Math 6441 - Spring 2020
Homework 3

Read Chapter 1.2 and 1.3 in Hatcher.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 9, 10, 11, 14, 18, and 20.* **Due: In class on February 14.**

1. Let G_1 and G_2 be non-trivial groups. Show that $G_1 * G_2$ is non-abelian, has elements of infinite order, and has trivial center.
2. Prove that a finite order element of $G_1 * G_2$ is either an element in G_1, in G_2 or a conjugate of such an element.
3. Show that the maximal finite order of an element of $\mathbb{Z}_n * \mathbb{Z}_m$ is the maximum of n and m.
4. Suppose a group G has a presentation $\langle x_1, \ldots, x_n | r_1, \ldots r_m\rangle$ where the relators are
$$r_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{k_i}}^{s_{i_{k_i}}}$$
for $i = 1, \ldots m$, and the s_i are ± 1. Then show that if H is any other group and $h_1, \ldots, h_n$ are any elements of H that satisfy
$$h_{i_1}^{s_{i_1}} \cdots h_{i_{k_i}}^{s_{i_{k_i}}} = e_h,$$
where e_H is the identity element in H, then there is a unique homomorphism $\phi : G \to H$ such that $\phi(x_i) = h_i$.
5. Prove that the groups $\langle x, y | x^2y^2\rangle$ and $\langle a, b | baba^{-1}\rangle$ are isomorphism.
6. Show that $\mathbb{Z} \oplus \mathbb{Z}$ has presentation $\langle x, y | xyx^{-1}y^{-1}\rangle$.
7. Consider the rational numbers $\mathbb{Q}$ as a group under addition. Show that $\mathbb{Q}$ has presentation
$$\langle x_i, i = 1, 2, 3, \ldots | x_n^n = x_{n-1}, i > 1\rangle.$$
Hint: try to construct a map by sending x_i to $\frac{1}{n!}$.
8. Let X be the union of the unit sphere S^2 in $\mathbb{R}^3$ and the unit disk D^2 in the xy-plane. Compute $\pi_1(X)$.
9. Take two copies of the torus $S^1 \times S^1$ and let X be the space obtained by identifying $S^1 \times \{pt\}$ in one torus with $S^1 \times \{pt\}$ in the other torus using the identity map on S^1. Compute $\pi_1(X, x_0)$.
10. Given a map $f : X \to X$ the **mapping torus T_f of** f is the space obtained from $X \times [0, 1]$ by identifying $(x, 0)$ with $(f(x), 1)$ for all $x \in X$. If $X = S^1 \vee S^1$ and f is a base point preserving map, write a presentation for $\pi_1(T_f)$ in terms of the map $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$.
11. Let X and Y be two non-empty spaces. If X is path connected and Y has two path components then show that the join $X * Y$ simply-connected. You may assume the path components of Y are open, it is true in general and you might want to think how to handle the general case but you don't need to write this up. (This is true for arbitrary Y, maybe think about how you would prove this, but you only need to write up the case when Y has two path components.)
12. Let X_1 and X_2 be the torus $S^1 \times S^1$ with an open disk removed and $C'' = S^1 \times \{pt\}$ in X_1 (we assume the disk that is removed is disjoint from C''). Let X be $X_1 \cup X_2$ with the boundaries glued together. So X is a genus 2 surface. Let C be the boundary of X_1 sitting in X and C' be C'' thought of as a subset of X. Show there is no retraction from X to C but there is a retraction from X to C'. (Hint: what is C in the abelianization of $\pi_1(X)$?)

13. Use covering spaces to show that the free group of rank 2 has a normal subgroup of index 3 and a non-normal subgroup of index 3. If the free group is generated by elements a and b the give explicit generators for the subgroups.
14. If X is a space with contractible universal cover then show that any map $S^n \to X$, $n \geq 2$, can be extended to a map $D^{n+1} \to X$.
15. If X is space with contractible universal cover and Y is any path connected CW-complex with a vertex labeled y_0, then show that any homomorphism $\phi : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is induced by a continuous map $f : Y \to X$. That is f takes y_0 to x_0 and $f_* = \phi$.
    Hint: Consider the previous problem.
16. If F_n is the free group on n generators and G is a subgroup in index k. Then you know from Hatcher Appendix 1.A that G is a free group. How many generators is it a free group on?
17. Let X be a connected, locally pathwise connected, and semi-locally simply connected topological space (so a space with a universal cover). Prove that a connected n-fold covering spaces of X correspond to representations of $\pi_1(X)$ to the symmeric group S_n that acts transitively on $\{1, \ldots, n\}$.
    More specifically, show that given a connected covering space $\widetilde{X} \to X$ there is an associated homomorphism $h : \pi_1(X) \to S_n$ (such that for any i and j there is some $g \in \pi_1(X)$ such that $h(g)(i) = j$) and conversely give such a homomorphism there is a connected covering space realizing this homomorphism.
18. Using the previous problem you can describe a degree n covering space of $S^1 \vee S^1$ by labeling the loops with elements of S_n (so that the two elements generate a transitive subgroup of S_n). Draw the covering spaces corresponding to the following labelings
    (a) Label with elements (1 2) and id in S_2.
    (b) Label with elements (1 2 3) and id in S_3.
    (c) Label with elements (1 2) and (2 3) in S_3.
19. Using the above idea list all 3 fold covers of $S^1 \vee S^1$.
20. Let X be a path connected, locally path connected space with $\pi_1(X, x_0)$ finite, show that any map $X \to S^1$ is nullhomotopic.
21. If F is a free group and N is a non-trivial infinite index normal subgroup of F, then show using covering spaces that N is not finitely generated. In particular show that if F is a free group on more than one generator then the commutator subgroup of F is not finitely generated.
22. Show that a finitely generated group has only a finite number of index n subgroups for a fixed n. Hint: Consider the case of free groups using graphs and covering space theory. Then prove the general result by the fact that any group is a quotient group of a free group.
23. Let $p : X \to Y$ and $q : Y \to Z$ be covering spaces. If each are of finite degree show that $q \circ p : X \to Z$ is a covering space.
24. Show that if in the previous problem p is of infinite degree then the composition does not have to be a covering space. For this consider $Z = \cup_{i=1}^{\infty} C_i$, where C_i is the circle of radius $1/i$ centered at $(0, 1/i)$ in $\mathbb{R}^2$. Let Y be the union of the x-axis and $C = \cup_{i=2}^{\infty} C_i$ and all the translates of C by vectors $(n, 0)$. Show there is an obvious infinite cover of Y over Z. Now find a 2 fold cover of Y such that the composition with the infinite cover is not a covering map.