Math 500 - Fall 2001
Final Exam

You may use your book and notes on this test. Turn in the exam by noon on Monday December 17th.

1) Let X be any Hausdorff topological space and X^* the topological space defined as follows: as a set of points $X^* = X \cup \{\infty\}$ (here ∞ is just a point that is not in X that we are denoting ∞) and a set U is open in X^* if it is either an open set in X or $U = X^* \setminus K$ where K is a compact set in X. The space X^* is called the *one point compactification* of X.

a) Show that we have defined a topology on X^*.

b) Show X^* is compact and X is open in X^*.

c) Show X is dense in X^* if and only if X is not compact.

2) Prove the one point compactification of $\mathbb{R}$ is S^1. Also show the one point compatification of the natural numbers is homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} | n = 1, 2, 3, \ldots\}$ of $\mathbb{R}$.

3) A topological space X is called *locally path connected at x* if for every open set U containing x there is a path connected open set V such that $x \in V \subset U$. The space X is called *locally path connected* if it is locally path connected at each of its points. Show a connected, locally path connected space is path connected.

4) Let Σ be a connected, compact, oriented surface and $f : S^1 \to \Sigma$ be an embedding of S^1. Assume that $f(S^1)$ does not disconnect Σ. We can extend f to a map $F : S^1 \times (-\epsilon, \epsilon) \to \Sigma$. Consider $\Sigma^0 = \Sigma \setminus F(S^1 \times (-\epsilon, \epsilon))$. Note Σ^0 has two boundary components. Glue a disk to each of these and call the resulting surface Σ_s. We say that Σ_s is obtained from Σ by *surgery along $f(S^1)$*.

a) Find a formula relating the Euler characteristic of Σ_s to the Euler characteristic of Σ.

b) What surface would we obtain if $f(S^1)$ separated Σ?

5) Let $f : \Sigma \to \Sigma$ be a homeomorphism from a connected compact surface Σ to itself with the following property: for each point $x \in \Sigma$ there is an open set U such that $x \in U$ and $f(U) \cap U = \emptyset$. Moreover, assume that $f^2(x) = x$. Let Σ' be the decomposition space $\{\{x, f(x)\} | x \in \Sigma\}$. As alway there is a natural map $p : \Sigma \to \Sigma'$. In this situation p is called a *covering map* and Σ is said to *cover* Σ'.

a) Show that Σ' is a surface.

b) Find a relation between the Euler characteristic of Σ and Σ'.

c) Can a surface of genus 3 cover a surface of genus 4? Can a surface of genus 3 cover a surface of genus 1?

d) Show a surface of genus 3 covers a surface of genus 2 by explicitly finding an f as above.