Math 4318 **Practice Midterm Exam 2** **Spring 2011**

Solutions

1) Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Define

$$G : C^0([a,b]) \to C^0([a,b]) : f \mapsto g \circ f.$$

If $C^0([a,b])$ has the sup-norm show that G is uniformly continuous.

Solution: Given any $\epsilon > 0$ we can us the uniform continuity of g to find a $\delta > 0$ so that

$$|x - y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon/2.$$

Now if $f_1, f_2 \in C^0([a,b])$ and $\|f_1 - f_2\|_\infty < \delta$ then for all $x \in [a,b]$ we know that $|f_1(x) - f_2(x)| < \delta$ so $|G(f_1)(x) - G(f_2)(x)| = |g(f_1(x)) - g(f_2(x))| < \epsilon/2$. Thus

$$\|G(f_1) - G(f_2)\|_\infty = \sup_{x\in[a,b]} \{|G(f_1)(x) - G(f_2)(x|\} \le \epsilon/2 < \epsilon.$$

So we have shown that given any $\epsilon > 0$ there is a $\delta > 0$ so that $\|f_1 - f_2\|_\infty < \delta$ implies that $\|G(f_1) - G(f_2)\|_\infty < \epsilon$. Or in other words that G is uniformly continuous.

2) Show that if $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions that converge uniformly to f and the sequence f_n' converges uniformly to g then $f' = g$.

Solution: We need to show that $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = g(x)$. To this end consider

$$\left| g(x) - \frac{f(y) - f(x)}{y - x} \right| = |g(x) - f_n'(x)| + \left| f_n'(x) - \frac{f_n(y) - f_n(x)}{y - x} \right| + \left| \frac{f_n(y) - f_n(x)}{y - x} - \frac{f(y) - f(x)}{y - x} \right|.$$

We must bound each term on the left. Notice that the mean value theorem gives us that for some t between x and y we have

$$(f_m(x) - f_n(x)) - (f_m(y) - f_n(y)) = (f_m'(t) - f_n'(t))(x - y)$$

which gives

$$\left| \frac{f_m(x) - f_m(y)}{x - y} - \frac{f_n(x) - f_n(y)}{x - y} \right| = |f_m'(t) - f_n'(t)|.$$

Since $\{f_n'\}$ converges, it is Cauchy. Thus given $\epsilon > 0$ there is an N such that $m, n \ge N$ implies that $|f_m'(x) - f_n'(x)| < \epsilon/3$ for all $x \in [a,b]$. Thus for $m, n \ge N$ we have

$$\left| \frac{f_m(x) - f_m(y)}{x - y} - \frac{f_n(x) - f_n(y)}{x - y} \right| < \epsilon/3.$$

Letting $m \to \infty$ gives

$$\left| \frac{f_n(y) - f_n(x)}{y - x} - \frac{f(y) - f(x)}{y - x} \right| < \epsilon/3$$

for all $n \ge N$. Now since f_n is differentiable there is a $\delta > 0$ such that $\left| f_n'(x) - \frac{f_n(y)-f_n(x)}{y-x} \right| < \epsilon/3$ for all $|x - y| < \delta$. Finally since $\{f_n'\}$ converges uniformly to g there is some N' such that $n \geq N'$ implies that $|g(x) - f_n(x)| \leq \epsilon/3$ for all x. Putting this into the first equation able gives

$$\left| g(x) - \frac{f(y) - f(x)}{y - x} \right| < \epsilon$$

for $n \geq \max N, N'$ and $|x - y| \leq \delta$. But since the inequality does not involve n we see that $|g(x) - \frac{f(y)-f(x)}{y-x}| < \epsilon$ for all $|x - y| < \delta$. That is we have shown that $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x} = g(x)$.

3) Let $\mathcal{F}$ be a family of equicontinuous functions from an interval $[a, b]$ to $\mathbb{R}$ that are bounded in the sup-norm. Define

$$F(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

for all $x \in [a, b]$. Show that F is continuous.

Solution: Given $\epsilon > 0$ the equicontinuity of $\mathcal{F}$ implies there is some $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon/2$ for all $f \in \mathcal{F}$. Thus for $|x - y| < \delta$ we have

$$F(x) = \sup\{f(x) : f \in \mathcal{F}\} \leq \sup\{f(y) + \epsilon/2 : f \in \mathcal{F}\} = \epsilon/2 + \sup\{f(y) : f \in \mathcal{F}\} = \epsilon/2 + F(y).$$

Similarly

$$F(y) \leq \epsilon/2 + F(x).$$

Thus $|F(x) - F(y)| \leq \epsilon/2 < \epsilon$ if $|x - y| < \delta$.

4) a) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $b \in \mathbb{R}^m$. Define $f(x) = Lx + b$. What is the derivative of f at $y \in \mathbb{R}^n$?

Solution: The derivative is $Df(x) = L$.

b) Using the definition prove that your answer from a) is the derivative of f.

Solution: Indeed we have

$$\frac{\|f(y) - (f(x) + L(y - x))\|}{\|y - x\|} = \frac{\|Ly + b - (Lx + b + Ly - Lx)\|}{\|y - x\|} = \frac{0}{\|y - x\|} = 0.$$

So the limit as $y \to x$ of $\frac{\|f(y)-(f(x)+L(y-x))\|}{\|y-x\|}$ is zero.

c) Compute the derivative of $f(x, y, z) = (xy^2, x + y, xy, 5 + x)$.

Solution: Since the partial derivatives of the component functions are all continuous we know f is differentiable and we can represent $Df(x, y, z)$ (in the standard basis on $\mathbb{R}^3$ and $\mathbb{R}^4$) by the Jabobian matrix:

$$Df(x, y, z) = \begin{bmatrix} y^2 & 2xy & 0 \\ 1 & 1 & 0 \\ y & x & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

5) Answer the following questions **T**rue or **F**alse. Circle either **T** or **F** to indicate your answer. You do not need to justify your answer.

1. Analytic functions are smooth.

   T This is true since power series are infinitely differentiable in there interval of convergence.

2. A set in $C^0([a,b])$ is compact if and only if it is closed and bounded. (Here we are using the sup-norm on C^0.)

   F You also need equicontinuity.

3. If the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ exists then all the partial derivatives of the coordinate functions exist.

   T A theorem from class.

4. The derivative of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by the Jacobian matrix (in the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$).

   T A theorem from class.

5. A sequence of differentiable functions $\{f_n\}$ on a compact interval always has a subsequence that converge uniformly to some function on the interval.

   F Consider $f_n(x) = n$.

6. If a sequence of continuous functions $\{f_n\}$ converges uniformly to f on $[a,b]$ then the sequence is equicontinuous.

   T This was a homework problem.

7. The derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ can be represented by a 3×3 matrix.

   F It can be represented as a 3×2 matrix.

8. The normed vector space $(C^n([a,b]), \|\cdot\|_{C^n})$ is a Banach space.

   T Theorem from class.

9. If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ are differentiable functions then $g \circ f$ does not have to be differentiable.

   F The chain rule says the composition will be differentiable.

10. If a sequence $\{f_n\}$ of functions on a compact set is bounded in the sup-norm and equicontinuous, then the sequence converges uniformly.

    F You only know a subsequence converges uniformly.