# Math 4431 - Fall 2016
# Homework 3

*Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in* **only** *problems 2, 4, 6, 7, 10, 12.* **Due: October 17**

1) Show that the any subspace of a separable metric space is separable.

2) Show the continuous image of a separable space is separable.

3) Let $U_i$ be a open dense set in $X$ for each $i = 1 \dots n$. Prove $\cap_{i=1}^n U_i$ is dense.

4) A subset $S$ of a topological space $X$ is called a **zero set** if there is a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = S$. Prove:

a) zero sets are closed.

b) every closed non-empty subset of a metric space is a zero set.

c) if every closed subset of a second countable $T_1$-space $X$ is a zero set then $X$ is metrizable. (Hint: prove such and $X$ is regular then appeal to a metrization theorem.)

A space $X$ is called *Lindelöf* if every open cover $X$ has a countable subcover.

5) A regular, Lindelöf space is normal.

6) Let $X$ be a second countable space, then $X$ is Lindelöf.

7) For a metric space $X$ the following are equivalent: (a) $X$ is second countable, (b) $X$ is Lindelöf and (c) $X$ is separable.

8) Is there a continuous surjective map $f : [0,1] \to \mathbb{R}^2$? Why or why not.

9)Let $X$ be a Hausdorff space and $C_i$ compact subspaces such that $C_{i+1} \subset C_i$. Show that

a) $C = \cap_{i=1}^\infty C_i$ is non-empty.

b) $C$ is compact.

c) if the $C_i$ are connected show that $C$ is connected. (Hint: It will help to think of everything as a subset of $C_1$ (since you will then be working in a compact space). Assume not and try to prove there are disjoint open sets in $C_1$ separating the components of $C$. Now try to get a contradiction to the fact that the $C_i$'s are connected.)

10) Let $X$ be any Hausdorff topological space and $X^*$ the topological space defined as follows: as a set of points $X^* = X \cup \{\infty\}$ (here $\infty$ is just a point that is not in $X$ that we are denoting $\infty$) and a set $U$ is open in $X^*$ if it is either an open set in $X$ or $U = X^* \setminus K$ where $K$ is a compact set in $X$. The space $X^*$ is called the *one point compactification* of $X$.

a) Show that we have defined a topology on $X^*$.

b) Show $X^*$ is compact and $X$ is open in $X^*$.

c) Show $X$ is dense in $X^*$ if and only if $X$ is not compact.

11) Prove the one point compactification of $\mathbb{R}$ is $S^1$. Also show the one point compatification of the natural numbers is homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} | n = 1, 2, 3, \dots\}$ of $\mathbb{R}$.

12) Suppose $X$ is a compact Hausdorff space and $f : X \to Y$ is a quotient map. Show the following are equivalent: (a) $Y$ is Hausdorff, (b) $f$ takes closed sets in $X$ to closed sets in $Y$ and (c) the set $\{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\}$ is closed in $X \times X$.
Hint: You may use the following fact without proving it (though you should try!): if $f : X \to Y$ is a quotient map then $f$ takes closed sets to closed sets if and only if for all $y \in Y$ and $U$ open in $X$ containing $f^{-1}(y)$ then there is an open set $V$ in $Y$ such that $f^{-1}(y) \subset f^{-1}(V) \subset U$.
For (c) implies (b) for a closed set $C$ in $X$ consider $C \times X$ in $X \times X$ and consider projections for $X \times X$ to $X$.