Math 6441 - Fall 2009
Homework 3

Please work all the problems, but carefully write up problems 3, 4, 6, 7, and 10.

1. Let M be a closed oriented n manifold. Show $H_{n-1}(M;\mathbb{Z})$ is torsion free.
2. If M is a closed orientable 3 manifold with $H_1(M;\mathbb{Z}) = 0$ show $H_*(M;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ (that is show M is a homology sphere). If you prefer a little challenge show that a closed oriented 3 manifold with $\pi_1(M) = 1$ is a homotopy sphere.
3. Let M be a compact orientable 3 manifold with boundary having $H_1(M)$ finite. Show ∂M is a union of spheres.
4. If M is a closed orientable $4k$ manifold then the cup product pairing on $H^{2k}(M;\mathbb{Z})$ is a symmetric nondegenerate paring. Picking a basis for $H^{2k}(M)$ the pairing can be represented by a matrix. Such a symmetric matrix can be diagonalized over $\mathbb{R}$. After diagonalizing the number of positive elements down the diagonal is called b^{2k}_+ and the number of negative elements is b^{2k}_-. The signature of the pairing is $\sigma(M) = b^{2k}_+ - b^{2k}_-$. Prove that if $M = \partial W$ for a compact oriented $4k+1$ manifold then $\sigma(M) = 0$. Is $\mathbb{C}P^2\#\mathbb{C}P^2$ the boundary of a compact oriented 5 manifold? For a challenge what about $\mathbb{C}P^2\# -\mathbb{C}P^2$? Where $-\mathbb{C}P^2$ is $\mathbb{C}P^2$ with the reversed orientation.
5. Show X_1 and X_2 are homotopy equivalent if and only if for every space Y there is a one-to-one correspondence $\psi_Y : [X_1, Y] \to [X_2, Y]$ such that for every continuous map $h : Y \to Y'$ we have
$$h_* \circ \psi_Y = \psi_{Y'} \circ h_*.$$
6. Let $\pi : \widetilde{X} \to X$ be a covering space. Show $\pi_*\colon \pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for all $n \geq 2$. Compute $\pi_n(S^1)$ for all n. Compute $\pi_n(\Sigma)$ for all n, where Σ is an oriented surface of genus 2.
7. Given CW pairs (X, A) and (Y, B) and a cellular map $f : (X, A) \to (Y, B)$ show that
$$f_* \circ h_n = h_n \circ f_*$$
where h_n is the Hurewicz map $h_n : \pi_n(X, A) \to H_n(X, A)$ or $h_n : \pi_n(Y, B) \to H_n(Y, B)$.
8. With notation as in the previous problem show
$$\partial \circ h_n = h_{n-1} \circ \partial$$
where $\partial : \pi_n(X, A) = \pi_{n-1}(A)$ on the left hand side of the equation and on the right $\partial : H_n(X, A) \to H_{n-1}(A)$.
9. Compute the homotopy groups of $\mathbb{C}P^\infty$. The easiest way to do this is to think of S^∞ as an S^1 bundle over $\mathbb{C}P^\infty$. Note this shows that $\mathbb{C}P^\infty$ is a $K(\pi, n)$ for some π and n.
10. Show that any map $f : X \to Y$ from an n-dimensional CW complex to an n-connected space is null-homotopic.
11. A space X is *aspherical* if $\pi_n(X) = 0$ for $n > 1$. If Y is an aspherical space then show that for any homomorphism $\psi : \pi_1(X) \to \pi_1(Y)$ there is a map $f_\psi : X \to Y$ that induces ψ on π_1. In other words there is a one to one correspondence
$$[X, Y]_0 = Hom(\pi_1(X), \pi_1(Y)).$$