## Math 4441 - Fall 2016 Extra Practice Problems

1. Let $\Sigma$ be a regular, compact, orientable surface in $\mathbb{R}^{3}$ which is not homeomorphic to a sphere. Prove that there are points on $\Sigma$ where the Gaussian curvature is positive, negative and zero.
2. Determine the Christoffel symbols of a surface represented in the form $z=f(x, y)$.
3. Write down the differential equations that determine a geodesic on the surface given by $z=f(x, y)$ where $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is any function. That is if $\boldsymbol{\alpha}(t)=\boldsymbol{f}(a(t), b(t))$ is a geodesic. What equations must $a$ and $b$ satisfy?
4. Can there be a smooth closed geodesic curve bounding a disk on a surface with Gauss curvature is (a) strictly positive? (b) strictly negative? (c) zero? Prove your answer.
5. Let $p \in \Sigma$ and $S_{r}(\boldsymbol{p})$ be the geodesic circle with center $\boldsymbol{p}$ and radius $r$. Let $L$ be the length of $S_{r}(\boldsymbol{p})$ and $A$ the area of the region bounded by $S_{r}(\boldsymbol{p})$. Prove that

$$
4 \pi A-L^{2}=\pi^{2} K(\boldsymbol{p}) r^{4}+R
$$

where $R$ is a function of $r$ satisfying

$$
\lim _{r \rightarrow 0} \frac{R}{r^{4}}=0 .
$$

6. Let $\Sigma$ be the surface parameterized by $\boldsymbol{f}(u, v)=\left(u \cos v, u \sin v, u^{2}\right)$ for $u \geq 0$ and $0 \leq v \leq 2 \pi$. Let $\Sigma_{r}$ be the portion of the surface with $0 \leq u \leq r$.
(a) Calculate the geodesic curvature of the boundary circles of $\Sigma_{r}$ and also compute $\int_{\partial \Sigma_{r}} \kappa_{g}(s) d s$.
(b) What is $\chi\left(\Sigma_{r}\right)$ ?
(c) Use the Gauss-Bonnet Theorem to compute $\int_{\Sigma_{r}} K d A$. Compute the limit as $r \rightarrow$ $\infty$.
(d) Compute $K$ directly.
(e) Use the previous computation to explicitly compute $\int_{\Sigma_{r}} K d A$.
