Math 4432 - Spring 2017
Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 6, 7, 8, 10, and 11.*
Due: March 28

1. Given a continuous map $f : S^n \to X$, show that f is homotopic to a constant map if and only if f extends to a map $F : D^{n+1} \to X$.

2. Let D^2 be the unit disk in $\mathbb{R}^2$ and $S^1 = \partial D^2$. Show there does not exist a map $f : D^2 \to S^1$ that is the identity on S^1. (That is, that satisfies $f(x) = x$ for all $x \in S^1 \subset D^2$.)
   Hint: Think about the natural inclusion $g : S^1 \to D^2$.

3. Let X and Y be two path connected topological spaces. Write down a natural map $\psi : \pi_1(X) \oplus \pi_1(Y) \to \pi_1(X \times Y)$ and prove it is an isomorphism.

4. Compute $\pi_1(T^2)$ using problem 3, where T^2 is the torus. More generally, let $T^n = S^1 \times T^{n-1}$ (so, for example, $T^3 = S^1 \times S^1 \times S^1$). Compute $\pi_1(T^n)$.

5. We can regard $\pi_1(X, x_0)$ as base point preserving homotopy classes of maps of (S^1, pt) to (X, x_0). Let $[S^1, X]$ be the set of homotopy classes of maps S^1 to X (not necessarily base point preserving). There is a natural map
   $$\Psi : \pi_1(X, x_0) \to [S^1, X]$$
   that just ignores the base points. Show that Ψ is onto if X is path connected. Also show that $\Psi([\gamma]) = \Psi([\lambda])$ if and only if there is some $g \in \pi_1(X, x_0)$ such that $[\gamma] = g^{-1}[\lambda]g$.

6. The infinite dihedral group D_∞ is given by the presentation $\langle x, y | y^2, xyxy \rangle$. Show that D_∞ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.
   Hint: From class we know $\mathbb{Z}_2 * \mathbb{Z}_2$ has presentation $\langle a, b | a^2, b^2 \rangle$. Notice that x and ab both have infinite order.

7. Suppose that neither G or H is the trivial group. Show that $G * H$ is infinite and non-abelian.

8. Use the Seifert-Van Kampen Theorem to compute the fundamental group of the unit n-sphere S^n for $n > 1$.
   Hint: You might want to induct on n.

9. Recall we can think of the torus T^2 as the unit square in the first quadrant of $\mathbb{R}^2$ with opposite edges identified. Show that T^2 is also $\mathbb{R}^2$ modulo the equivalence relation:

$$(x, y) \sim (x', y') \iff (x, y) - (x', y') = (n, m),$$

   where n, m are any two integers. Another way to say this is that T^2 is $\mathbb{R}^2$ modulo the group of translations that preserve the integer lattice in $\mathbb{R}^2$. (This is just like thinking of S^1 as the interval with endpoints identified and as $\mathbb{R}^1$ modulo unit translation.)

10. Let A be a 2×2 matrix with integer entries

$$A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

   This matrix takes the integer lattice in $\mathbb{R}^2$ into the integer lattice. Show this induces a continuous map

$$\phi_A : T^2 \to T^2.$$

   Compute $(\phi_A)_* : \pi_1(T^2) \to \pi_1(T^2)$. (By this I mean choose a basis for $\pi_1(T^2)$ and give a matrix expressing $(\phi_A)_*$.)

11. If $\det A = \pm 1$ then ϕ_A is a homeomorphism (you don't have to prove this unless you want to). Assume $\det A = -1$. Let M be the quotient space of $D^2 \times S^1 \cup D^2 \times S^1$ by the relation $(\theta, \psi) \in \partial(D^2 \times S^1)$ in the first $D^2 \times S^1$ is identified with $\phi_A(\theta, \psi)$ in the second $D^2 \times S^1$. This is a 3-manifold (you don't have to prove this unless you want to). Compute $\pi_1(M)$. Hint: Your answer should only depend on q.