Math 4441 - Fall 2016
Formulas

1. If $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is a regular parameterization of a curve C then the length of C is $\int_a^b \|\boldsymbol{\alpha}'(t)\|\, dt$.

2. If $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is a regular parameterization of a curve then $f(t) = \int_a^t \|\boldsymbol{\alpha}'(x)\|\, dx$ has an inverse $g : [0, l] \to [a, b]$ so that $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(g(s))$ is a parameterization of the same curve with $\|\boldsymbol{\beta}'(s)\| = 1$ (that is $\boldsymbol{\beta}$ is an arc length parameterization).

3. If $\boldsymbol{\alpha}$ is an arc length parameterization of a curve C, then $\boldsymbol{T}(s) = \boldsymbol{\alpha}'(s)$ is a unit tangent vector and $\boldsymbol{T}'(s)$ is perpendicular to $\boldsymbol{T}(s)$. The curvature of C at $\boldsymbol{\alpha}(s)$ is
$$\kappa(s) = \|\boldsymbol{T}'(s)\|$$
and the normal vector is $\boldsymbol{N}(s) = \frac{\boldsymbol{T}'(s)}{\|\boldsymbol{T}'(s)\|}$.

4. If $\boldsymbol{\alpha}$ is a regular parameterization of a curve C (but not necessarily an arc length parameterization), then the curvature of C at $\boldsymbol{\alpha}(t)$ is
$$\kappa(t) = \left\| \left(\frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|} \right)' \frac{1}{\|\boldsymbol{\alpha}'(t)\|} \right\|.$$

5. If $\boldsymbol{\alpha} : [0, l] \to \mathbb{R}^2$ is an arc length parameterization of a plane curve, then we define $\widehat{\boldsymbol{N}}(s)$ to be $\boldsymbol{\alpha}'(s)$ rotated by $\pi/2$ counter-clockwise, then we define the signed curvature to be
$$\kappa_\sigma(s) = \widehat{\boldsymbol{N}}(s) \cdot \boldsymbol{\alpha}''(s) = \widehat{\boldsymbol{N}}(s) \cdot \boldsymbol{T}'(s).$$

6. If $\boldsymbol{\alpha} : [0, l] \to \mathbb{R}^2$ is an arc length parameterization of a plane curve C, then there is a function $\theta(s)$ such that
$$\boldsymbol{\alpha}'(s) = (\cos\theta(s), \sin\theta(s)),$$
and in particular
$$\boldsymbol{\alpha}(s) = (a + \int_0^s \cos\theta(t)\, dt, b + \int_0^s \sin\theta(t)\, dt),$$
where $\boldsymbol{\alpha}(0) = (a, b)$. The signed curvature is $\kappa_\sigma(s) = \theta'(s)$.

7. With the notation above the rotation number of a curve C is
$$R(C) = \frac{1}{2\pi}(\theta(l) - \theta(0)).$$

8. With notation above the total signed curvature of a curve C is $TK(C) = \int_0^l \kappa_\sigma(t)\, dt$.

9. If $\boldsymbol{\alpha}$ is a curve in $\mathbb{R}^3$ then the binormal vector is $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$ and the torsion is
$$\tau(s) = -\boldsymbol{B}'(s) \cdot \boldsymbol{N}(s).$$

10. For a surface Σ in $\mathbb{R}^3$ in local coordinates
$$\boldsymbol{f} : V \to \Sigma$$
where V is an open subset of $\mathbb{R}^2$ with coordinates (u, v) we have the first fundamental form in the basis $\{\boldsymbol{f}_u, \boldsymbol{f}_u\}$ is given by the matrix with entries $g_{11} = \boldsymbol{f}_u \cdot \boldsymbol{f}_u$, $g_{12} = g_{21} = \boldsymbol{f}_u \cdot \boldsymbol{f}_v$ and $g_{22} = \boldsymbol{f}_v \cdot \boldsymbol{f}_v$. The second fundamental from if given by $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ where $A = S_{\boldsymbol{p}}(\boldsymbol{f}_u) \cdot \boldsymbol{f}_u, B = S_{\boldsymbol{p}}(\boldsymbol{f}_u) \cdot \boldsymbol{f}_v$, and $C = S_{\boldsymbol{p}}(\boldsymbol{f}_v) \cdot \boldsymbol{f}_v$, where $S_{\boldsymbol{p}}$ is the shape operator. And if $\boldsymbol{N}$ is a unit normal vector field to the surface, then the shape operator applied to a tangent vector $\boldsymbol{v}$ in $T_{\boldsymbol{p}}\Sigma$ is $S_{\boldsymbol{p}}(\boldsymbol{v}) = -\boldsymbol{N}_{\boldsymbol{v}}(\boldsymbol{p})$, that is the directional derivative of $\boldsymbol{N}$ in the direction $\boldsymbol{v}$.

