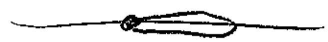
so ϕ is a homomorphism

now to show ϕ is 1-1 we just check $\ker\phi = \{e\}$

if $\gamma \in \ker\phi$ then the lift $\tilde\gamma$ of γ with $\tilde\gamma(0) = 0$ has $\tilde\gamma(1) = 0$

so $\tilde\gamma : [0,1] \to \mathbb{R}$ is a loop in $\mathbb{R}$



let $\tilde H(s,t) = s\,\tilde\gamma(t)$

note: $\tilde H(s,0) = 0$

$\tilde H(s,1) = \tilde\gamma(t)$

$\tilde H(0,t) = H(1,t) = 0$

$\tilde H$ is homotopy of $\tilde\gamma$ to constant loop

let $H = p \circ \tilde H$

H is a homotopy of γ to constant loop e. ✓

C. Applications

given a map $f : S^1 \to S^1$

let $\bar f : [0,1] \to S^1$ be a map st.

$$p \circ \bar f = f \qquad \begin{tikzcd} {[0,1]} \arrow[d, "p"'] \arrow[dr, "\bar f"] & \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd}$$

lemma 16 says there is a unique lift $\tilde f : [0,1] \to \mathbb{R}$ of $\bar f$ once we choose where to lift $\bar f(0)$

choose any such lift

define: the **degree** of $f : S^1 \to S^1$ is the number

$$\deg f = \tilde f(1) - \tilde f(0)$$

note: if $\tilde{\tilde f}$ is another lift then $\tilde{\tilde f}(s) = \tilde f(s) + k$ for some k so

$$\tilde{\tilde f}(1) - \tilde{\tilde f}(0) = \tilde f(1) + k - (\tilde f(1) + k) = \tilde f(1) - \tilde f(0)$$

so $\deg f$ is well defined

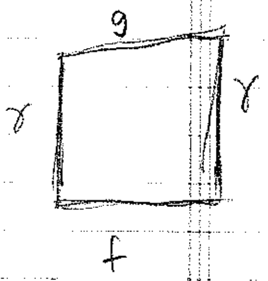
Th^m 19:

$$\boxed{\begin{array}{c} f: S^1 \to S^1 \text{ and } g: S^1 \to S^1 \text{ are homotopic} \\ \text{iff} \\ \deg f = \deg g \end{array}}$$

Proof: ($\Rightarrow$)

let $F: S^1 \times I \to S^1$ be the homotopy
let $\bar{F}: I \times I \to S^1$ be s.t. $\bar{F}(p(s), t) = F(s,t)$

$$\begin{tikzcd} {[0,1] \times [0,1]} \arrow[d, "p \ \downarrow id"'] \arrow[dr, "\bar{F}"] & \\ S^1 \times [0,1] \arrow[r, "F"] & S^1 \end{tikzcd}$$



by Th^m 18 $\exists$ lift of $\bar{F}$ to $\tilde{F}: I \times I \to \mathbb{R}$

now $\tilde{F}_0(s) = \tilde{f}(s)$ & $\tilde{F}_1(s) = \tilde{g}(s)$

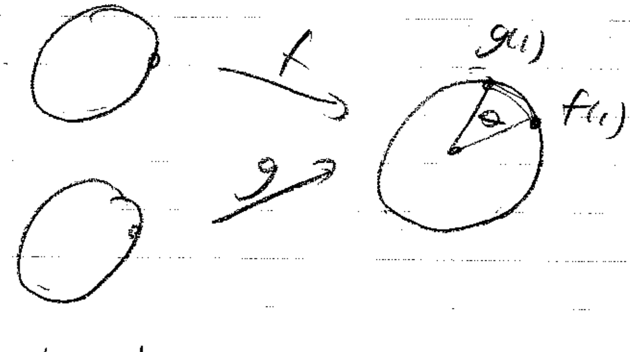
so $\deg f = \tilde{F}(1,0) - \tilde{F}(0,0)$
$\deg g = \tilde{F}(1,1) - \tilde{F}(0,1)$

