Math 6441 - Spring 2020
Homework 5

Read Chapter 3.1 and 3.2 in Hatcher's book.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 5, 6, 8, 9, 11, and 13.* **Due: April 15.**

1. Given a homomorphism $f : G \to H$ of abelian group show there is an induced homomorphism $f_* : H_n(X; G) \to H_n(X; H)$.
2. If $0 \to G_1 \xrightarrow{a} G_2 \xrightarrow{b} G_3 \to 0$ is a short exact sequence of abelian groups then for any topological space there is a long exact sequence

   $$\ldots \to H_n(X; G_1) \xrightarrow{a_*} H_n(X; G_2) \xrightarrow{b_*} H_n(X; G_3) \xrightarrow{\beta} H_{n-1}(X; G_1) \ldots$$

   The map β is called the Bockstein map.
3. Let X be a CW complex of dimension n with a finite number of cells. Let l_k be the number of k cells in X. Show

   $$\sum_{i=0}^{n} (-1)^i l_k = \sum_{i=0}^{n} (-1)^i \, \text{rank} H_i(X).$$

   This number is called the Euler characteristic of X and denoted $\chi(X)$. Note the formula above implies that you can (1) compute $\chi(X)$ easily from the cell structure on X, but it is an invariant of the homotopy type of X.
4. If X is a CW complex that is the union of Y and Z where Y, Z, and $Y \cap Z$ are subcomplexes, then

   $$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$
5. Show that if $\widetilde{X}$ is a n fold covering space of the CW complex X then $\chi(\widetilde{X}) = n\chi(X)$. Hint: you need to say something about the CW structure on $\widetilde{X}$. For this problem you just need to describe the structure, but don't need to prove the structure is correct (though you should think through this).
6. Show that a surface of genus g can be a covering space of a surface of genus h if and only if $g = n(h-1)+1$ for some integer n. (Here all surfaces are assumed to be oriented.)
7. Compute the homology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$.
8. Given a sequence of abelian groups $G_0 = \mathbb{Z}, G_1, \ldots, G_n$ show there is a CW complex X such that $H_i(X) \cong G_i$.
9. A cochain $\phi \in C^1(X; G)$ can be thought of as a function from paths in X to G. Show that if $\delta\phi = 0$ then
   (a) $\phi(\gamma * \eta) = \phi(\gamma) + \phi(\eta)$,
   (b) if γ is homotopic to η rel endpoints, then $\phi(\gamma) = \phi(\eta)$,
   (c) ϕ is a coboundary if and only if $\phi(f)$ only depends on the endpoints of f, for all f,
   (d) Show that the discussion above gives a homomorphism $H^1(X; G) \to Hom(\pi_1(X); G)$ Note: the universal coefficients theorem says that this map is an isomorphism if X is path connected.

10. Show that a map $f : S^n \to S^n$ has degree n if and only if $f^* : H^n(S^n; \mathbb{Z}) \to H^n(S^n; \mathbb{Z})$ is multiplication by n.
11. In Hatcher's book, the cohomology of $\mathbb{R}P^n$ is computed. Use this computation (you do not have to reproduce it) to show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ if $n > m$.
12. Let M_g be a surface of genus g. Let X be the wedge product of g copies of the torus T^2. Problem 1 in Section 3.2 of Hatcher's book gives a quotient map $M_g \to X$. Compute the map induced on cohomology and determine the cup product structure on M_g in terms of the cup produce structure of T^2 (which we worked out in class).
13. Show that any map $S^4 \to S^2 \times S^2$ must induce the zero map on H^4. Show that this is not necessarily true for maps $S^2 \times S^2 \to S^4$.
14. For $S^n \times S^m$ compute all cap products (don't forget the case when $n = m$).