Math 4431 - Fall 2016
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 5, 6, 7, 8, 9, and 10.* **Due: September 30**

1) Given a collection of topological spaces $\{X_\alpha\}_{\alpha \in J}$ show that the product topology on $\prod_{\alpha \in J} X_\alpha$ is the smallest topology for which each of the projection maps is continuous.

2) Let D^2 be the unit disk in $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$. Show that the upper hemisphere of S^2 is homeomorphic to D^2 and similarly for the lower hemisphere. Use this to show that S^2 is homeomorphic to $D^2 \cup_g D^2$ for some homomorphism $g : S^1 \to S^1$ where $S^1 = \partial D^2$. Determine g.

3) Let D^2 be the unit disk in $\mathbb{R}^2$. Let $\mathcal{D}$ be a decomposition of $\mathbb{R}^2$ whose only non-trivial set is D^2. Show $\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$.

4) Let $X = [0, 1], A = \{0, 1\} \subset X$ and $Y = [4, 5]$. Define the map $f : A \to Y$ by $f(0) = 4$ and $f(1) = 5$. Show that $X \cup_f Y$ is homeomorphic to S^1.

5) Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$. Show that the decomposition space of X defined as

$$\mathcal{D} = \{S_r | r > 0\},$$

where $S_r = \{(x, y)|x^2 + y^2 = r^2\}$, is homeomorphic to $\mathbb{R}$.

6) Show that S^1 is not homeomorphic to $[0, 1]$.
Hint: Think connectedness.

7) Show that if U is an open connected subset of $\mathbb{R}^2$, then it is path connected. Hint: Fix an $x_0 \in U$ and show that the set of points in U that can be joined to x_0 by a path is both open and closed in U.

8) Show that a closed subspace of a normal space is normal.

9) Show that a connected normal space having more than one point is uncountable. (Hint: use Urysohn's Lemma)

10) Show that a space X is Hausdorff if and only if $\Delta = \{(x, x)|x \in X\}$ is closed in $X \times X$.

11) Show that a space X is Hausdorff if and only if $\{x\}$ is equal to the intersection of all closed neighborhoods containing x. A closed neighborhood of a point x is a closed set C that contains an open set U such that $x \in U \subset C$.

12) The Tietze extension theorem is

Theorem 1 *Let X be a normal space and A a closed subspace of X. Then any continuous function $f : A \to [0, 1]$ can be extended to a continuous function $f : X \to [0, 1]$.*

Show that Urysohn's Lemma follows from the Tietze extension theorem. (Remark: It is also true but harder that the Tietze extension theorem follows from Urysohn's Lemma. See the book for a proof.)

A *topological vector space* is a vector space V together with a topology $\mathcal{T}$ so that the vector space operations are continuous. That is

$$+ : V \times V \to V : (v, w) \mapsto v + w$$

and

$$* : \mathbb{R} \times V \to V : (r, v) \mapsto rv$$

are continuous.

13) Show that for any fixed $v_0 \in V$ the map $f_{v_0} : V \to V$ given by $f_{v_0}(v) = v + v_0$ is a homeomorphism. Similarly for $r \neq 0$ in $\mathbb{R}$ the map $f_r : V \to V$ given by $f_r(v) = rv$ is a homeomorphism.

Give a vector $v \in V$ and a set $S \subset V$ then $S + v$ is defined to be $\{s + v : s \in S\}$.

14) Let V be a topological vector space. If $\mathcal{B}_0$ is a neighborhood basis of $0 \in V$ then show that $\{U + v : U \in \mathcal{B}_0, v \in V\}$ is a basis for V.

15) Show that a topological vector space V is Hausdorff if and only if $\{0\}$ is closed.
Hint: Show that if O is any open set in V containing 0 then there is an open set $U \subset O$ such that $0 \in U$ and $U + U \subset O$. (Here $S + S = \{s + t : s \in S, t \in S\}$.)