$F'(0,1) - F'(0,0)$

note: $\tilde{F}(0,t), \tilde{F}(1,t) : I \to \mathbb{R}$ are lifts of $\bar{F}(1,t) = \bar{F}(0,t)$
ie lifts of some path in S^1
so $\tilde{F}(0,t) = \tilde{F}(1,t) + k$ some k

$\therefore \deg f = \tilde{F}(1,0) - \tilde{F}(0,0) = k = \tilde{F}(1,1) - \tilde{F}(0,1) = \deg g$ ✓

($\Leftarrow$) assume $\deg f = \deg g$
we want to homotop f to equal g
let's get them to match up at a pt.
let θ = angel between $g(1)$ & $f(1)$



let $R_t: S^1 \to S^1$ be rotation through angle t

$H(s,t) = R_{t\theta} \circ f(s)$

so $H(s,0) = f(s)$, $H(s,1) = R_\theta \circ f(s)$

so $H(1,1) = R_\theta \circ f(1) = g(1)$

so after a homotopy we can assume $f(1) = g(1)$
(further, after another homotopy can assume $f(1) = g(1) = 1$)

now let $\bar{f} : [0,1] \to S^1$, $\bar{g} : [0,1] \to S^1$
let $\tilde{f}$ be lift of $\bar{f}$ s.t. $\tilde{f}(0) = 0$
$\tilde{g}$ ——— $\bar{g}$ — $\tilde{g}(0) = 0$
now $\deg(f) = \deg(g) \Rightarrow \tilde{f}(1) = \tilde{g}(1)$

set $\tilde{H}(s,t) = t\,\tilde{f}(s) + (1-t)\,\tilde{g}(s)$
so

$\tilde{H}(s,0) = \tilde{g}(s)$
$\tilde{H}(s,1) = \tilde{f}(s)$

and $\tilde{H}(0,t) = t\,\tilde{f}(0) + (1-t)\,\tilde{g}(0) = t(0) + (1-t)(0) = 0$
$\tilde{H}(1,t) = t\,\tilde{f}(1) + (1-t)\,\tilde{g}(1) = \tilde{f}(1) = \tilde{g}(1)$

so $p \circ \tilde{H}_t(s) : [0,1] \to S^1$
decends to a map $H_t : S^1 \to S^1$ for all t.
with $H_0 = f$ & $H_1 = g$

$\therefore f \simeq g$ ✓

<u>Simple facts:</u> 1) a constant map $f : S^1 \to S^1$ has deg 0.

2) $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ is multiplication by $\deg f$

ie. $\mathbb{Z} \longrightarrow \mathbb{Z}$
$[\gamma] \longmapsto (\deg f)\,[\gamma]$

> these 2 facts $\Rightarrow$ a map $f : S^1 \to S^1$ is homotopically trivial iff it induces the trivial map on π_1

this is not in general true for other spaces $X \neq S^1$

<u>exercise</u>: prove these facts

<u>lemma 20</u>: $\boxed{f: S^1 \to S^1 \text{ extends to a map } F: D^2 \to S^1 \text{ iff } \deg f = 0}$

<u>Proof</u>: ($\Rightarrow$) let $P: [0,1] \times S^1 \to D^2 : (r,\theta) \mapsto (r,\theta)$

note $H(s,t) = F \circ P(t,s)$ is a homotopy

$H(s,0) = F \circ (0,s) = F(0,s) = F(0,0) = pt$

origin (angle does not matter)

so $H_0(s)$ constant map

$H(s,1) = F(1,s) = F|_{S^1 = \partial D^2}(s) = f(s)$

so $f \simeq$ constant $\therefore$ $\deg f = \underline{0}$ ✓

($\Leftarrow$) if $\deg f = 0$ then $\exists$ homotopy $H(s,t): S^1 \times [0,1] \to S^1$ st. $H(s,1) = f$ & $H(s,0) = pt$

$$\begin{array}{ccc} S^1 \times [0,1] & \xrightarrow{H(s,t)} & S^1 \\ \downarrow \text{quotient map} & \nearrow & \\ D^2 & & \end{array}$$

so H induces map on quotient space

$$D^2 = S^1 \times [0,1] / S^1 \times \{0\}$$

call map F ✓

<u>lemma 21</u>: $\boxed{f: S^1 \to S^1 \text{ continuous map with } f(-x) = -f(x) \text{ then } \deg f \text{ is odd}}$

