuming the later we have for $c < x < c + \delta$ that

$$|f'(x) - f'(c)| \geq |L - f'(c)| - |f'(x) - L| > |L - f'(c)| - \epsilon = \frac{|f'(c) - L|}{2} = \epsilon > 0.$$

Thus $f'((c, c+\delta])$ is disjoint from $(f'(c) - \epsilon, f'(c) + \epsilon)$, but this contradicts the intermediate value theorem for derivatives.

3) Let $f : [a, b] \to \mathbb{R}$ be an integrable function with $f(x) \geq 0$ for all $x \in [a, b]$.

a) If f is continuous at $c \in (a, b)$ and $f(c) > 0$ show that

$$\int_a^b f(x)\, dx > 0.$$

Solution: Since f is continuous at c there is some $\delta > 0$ such that for all $x \in [a, b]$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \frac{f(c)}{2}$. We can moreover assume that δ is small enough so that $I = (c - \delta, c + \delta) \subset [a, b]$. Now let χ_I be the characteristic function of I (that is $\chi_I(x) = 1$ if $x \in I$ and zero otherwise). Then we know that for $|x - c| < \delta$ we have $f(x)/2 > f(c)/2$ Thus the function $g(x) = \frac{f(c)}{2}\chi_I$ satisfies $g(x) \leq f(x)$ for all x. Now

$$\int_a^b f(x)\, dx \geq \int_a^b g(x) = \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\, dx = \frac{f(c)}{2}(2\delta) > 0.$$

(You could also prove this using the definition of integral, but choosing partitions appropriately. For example show that the lower Darboux integral is greater than zero.)

b) If the set $C = \{x \in [a, b] : f(x) = 0\}$ has measure zero show that

$$\int_a^b f(x)\, dx > 0.$$

Solution: Since f is integrable we know the set D of points where f is discontinuous has measure zero. If $[a,b]-D$ had measure zero then so would $[a,b] = D \cup ([a,b]-D)$ (since the union of two sets of measure zero have measure zero). But it is easy to see that $[a,b]$ does not have measure zero (argument below). Thus $[a,b]-D$ does not have measure zero. From this we can conclude that $([a,b]-D)$ is not a subset of C (since subsets of measure zero have measure zero). But since $[a,b]-D$ is the set of points where f is continuous it is clear that there is a point $c \in (a,b)$ where f is continuous and not in C, that is $f(c) > 0$. Thus we are done by part a).

Proof that $[a,b]$ does not have measure zero. Now suppose $\{U_i\}$ is an cover of $[a,b]$ by open intervals. Since $[a,b]$ is compact there are a finite number of U_i that cover $[a,b]$, say $U_{i_1}, \ldots, U_{i_k}$. We can assume that each U_{i_j} intersects $[a,b]$ (or else we could through it out and still have a cover). It is clear that $U = \cup_{j=1}^k U_{i_j}$ is connected (if not then since $[a,b]$ is connected it would be in one of the components of U and thus there would be some U_{i_j} that don't intersect $[a,b]$). Thus U is an open interval that contains $[a,b]$ it is clear that the length of U is larger than the length of $[a,b]$, that is $b-a$. Thus the total length of the U_{i_j} is bigger than $b-a$ and hence the total length of the U_i is bigger than $b-a$. In particular we cannot find a cover of $[a,b]$ with total length less than, say, $\frac{1}{2}(b-a)$. So $[a,b]$ does not have measure zero.

4) Let $f : [0,1] \to \mathbb{R}$ be the function that is 0 for all irrational numbers and $f(x) = x$ for all rational numbers. Prove that f is not integrable. Hint: Show that the upper and lower Darboux integrals cannot be the same. Specifically show that any upper sum is bounded below by $\frac{1}{2}$.

Solution: If $\mathcal{P}$ is any partition of $[0,1]$ then notice since each interval in the partition contains an irrational number we know that the minimum of f on each of these intervals is zero. Thus

$$L(f, \mathcal{P}) = 0$$

for all $\mathcal{P}$ and so $\underline{\int_0^1} f(x)\,dx = 0$. Now if $I = [x_{i-1}, x_i]$ is an interval of the partition $\mathcal{P}$ then there is a sequence of rational numbers approaching x_i and thus the maximum of f on I is x_i. Noting that $\frac{x_i + x_{i-1}}{2} < x_i$ (because $x_i > x_{i-1}$) we have

$$\begin{aligned} U(f, \mathcal{P}) &= \sum_{i=1}^n x_i(x_i - x_{i-1}) > \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \frac{1}{2}\sum_{i=1}^n x_i^2 - x_{i-1}^2 = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(1-0) = \frac{1}{2}. \end{aligned}$$

Thus for any partition $\mathcal{P}$ the upper sum of f is bounded below by $\frac{1}{2}$ and hence

$$\overline{\int_0^1} f(x)\,dx \geq \frac{1}{2}.$$

In particular the upper and lower sums are not the same and hence f is not integrable.

5) Answer the following questions **T**rue or **F**alse. Circle either **T** or **F** to indicate your answer. You do not need to justify your answer.

I am providing reasons for the answers but you do not need to do so.

1. If $|f|$ is integrable on $[a, b]$ then f is integrable on $[a, b]$.

   F If f is 1 for irrational and -1 for rational numbers on $[0, 1]$ then $|f|$ is a constant function and hence integrable, but f is discontinuous everywhere so it is not integrable.

2. If f is not integrable on $[a, b]$ then there are partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$ such that $L(f, \mathcal{Q}) > U(f, \mathcal{P})$

   F The upper sum is always larger than the lower sum for any partition.

3. If a function is differentiable on an open interval I then it is continuous on I.

   T A theorem from class.

4. Sets of measure zero must be countable.

   F Countable sets have measure zero, but there are uncountable sets (like the middle thirds Cantor set) that also have measure zero.

5. If a function if differentiable on an open interval I then its derivative is continuous on I.

   F The function $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and 0 for $x = 0$ is differentiable on all of $\mathbb{R}$ but its derivative is not continuous at 0.

6. If a function has bounded derivative on an interval then it is uniformly continuous on the interval.

   T If the derivative of a function is bounded on an interval then the function is Lipschitz (see problem 1). Lipschitz functions are uniformly continuous.

7. Every integrable function has an anti-derivative.

   F Derivatives satisfy the intermediate value property. Since not every integrable function satisfies this, not every function can be a derivative (in particular have an anti-derivative).

8. The set of integrable functions form a vector space.

   T Theorem from class.

9. The product of integrable functions is integrable.

   T Theorem from class (also easily follows from the Riemann-Lebesgue theorem).

10. The composition of integrable functions is integrable.

    F We had a counterexample to this on homework 2.