

## Math 4432 - Spring 2017 Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 2, 3, 4, 6, 7, and 11. **Due: February 7**

1. Show that if  $U$  is an open connected subset of  $\mathbb{R}^2$ , then it is path connected.  
Hint: Fix an  $x_0 \in U$  and show that the set of points in  $U$  that can be joined to  $x_0$  by a path is both open and closed in  $U$ .

2. Show that  $S^1$  is not homeomorphic to  $[0, 1]$ .

A collection of sets  $\mathcal{C} = \{C_\alpha\}_{\alpha \in I}$  has the **finite intersection property** if for every finite sub-collection  $\{C_{\alpha_1}, \dots, C_{\alpha_n}\}$  of  $\mathcal{C}$  the intersection  $\bigcap_{i=1}^n C_{\alpha_i}$  is non-empty.

3. Show a space  $X$  is compact if and only if every collection of closed sets  $\{C_\alpha\}_{\alpha \in I}$  having the finite intersection property has  $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$ .  
Hint: Think about the complements of the  $C_\alpha$ 's.

4. Is there a continuous surjective map from  $[0, 1]$  to  $\mathbb{R}^2$ ? Prove your answer.

5. Let  $X = [0, 1]$ ,  $A = \{0, 1\} \subset X$  and  $Y = [4, 5]$ . Define the map  $f : A \rightarrow Y$  by  $f(0) = 4$  and  $f(1) = 5$ . Show that  $X \cup_f Y$  is homeomorphic to  $S^1$ .

6. Let  $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ . Show that the decomposition space of  $X$  defined as

$$\mathcal{D} = \{S_r \mid r > 0\},$$

where  $S_r = \{(x, y) \mid x^2 + y^2 = r^2\}$ , is homeomorphic to  $\mathbb{R}$ .

7. Let  $X = S^1 \times [0, 1]$  and consider the decomposition space

$$\mathcal{D} = \{\{(x, t)\} : t \in (0, 1] \text{ and } x \in S^1\} \cup \{(x, 0) : x \in S^1\},$$

that is the only non-trivial set in the decomposition is  $S^1 \times \{0\}$ . Prove that  $\mathcal{D}$  is homeomorphic to  $D^2$ .

**Remark:** You might also want to try to show that if  $S^1$  is replaced by  $S^n$  then the analogous decomposition space is homeomorphic to  $D^{n+1}$ .

8. Let  $D^2$  be the unit disk in  $\mathbb{R}^2$  and  $S^2$  be the unit sphere in  $\mathbb{R}^3$ . Show that the upper hemisphere of  $S^2$  is homeomorphic to  $D^2$  and similarly for the lower hemisphere. Use this to show that  $S^2$  is homeomorphic to  $D^2 \cup_g D^2$  for some homeomorphism  $g : S^1 \rightarrow S^1$  where  $S^1 = \partial D^2$ . Determine  $g$ .
9. Let  $D^2$  be the unit disk in  $\mathbb{R}^2$ . Let  $\mathcal{D}$  be a decomposition of  $\mathbb{R}^2$  whose only non-trivial set is  $D^2$ . Show  $\mathcal{D}$  is homeomorphic to  $\mathbb{R}^2$ .
10. Let  $D$  be a disk and  $I$  be an interval in  $\partial D$ . If  $\Sigma$  is a surface and  $f : I \rightarrow \partial \Sigma$  is an embedding, then show the surface

$$\Sigma \cup_f D$$

is homeomorphic to  $\Sigma$ .

Hint: It might be good to try to show that the space obtained from a disk and an annulus by gluing them along intervals in their boundary is homeomorphic to an annulus. You may assume, as discussed in class, that given any connected component  $B$  of  $\partial\Sigma$  there is an open set  $U$  in  $\Sigma$  that contains  $B$  and is homeomorphic to  $S^1 \times [0, 1)$ .

11. Show that for any connected surface  $\Sigma$  and points  $p$  and  $q$  in  $\Sigma$  there is a homeomorphism  $h : \Sigma \rightarrow \Sigma$  such that  $h(p) = q$ .

Hint: Fix  $p$  and consider the set

$$S = \{q \in \Sigma \text{ such that there is a homeomorphism sending } p \text{ and } q\}.$$

**Remark:** Notice that from this problem you know for any non-empty surface there are an uncountable number of different homeomorphisms  $\Sigma \rightarrow \Sigma$ . Also, there are two extensions of this problem you might want to consider. 1) If  $M$  is a connected  $n$ -manifold and points  $p$  and  $q$  in  $M$  there is a homeomorphism  $h : M \rightarrow M$  such that  $h(p) = q$ . 2) If  $M$  is a connected  $n$ -manifold, for  $n > 1$ , and  $\{p_1, \dots, p_k\}$  and  $\{q_1, \dots, q_k\}$  are two collections of distinct points in  $M$ , then there is a homeomorphism  $h : M \rightarrow M$  such that  $h(p_i) = q_i$  for  $i = 1, \dots, k$ . What happens for  $n = 1$ ? Is there any similar statement that can be made?