<u>**Proof:**</u> $f: S^1 \to S^1$

write as $\bar{f}: [0,1] \to S^1$

let $a = \bar{f}(0)$

let $\{\tilde{a}_i\}_{i \in \mathbb{Z}} = p^{-1}(a)$ $\qquad p: \mathbb{R} \to S^1$

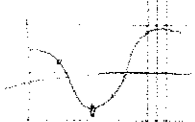
$$\tilde{a}_i = \tilde{a}_0 + i$$

$f(-1) = -f(1) = -a$

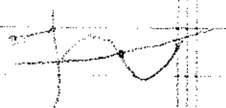
$\overset{\shortparallel}{\bar{f}(\tfrac{1}{2})}$ $\qquad$ so $p^{-1}(-a) = \{\tilde{b}_i\}_{i \in \mathbb{Z}}$

$$\tilde{b}_i = \tilde{a}_i + \tfrac{1}{2}$$

$$[\underset{\frac{1}{2}}{\text{—+—}}] \xrightarrow{f} S^1$$



$f_1 = f|_{[0,\frac{1}{2}]} \qquad f_2 = f|_{[\frac{1}{2},1]}$



note $f(x) = f(-(-x)) = -f(-x)$ so $f_2(x) = -f_1(x - \tfrac{1}{2})$

so if $\tilde{f}_1$ is a lift of f_1 starting at $\tilde{a}_0$ then $f_1(\tfrac{1}{2}) = \tilde{b}_i$ some i

$\tilde{f}_1(x - \tfrac{1}{2}) + \tfrac{1}{2}$ is a lift of f_2 starting at $\tilde{a}_0 + \tfrac{1}{2}$

$\cos(x+\pi), \sin(x+\pi)$

$\cos \cdot \cos\pi - \sin \cdot \sin\pi$

$= -\cos x \qquad \sin x \cos\pi$

so $\tilde{f}_2(x) = \tilde{f}_1(x - \tfrac{1}{2}) + i + \tfrac{1}{2}$ is a lift of f_2 starting at $\tilde{f}_2(\tfrac{1}{2}) = \tilde{f}_1(0) + i + \tfrac{1}{2} = \tilde{a}_0 + i + \tfrac{1}{2} = \tilde{b}_i$

$$\tilde{f}_2(1) = \tilde{b}_i + i + \tfrac{1}{2} = \tilde{a}_0 + 2i + 1$$

so $\tilde{f}_2 * \tilde{f}_1$ is a lift of f

$$\deg f = \tilde{f}_2 * \tilde{f}_1(1) - \tilde{f}_2 * \tilde{f}_1(0) = \tilde{a}_0 + 2i + 1 - \tilde{a}_0$$
$$= 2i + 1$$

Theorem 22 (Borsuk-Ulam I)

$\nexists$ a continuous map $f: S^2 \to S^1$ sending antipodal pts to antipodal pts

Proof: if $f: S^2 \to S^1$ is such a map
then let $S^1 \subset S^2$ be the equator
$f|_{S^1}: S^1 \to S^1$ is a map s.t. $f(-x) = -f(x)$
so $\deg f$ is odd
but $f|_{S^1}$ extends over northern hemisphere
so $\deg f = 0$ $\otimes$

Theorem 23 (Borsuk-Ulam II)

any continuous map $f: S^2 \to \mathbb{R}^2$ must send some pair of antipodal pts to the same pt

