# Math 4431 - Fall 2016
# Homework 4

*Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in* **only** *problems 1, 2, 5, 7, 10 and 12.* **Due: November 2**

1) Show a compact metric space is second countable and separable.
Hint: Think totally bounded.
This actually follows immediately from Problem 7) on the previous homework assignment, but do not use this. Come up with an argument using total boundedness of a compact metric space.

2) Let $(X, d)$ be a compact metric space. If every bounded closed subset of $C(X, \mathbb{R})$ is compact, then show that $X$ consists of a finite number of points.
Hint: In every infinite metric space there is an infinite sequence such that no point of the sequence is a limit point of the sequence (you need to prove this). Now think about the Tietze extension theorem.

3) Let $X$ be a compact space. Show that any metric on $X$ that induces the topology on $X$ is complete.

4) Show that if $X$ is a non-compact metrizable space, then there is a metric on $X$ inducing the given topology that is not complete.

5) Let $X$ be a compact metric space and $Y$ a Hausdorff space. If there is a surjective continuous map $f : X \to Y$ then $Y$ is also a compact metric space.
Hint: Try to find an countable basis for $Y$.

6) Show any regular connected space has uncountably many points.
Hint: Show that a countable regular space is not connected. It might be good to know that a regular Lindelöf space is normal (this is problem 6 on the last homework, while you should try to prove this you don't have to when you write up the solution to this problem).

7) Let $A$ be an open subset of the complete metric space $(X, d)$. Show that there is some metric on $A$ inducing the subspace topology on $A$ that is complete.
Hint: There is a continuous function $f : A \to \mathbb{R}$ defined by $f(x) = \frac{1}{d(x, X-A)}$ and the map $F : A \to X \times R$ defined by $F(x) = (x, f(x))$.

A subset $A$ of a space $X$ is called $G_\delta$ if there is a countable collection $\{U_i\}_{i=1}^\infty$ of open sets such that $A = \cap_{i=1}^\infty U_i$.

8) Show that a set $A$ in a complete metric space $(X, d)$ is a complete metric space (possibly with a new metric inducing its subspace topology) if it is a $G_\delta$ set in $X$. Conclude that the topology on the irrational numbers as a subset of $\mathbb{R}$ can be given a complete metric.
Hint: Consider a function $f_i(x)$ as in problem 7) for each $U_i$. Show any sequence $\{x_n\}$ in $A$ that is Cauchy in $d$ and for which $\{f_i(x_n)\}$ is a bounded sequence for each $i$ converges to a point in $A$. Now consider $d'(x, y) = d(x, y) + \sum_{i=1}^\infty \frac{1}{2^i} \min\{1, |f_i(x) - f_i(y)|\}$.

9) A closed set in a metric space is $G_\delta$.

10) If $X$ is compact then show that for every topological space $Y$ the projection map $\pi : X \times Y \to Y$ is a closed map (that is takes closed sets to closed sets).

11) The projection map $\pi : X \times Y \to Y$ is a closed map if and only if for all open sets $U$ in $X \times Y$ the set $\{y \in Y : X \times \{y\} \subset U\}$ is open in $Y$.

12) Show that if for every topological space $Y$ the projection map $\pi : X \times Y \to Y$ is a closed map then $X$ is compact.
Hint: Given an open cover $\mathcal{O}$ let $\mathcal{O}'$ be $\mathcal{O}$ together with all finite unions of open sets in $\mathcal{O}$. So $\mathcal{O}'$ is closed under finite unions and $\mathcal{O}$ has a finite subcover if and only if $X \in \mathcal{O}'$. Let $Y$ be the space whose points are open sets in $X$ and the open sets in $Y$ are collections of opens sets $\mathcal{C}$ such that (1) if $U \in \mathcal{C}$ and $U \subset V$ with $V$ open in $X$ then $V \in \mathcal{C}$ and (2) $\mathcal{C} \cap \mathcal{O}' \neq \emptyset$ unless $\mathcal{C} = \emptyset$. Show this gives $Y$ a topology. Use $Y$ to show $\mathcal{O}$ has a finite subcover (consider the set $\{(x, U) \in X \times Y : x \in U\}$ and use problem 11)).

13) If $C$ is a Cantor set then show that any two non-empty compact open subsets of $C$ are homeomorphic.

14) If $C$ is a Cantor set then show that any two non-compact open subsets of $C$ are homeomorphic.

15) Let $X$ be a compact metric space. Show that $X$ is a Cantor set if and only if $X$ satisfies the following properties:

a. $X$ has at least one non-empty compact open subset and $X$ has at least one non-compact open subset,

b. any two non-empty compact open subsets are homeomorphic, and

c. any two non-compact open subsets are homeomorphic.

Hint: You might need to know that any compact connected metric space has non-cut points (we say a point $x$ in a connected set $X$ is a cut point if $X - \{x\}$ is disconnected). You also might like to try to prove this.