11. With notation above the Gauss and mean curvature is give by
$$K = \frac{AC - B^2}{g_{11}g_{22} - g_{12}^2} \quad \text{and} \quad H = \frac{1}{2}\frac{Ag_{22} - 2Bg_{12} + Cg_{11}}{g_{11}g_{22} - g_{12}^2}.$$

12. Given a Riemannian metric $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ on a surface that satisfies $g_{12} = g_{21} = 0$ then the Gauss curvature is expressed by

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}} \left\{ \left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}} \right)_v + \left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}} \right)_u \right\}.$$

13. With the notation from 12 we can express the Christoffel symbols as

$$\Gamma^i_{jk} = \sum_{l=1}^{2} g^{il} \frac{1}{2} ((g_{kl})_{u_j} + (g_{lj})_{u_k} - (g_{jk})_{u_l})$$

where the local coordinates are (u_1, u_2) and the g^{ij} are the entries in the inverse to the matrix (g_{ij}).

14. Using the notation above the Gauss curvature can also be expressed

$$K = \frac{1}{g_{11}} [(\Gamma^2_{11})_{u_2} - (\Gamma^2_{12})_{u_1} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{21} - (\Gamma^2_{12})^2]$$

15. If $\boldsymbol{f} : V \to \Sigma$ is a local coordinate system on Σ and a curve $\boldsymbol{\alpha}(t)$ is expressed as $\boldsymbol{f}(u(t), v(t))$ and a vector field along $\boldsymbol{\alpha}$ is expressed by $\boldsymbol{w}(t) = a(t)\boldsymbol{f}_u(u(t), v(t)) + b(t)\boldsymbol{f}_v(u(t), v(t))$, then $\boldsymbol{w}$ is parallel along $\boldsymbol{\alpha}$ if

$$a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{21}bu' + \Gamma^1_{22}bv' = 0 \text{ and}$$
$$b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{21}bu' + \Gamma^2_{22}bv' = 0.$$

16. If $\boldsymbol{f} : V \to \Sigma$ is a local coordinate system on Σ and a curve $\boldsymbol{\alpha}(t)$ is expressed as $\boldsymbol{f}(a(t), b(t))$ then $\boldsymbol{\alpha}$ is a geodesic if

$$a'' + \Gamma^1_{11}(a')^2 + 2\Gamma^1_{12}a'b' + \Gamma^1_{22}(b')^2 = 0 \text{ and}$$
$$b'' + \Gamma^2_{11}(a')^2 + 2\Gamma^2_{12}a'b' + \Gamma^2_{22}(b')^2 = 0.$$

17. If $\boldsymbol{\alpha}$ is a parameterization of a curve in a surface Σ with Riemannian metric g the its geodesic curvature κ_g is the length of $\nabla_{\boldsymbol{\alpha}'}\boldsymbol{\alpha}'$.

18. If Σ is a surface with piecewise smooth boundary $\partial\Sigma = C_0 \cup \ldots \cup C_{k-1}$ and Riemannian metric g, then let θ_i be the exterior angle between C_i and C_{i+1} and we have the formula

$$\sum_{i=0}^{k-1} \int_{C_i} \kappa_g(s)\, ds + \int_{\Sigma} K\, dA + \sum_{i=0}^{k-1} \theta_i = 2\pi\chi(\Sigma).$$

19. In geodesic polar coordinates $\boldsymbol{f} : V \to \Sigma$ centered at $\boldsymbol{p}$ the Riemannian metric (that is, first fundamental form) is $\begin{bmatrix} 1 & 0 \\ 0 & G(r,\theta) \end{bmatrix}$ where $\sqrt{G(r,\theta)} = r - \frac{1}{6}K(\boldsymbol{p})r^3 + R(r,\theta)$ where $\lim_{r\to 0} \frac{R(r,\theta)}{r^3} = 0$.