Proof: given $f: S^2 \to \mathbb{R}^2$
assume $f(x) \neq f(-x)$ for all $x \in S^2$
then define
$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Exercise: show g is continuous

note $g(-x) = -g(x)$ and $g: S^2 \to S^1 \subset \mathbb{R}^2$
$\otimes$ Thm 22

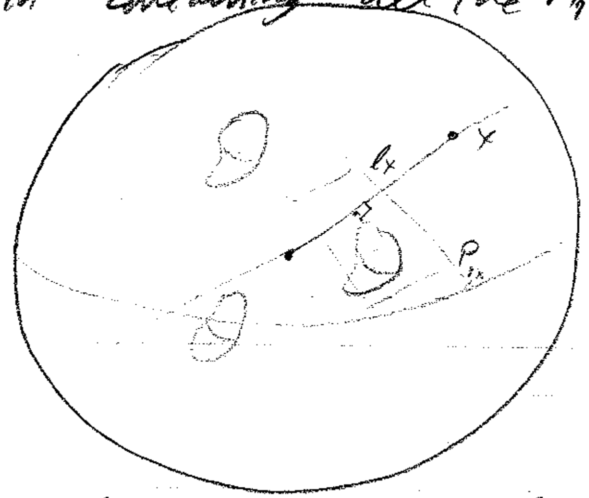
Remark: Thm 23 $\Rightarrow$ at any point in time there are antipodal pts on the earth with the same temperature & humidity! (or pick your favorite two continuously varying quantities).

"Ham sandwich Thm"

Thm 24:

> let R_1, R_2, R_3 be 3 connected open regions in $\mathbb{R}^3$ each of which is bounded and with finite volume. Then $\exists$ a plane which cuts them all in half by volume.

Proof: let $S^2 \subset \mathbb{R}^3$ be a large sphere about origin containing all the R_i's



given $x \in S^2$ let l_x = line through origin & x

for each R_i $\exists$ a plane P_{ix} perpendicular to l_x that cuts R_i in half

let $d_i(x)$ = distance of P_{ix} from origin (where $d_i(x) > 0$ if plane on same side of origin as x)

exercise: $d_i(x)$ is a continuous f^n $S^2 \to \mathbb{R}$
Hint: eq^{n}s of planes $\perp$ to l_x continuously vary with x and volume of half of R_i cut by a plane continuously changes w/ eq^n of plane.

clearly $d_i(-x) = -d_i(x)$
so set $f: S^2 \to \mathbb{R}^2: x \mapsto (d_1(x) - d_2(x), d_1(x) - d_3(x))$
Th^m 23 $\Rightarrow \exists\ x$ st $f(-x) = f(x)$ so
$d_1(-x) - d_2(-x) = -d_1(x) - d_2(x)$ & $d_1(-x) - d_3(-x) = d_1(x) - d_3(x)$

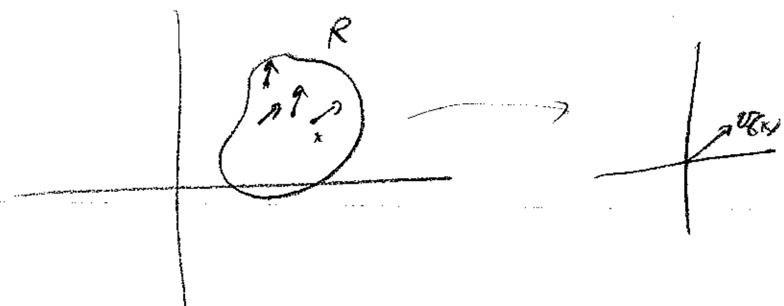
so $2d_1(x) = 2d_2(x) = 2d_3(x)$
so $d_1(x) = d_2(x) = d_3(x)$
so $\exists$ a plane $\perp$ to ℓ_x that cuts R_1, R_2, R_3 in half ✓

