Math 4431 - Fall 2016
Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 6, 7, 9, 11 and 15.* **Due: In class September 9**

1) Given two bases $\mathcal{B}_1$ and $\mathcal{B}_2$ generating topologies $\mathcal{T}_1$ and $\mathcal{T}_\in$, respectively, on a space X, show that $\mathcal{T}_1 \subset \mathcal{T}_2$ if and only if for each $p \in X$ and $U \in \mathcal{B}_1$ with $p \in U$, there is a set $V \in \mathcal{B}_2$ with $p \in V \subset U$.

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The **subspace topology** on A is the topology on A defined by $\mathcal{T}_A = \{U \cap A | U \in \mathcal{T}\}$.

2) Show $\mathcal{T}_A$ is a topology on A.

3) Let $\mathbb{R}$ be the x-axis in $\mathbb{R}^2$. Show the subspace topology on $\mathbb{R}$ is the same as the standard topology on $\mathbb{R}$ defined in class.

4) Show the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the standard topology on $\mathbb{R}^2$.

5) Show the projection onto either factor of a product space is continuous. That is show the maps $X \times Y \to X : (x, y) \mapsto x$ and $X \times Y \to Y : (x, y) \mapsto y$ are continuous.

6) Given a map $f : Z \to X \times Y$ you can always think of it as defined by $f(z) = (g(z), h(z))$ where $g : Z \to X$ and $h : Z \to Y$. Show that f is continuous if and only if g and h are both continuous.

7) The product of two Hausdorff spaces is Hausdorff.

8) Let X be any set with the finite complement topology. When is this space Hausdorff? Hint: this depends on the cardinality of X.

9) Finite sets in a Hausdorff space are closed.
Hint: First prove that sets with one point in them are closed.

10) Let $X = [0, 1)$ and $Y = S^1 \subset \mathbb{R}^2$ and define $f : X \to Y : t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Show f is a continuous bijection from X to Y. Show that the inverse of f is not continuous.

11) Let $Y = \{0, 1\}$ with the discrete topology. Let X be any topological space. Show the following are equivalent:

1. X has the discrete topology,
2. every function $f : X \to Y$ is continuous,

3. every function $f : X \to Z$, where Z is any topological space, is continuous.

12) Let $d : X \times X \to \mathbb{R}$ be a metric on X. Give X the topology induced by d. Show d is continuous.

13) Let d be a metric on X. Define $d' = \min\{d(x,y), 1\}$. Show d' is a metric and d' and d generates the same topology on X.

14) A set S in a topological space $(X, \mathcal{T})$ is called **dense** if $\overline{S} = X$.

1. Let X be any infinite set with the finite complement topology. Let S be an infinite subset of X. Show that S is dense in X.
2. Let $f, g : X \to Y$ be two continuous functions from X to Y. If Y is a Hausdorff space and $f(x) = g(x)$ on some dense set in X then show that $f = g$ on all of X.
   Hint: Show the subset of X on which $f = g$ is a closed set.

Let ρ be a bounded metric on $X = \mathbb{R}^n$ (by bounded I mean there is some constant C so that $\rho(x,y) < C$ for all $x, y \in X$). Given two sets A and B in X define

$$\rho(x, A) = \inf\{\rho(x,y) | y \in A\},$$

$$d_A(B) = \sup\{\rho(x, A) | x \in B\}$$

and

$$d(A, B) = \max\{d_A(B), d_B(A)\}.$$

Note: $d(\{x\}, \{y\}) = \rho(x, y)$.

15) Let $\mathcal{F}$ be the set of all nonempty closed and bounded sets in X. Show d is a metric on $\mathcal{F}$. This metric is called the Hausdorff metric on $\mathcal{F}$.
Hint: To establish the triangle inequality $d(A,C) \leq d(A,B) + d(B,C)$, first show $\rho(a, C) \leq \rho(a,b) + d(B,C)$ is true for any $b \in B$.

16) Given A and B in $\mathcal{F}$ show that $d(A,B) < \epsilon$ if and only if $A \subset U_\rho(B, \epsilon)$ and $B \subset U_\rho(A, \epsilon)$ where $U_\rho(A, \epsilon) = \{x \in X | \rho(x, A) < \epsilon\}$.