Math 6441 - Spring 2020
Supplement 2: Computing maps on cellular homology

Given two CW complexes X and Y we call a continuous map $f : X \to Y$ cellular if $f(X^{(n)}) \subset Y^{(n)}$ for all n. Here $X^{(n)}$ is the n skeleton of X. Any continuous map can be homotoped to be cellular.

Given a cellular map $f : X \to Y$ one can compute the induced map

$$f_* : C_k^{CW}(X) \to C_k^{CW}(Y)$$

and hence the map $f_* : H_k^{CW}(X) \to H_k^{CW}(Y)$ as follows. Suppose the k cells of X are e_i^k and the k cells of Y are f_j^k, recall these are the generators of the respective CW chain groups. Consider the sequence of maps

$$e_i^k \xrightarrow{f} Y^{(k)} \xrightarrow{q} Y^{(k)}/Y^{(k-1)} = \vee S^k \xrightarrow{q_j} S^k,$$

where q is the quotient map indicated, and q_j is another quotient map where all but the jth sphere is collapsed. (Recall there is one S^k in the wedge for each k cell in Y). Let $\overline{f}_{i,j}$ be the composition of these maps. So

$$\overline{f}_{i,j} : e_i^k \to S^k$$

and since ∂e_i^k is mapped to a single point by $\overline{f}_{i,j}$, $\overline{f}_{i,j}$ induces a map

$$f_{i,j} : e_i^k/\partial e_i^k \to S^k$$

and since $e_i^k/\partial e_i^k$ is homeomorphic to S^k we see that $f_{i,j}$ is a map between spheres. Now

$$f_* : C_k^{CW}(X) \to C_k^{CW}(Y)$$

is the map that is defined by sending e_i^k to $\sum_j \deg(f_{i,j}) f_j^k$. (Recall the chain groups are free so it suffices to define them on the generators.)

It is not too hard to prove this recall we showed that $H_k^{CW}(X) \cong H_k(X^{(k)}, X^{(k-1)})$ so you just need to see the map f induces on $H_k(X^{(k)}, X^{(k-1)})$. You can figure this out in essentially the same way we did for understanding ∂^{CW} in class.