D. Vector fields and the Euler Characteristic

$R \subset \mathbb{R}^2$ a region in the plane

a vector field on R is a map $v: R \to \mathbb{R}^2$

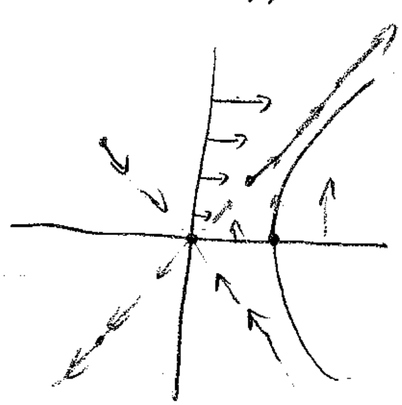
interpretation:

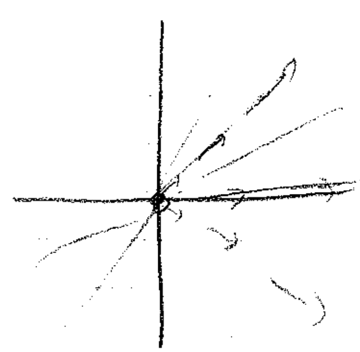


think of v as giving a vector at every point of R

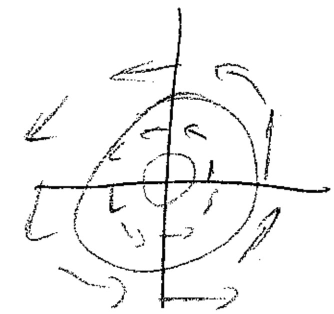
Ordinary Differential Equations tells you how to "integrate" a vector field to get "flow lines" i.e. trajectories of particles whose velocities are given by v

example 1) $v(x,y) = (y, x)$ 2) $v(x,y) = (x,y)$





3) $v(x,y) = (-y, x)$.



let $\gamma : S^1 \to \mathbb{R}^2$ be a circle in $\mathbb{R}^2$
and $v : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field

if $v(x) \neq 0$ for all $x \in \operatorname{im} \gamma$ then the <u>degree of v along γ</u> is the degree of the map

$$f(x) = \frac{v(\gamma(x))}{|v(\gamma(x))|} \qquad f : S^1 \to S^1$$

<u>lemma 25:</u>

> if $H : S^1 \times [0,1] \to \mathbb{R}^2$ is a homotopy
> and $v : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field s.t.
> $v(x) \neq 0$ for all $x \in$ image H
> then (degree of v along H_0) = (degree of v along H_1)

<u>Proof</u>: $F(x,t) = \dfrac{v(H(x,t))}{|v(H(x,t))|}$ is a homotopy

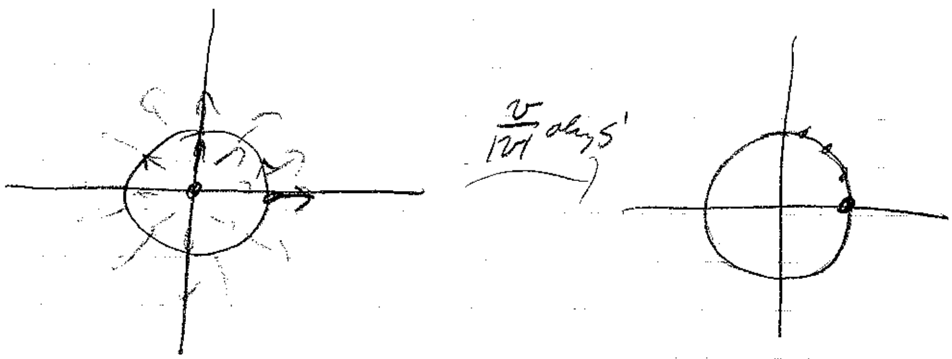
from $\dfrac{v(H(x,0))}{|v(H(x,0))|}$ to $\dfrac{v(H(x,1))}{|v(H(x,1))|}$: done by Th 19

let v be a v.f and x an <u>isolated zero</u> of v
ie $v(x) = 0$
$v(y) \neq 0$ for all $y \in B(x, \varepsilon) \setminus \{x\}$
some $\varepsilon > 0$
let $S^1 = \partial B(x, \varepsilon)$ (oriented counterclockwise)

the index of v at x is

$$\mathrm{ind}(v,x) = \text{degree of } v \text{ along } S^1 = \partial B(x,\varepsilon)$$

