Math 4441 - Fall 2016
Formulas for Midterm 1

1. If $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^n$ is a regular parameterization of a curve C then the length of C is $\int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt$.

2. If $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^n$ is a regular parameterization of a curve then $f(t) = \int_a^t \|\boldsymbol{\alpha}'(x)\|\,dx$ has an inverse $g : [0,l] \to [a,b]$ so that $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(g(s))$ is a parameterization of the same curve with $\|\boldsymbol{\beta}'(s)\| = 1$ (that is $\boldsymbol{\beta}$ is an arc length parameterization).

3. If $\boldsymbol{\alpha}$ is an arc length parameterization of a curve C, then $\boldsymbol{T}(s) = \boldsymbol{\alpha}'(s)$ is a unit tangent vector and $\boldsymbol{T}'(s)$ is perpendicular to $\boldsymbol{T}(s)$. The curvature of C at $\boldsymbol{\alpha}(s)$ is
$$\kappa(s) = \|\boldsymbol{T}'(s)\|$$
and the normal vector is $\boldsymbol{N}(s) = \frac{\boldsymbol{T}'(s)}{\|\boldsymbol{T}'(s)\|}$.

4. If $\boldsymbol{\alpha}$ is a regular parameterization of a curve C (but not necessarily an arc length parameterization), then the curvature of C at $\boldsymbol{\alpha}(t)$ is
$$\kappa(t) = \left\| \left(\frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|} \right)' \frac{1}{\|\boldsymbol{\alpha}'(t)\|} \right\|.$$

5. If $\boldsymbol{\alpha} : [0,l] \to \mathbb{R}^2$ is an arc length parameterization of a plane curve, then we define $\widehat{\boldsymbol{N}}(s)$ to be $\boldsymbol{\alpha}'(s)$ rotated by $\pi/2$ counter-clockwise, then we define the signed curvature to be
$$\kappa_\sigma(s) = \widehat{\boldsymbol{N}}(s) \cdot \boldsymbol{\alpha}''(s) = \widehat{\boldsymbol{N}}(s) \cdot \boldsymbol{T}'(s).$$

6. If $\boldsymbol{\alpha} : [0,l] \to \mathbb{R}^2$ is an arc length parameterization of a plane curve C, then there is a function $\theta(s)$ such that
$$\boldsymbol{\alpha}'(s) = (\cos\theta(s), \sin\theta(s)),$$
and in particular
$$\boldsymbol{\alpha}(s) = (a + \int_0^s \cos\theta(t)\,dt, b + \int_0^s \sin\theta(t)\,dt),$$
where $\boldsymbol{\alpha}(0) = (a,b)$. The signed curvature is $\kappa_\sigma(s) = \theta'(s)$.

7. With the notation above the rotation number of a curve C is
$$R(C) = \frac{1}{2\pi}(\theta(l) - \theta(0)).$$

8. With notation above the total signed curvature of a curve C is $TK(C) = \int_0^l \kappa_\sigma(t)\,dt$.