

Math 4432 - Spring 2017 Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 3, 4, 5, 6, 7, 10, 12, and 13. **Due: March 7**

1. Recall the mapping class group of a space X , denoted by $\text{Mod}(X)$, is the group of isotopy classes of homeomorphisms of X . Show $\text{Mod}(X)$ is a group.
2. Prove $\text{Mod}(S^1)$ is isomorphic to \mathbb{Z}_2 .
3. Let G and H be groups and $f : G \rightarrow H$ a homomorphism. Show that the image of f is a subgroup of H .
4. If $\phi : G \rightarrow H$ is a homomorphism, then show that the image of ϕ is isomorphic to $G/\ker \phi$.
5. Show $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not isomorphic to \mathbb{Z}_4 .
HINT: Think of the orders of elements.
6. Show that \mathbb{Z}_6 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.
7. Show $\mathbb{Z} \oplus \mathbb{Z}$ is not isomorphic to \mathbb{Z} .
HINT: Where would such an isomorphism send $1 \in \mathbb{Z}$?
8. Suppose a group G has a presentation $\langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$ where the relators are

$$r_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{k_i}}^{s_{i_{k_i}}}$$

for $i = 1, \dots, m$, and the s_i are ± 1 . Then show that if H is any other group and h_1, \dots, h_n are any elements of H that satisfy

$$h_{i_1}^{s_{i_1}} \cdots h_{i_{k_i}}^{s_{i_{k_i}}} = e_H,$$

where e_H is the identity element in H , then there is a unique homomorphism $\phi : G \rightarrow H$ such that $\phi(x_i) = h_i$.

9. Show that $\mathbb{Z} \oplus \mathbb{Z}$ has presentation $\langle x, y | xyx^{-1}y^{-1} \rangle$.
10. Show that the dihedral group D_n has presentation $\langle x, y | x^n, y^2, xyxy \rangle$.
11. Consider the rational numbers \mathbb{Q} as a group under addition. Show that \mathbb{Q} has presentation

$$\langle x_i, i = 1, 2, 3, \dots | x_n^n = x_{n-1}, i > 1 \rangle.$$

Hint: try to construct a map by sending x_i to $\frac{1}{n!}$.

12. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous maps, then show $(g \circ f)_* = g_* \circ f_*$, where f_* denotes the homomorphism induced on the fundamental group by f .
13. If A is a subspace of X we say X *deformation retracts to* A if there is a continuous function $F : X \times [0, 1] \rightarrow X$ so that $F_0 = id_X$, the image of F_1 is contained in A , and $F_1|_A = id_A$ for all t . Here $F_t : X \rightarrow X : x \mapsto F(x, t)$. (So the F_t “deform” X to A without moving A .) Show that if X deformation retracts onto A then X and A are homotopy equivalent.