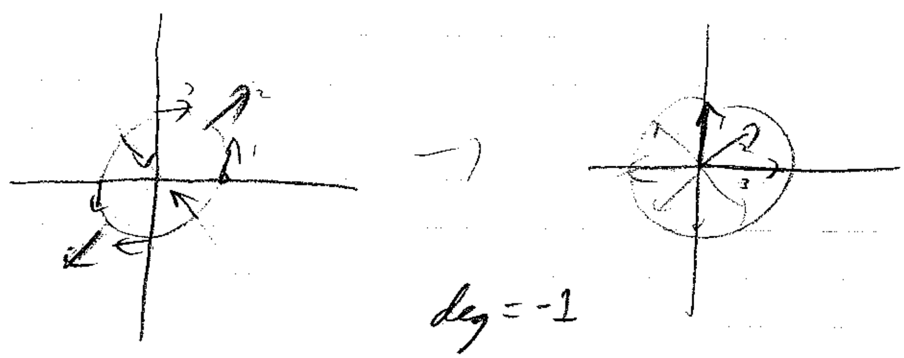
example: ① $v(x,y) = (x,y)$ $(0,0)$ isolated zero



has deg = 1

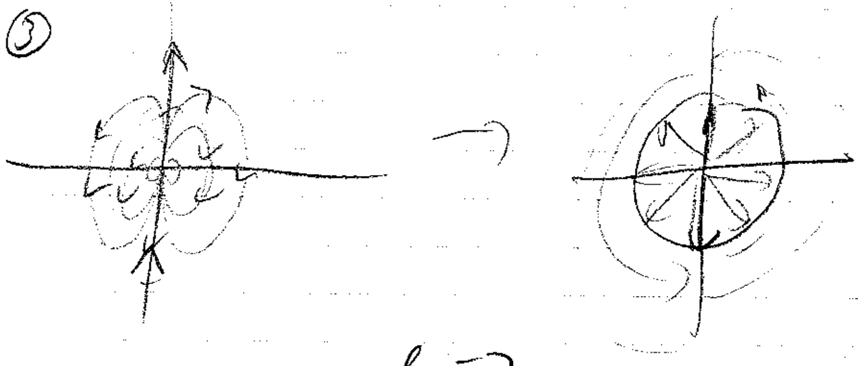
similarly $v(x,y) = (-x,-y)$ has degree 1 at $(0,0)$

② $v(x,y) = (y,x)$



deg = -1

③



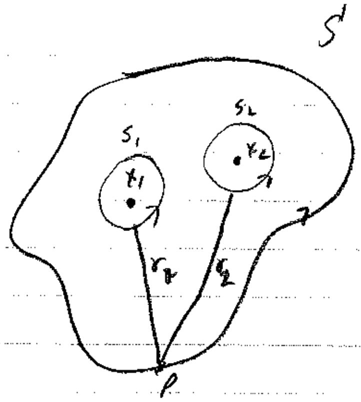
deg = 2

Thm 26:

v a vector field
$S^1 \subset \mathbb{R}^2$ an embedded loop oriented counterclockwise s.t. $v \neq 0$ on S^1
$x_1 \ldots x_n$ isolated zeros of v inside S^1
then

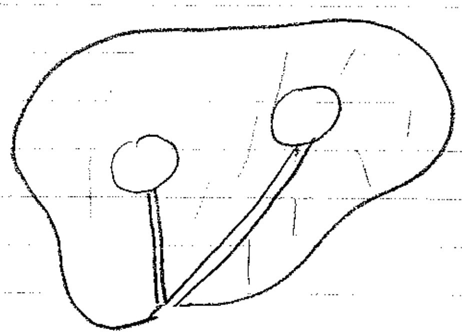
$$\deg \text{ of } v \text{ along } S^1 = \sum_{i=1}^{n} \operatorname{ind}(v, x_i)$$

Proof: example



index x_1 = deg v along S_1
— x_2 = — — S_2

pick γ_1 γ_2 as in picture



Consider loop l = start at p go around S'
then go along γ_1 then $-S_1$ then
back along γ_1 then γ_2 then $-S_2$
then $-\gamma_2$

$$= S' + \gamma_1 - S_1 - \gamma_1 + \gamma_2 - S_2 - \gamma_2$$

note l bounds a disk so

$$\deg \text{ of } v \text{ along } l = 0$$

exercise: $\deg \ell = \deg S^1 - \deg S_1 - \deg S_2$

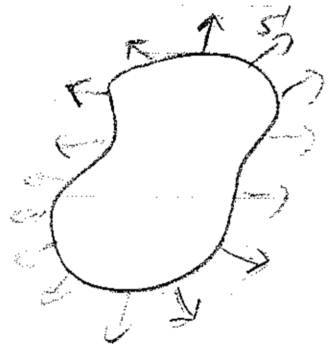
hint: the $\gamma_1', -\gamma_1$ cancel out & $\deg(-S_1) = -\deg(S_1)$

so $\deg S^1 = \deg S_1 + \deg S_2 = \operatorname{ind} x_1 + \operatorname{ind} x_2$

Cor 27:

If v is a vector field & $S^1 \subset \mathbb{R}^2$ an embedded loop s.t. v always points out of (or into) S^1 then v have some zeros inside S^1

pf:



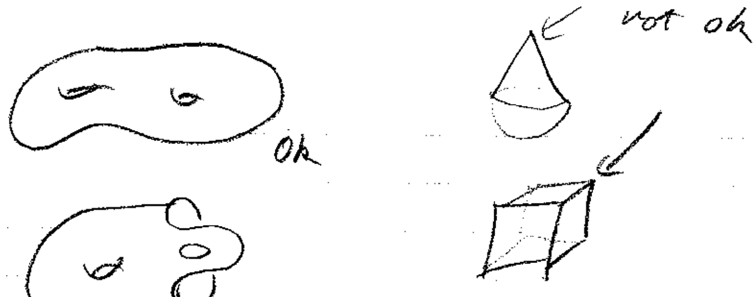
check $\deg v|_{S^1} = 1$

$\therefore$ must be zeros!

We want to talk about v.f's on surfaces to do this properly we need to develop lots of notation and machinery so we will do this "intuitively here". See the book for a rigorous treatment.

we will think of orientable surfaces as sitting in $\mathbb{R}^3$ with no corners or point

eg:

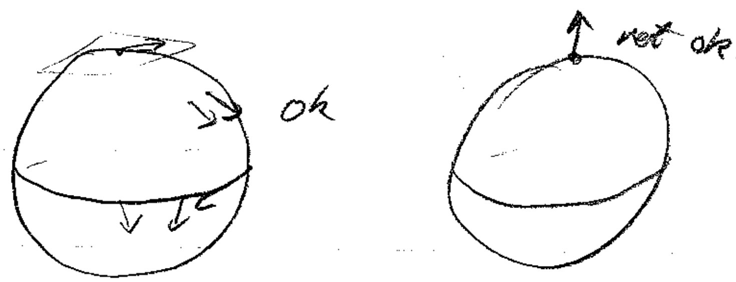


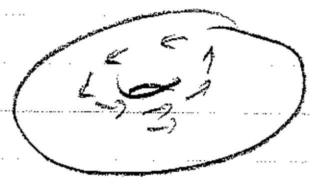
now given a surface $\Sigma \subset \mathbb{R}^3$

a vector field v on Σ is simply a map

$$v : \Sigma \to \mathbb{R}^3$$

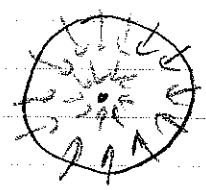
such that $v(x)$ is "tangent to Σ at x"





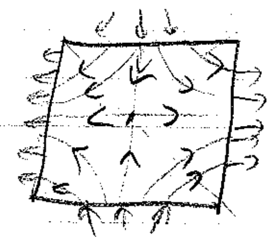
vector fields on handles

0-h



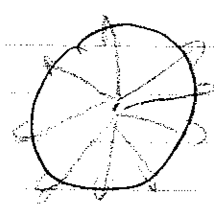
isolated zero
with index = 1

1-h



isolated zero
with index = -1

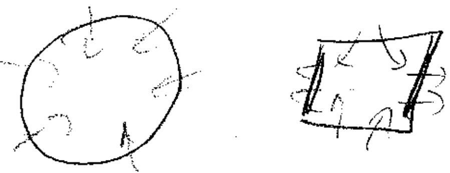
2-h



isolated zero
with index = 1

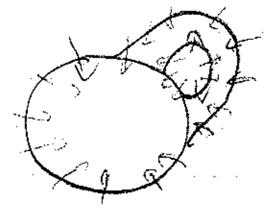
<u>note</u>: these vector fields were chosen so that they fit together

eg

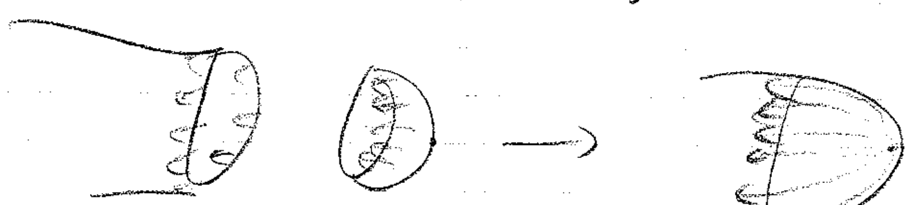


0-h

glue together get



(on all ∂ components v.f. pointing into the surface.)



2-h

so given a handle decomp of a surface get a v.f. on surface with
one zero for each 0-h w/index 1
1-h w/index -1
2-h 1

so Euler Characteristic is

$$\chi(S) = \#(\text{0-handles}) - \#(\text{1-handles}) + \#(\text{2-handles}) = \sum_{\substack{x \text{ a zero} \\ \text{of } v}} \text{index}(v, x)$$

<u>Big Thm</u>:

if v_1, v_2 are 2 vector fields on a surface Σ with isolated zeros then

$$\sum_{\substack{x \text{ zero} \\ \text{of } v_1}} \text{index}(v_1, x) = \sum_{\substack{y \text{ zero} \\ \text{of } v_2}} \text{index}(v_2, y)$$

Thm 28:

$\boxed{\begin{array}{l}\text{if } v \text{ is any vector field on } \Sigma \text{ with isolated zeros then} \\ \chi(\Sigma) = \sum_{\substack{x \text{ zero} \\ \text{of } v}} \text{index}(v, x)\end{array}}$

Cor 29:

$\boxed{\begin{array}{l}\text{if } \Sigma \not\cong T^2 \text{ or the Klein bottle} \\ \text{then any vector field on } \Sigma \text{ \underline{must} have \underline{zeros}}\end{array}}$

__Proof:__ only closed sfcs w/ $\chi = 0$ are T^2 & K^2 ✓

- so any time you try to draw a v.f. on S^2 there must be zeros!

  "Can't comb the hair on a coconut"

- if water covered the surface of the earth then there would have to be "stagnation pts"

  not true if we lived on a doughnut!

- if you want to build a "vector force field" that is tangent to a surface and has no "holes" (= zeros) then better build the field on a torus!

  (Tokamak)