## C. Applications

given a map $f: S^1 \to S^1$

let $\bar{f}: [0,1] \to S^1$ be the map $f \circ q = \bar{f}$

where

$$\begin{tikzcd}
{[0,1]} \arrow[d, "q"'] \arrow[dr, bend left, "\bar{f}"] & \\
S^1 \arrow[r, "f"'] & S^1
\end{tikzcd}$$

$$q(t) = (\cos 2\pi t, \sin 2\pi t)$$

Th$^{m}$ 8 says there is a unique lift $\tilde{f}: [0,1] \to \mathbb{R}$ of $\bar{f}$ once we choose lift of $\bar{f}(0)$

we now define the degree of $f$ to be the number

$$\deg f = \tilde{f}(1) - \tilde{f}(0)$$

note: if $\hat{f}$ is another such lift then $\hat{f}(s) = \tilde{f}(s) + k$ for some $k$

so $\hat{f}(1) - \hat{f}(0) = \tilde{f}(1) + k - (\tilde{f}(0) + k) = \tilde{f}(1) - \tilde{f}(0)$

and the degree is well-defined

## Th$^{m}$ 11:

$$\boxed{\begin{array}{c} f : S^1 \to S^1 \text{ and } g : S^1 \to S^1 \text{ are homotopic} \\ \Longleftrightarrow \\ \deg f = \deg g \end{array}}$$

**Proof:** $(\Rightarrow)$ let $F : S^1 \times [0,1] \to S^1$ be the homotopy

let $\bar{F} : [0,1]\times[0,1] \to S^1$ be the map such that

$$F(q(s),t) = \bar{F}(s,t)$$

$$\begin{tikzcd} {[0,1]\times[0,1]} \arrow[d, "q \times id"'] \arrow[dr, bend left, "\bar{F}"] & \\ S^1 \times [0,1] \arrow[r, "F"] & S^1 \end{tikzcd}$$

let $\tilde{f}, \tilde{g}$ be lifts of $\bar{f}$ ang $\bar{g}$ as above

by Th$^{m}$ 10 $\exists!$ lift of $\bar{F}$ to $\tilde{F} : [0,1]\times[0,1] \to \mathbb{R}$ with $\tilde{F}(0,0) = \tilde{f}(0)$

by uniqueness of path lifting we know $\tilde{F}(s,0) = \tilde{f}(s)$ since $\tilde{F}(s,0)$ is a lift of $\bar{f}$

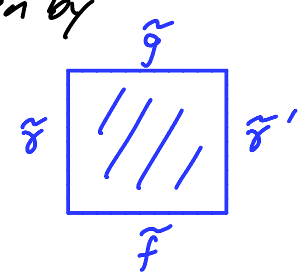
let $\gamma : [0,1] \to S^1 : t \longmapsto \bar{F}(0,t) = \bar{F}(1,t)$

set $\tilde{\gamma} : [0,1] \to \mathbb{R}$ lift of $\gamma$ s.t. $\tilde{\gamma}(0) = \tilde{f}(0)$

$\tilde{\gamma}' : [0,1] \to \mathbb{R}$ lift of $\gamma$ s.t. $\tilde{\gamma}'(0) = \tilde{F}(1)$

we can assume $\tilde{g}(0) = \tilde{\gamma}(1)$

note we have $\tilde{F}$ given by




so $\deg f = \tilde{F}(1,0) - \tilde{F}(0,0) = \tilde{\gamma}'(0) - \tilde{\gamma}(0)$

$\deg g = \tilde{F}(1,1) - \tilde{F}(0,1) = \tilde{\gamma}'(1) - \tilde{\gamma}(1)$

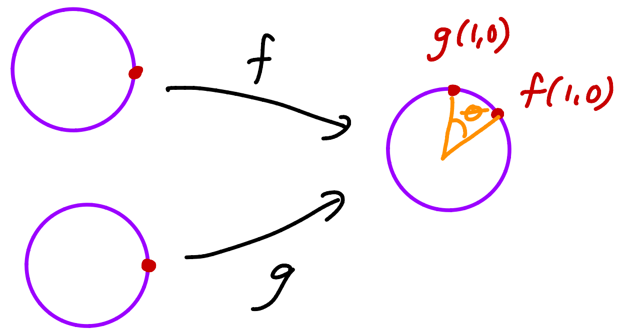
but $\tilde{\gamma}'(t) = \tilde{\gamma}(t) + k$ some $k$

and $\deg f = \deg g$ ✓

($\Leftarrow$) assume $\deg f = \deg g$

let $\theta$ be the angle between $f(1,0)$ and $g(1,0)$

let $R_t : S^1 \to S^1$ be rotation through angle $t$




set $H(s,t) = R_{t\theta} \circ f(s)$

so $H(s,0) = f(s)$

$H(s,1) = R_\theta \circ f(s)$ i.e. $H((1,0),1) = R_\theta \circ f((1,0)) = g((1,0))$

so after homotopy we can assume $f((1,0)) = g((1,0))$

(from ($\Rightarrow$) we know $\deg f$ unchanged under htpy.)

let $\bar{f}, \bar{g} : [0,1] \to S^1$ be as above (note $\bar{f}(0) = \bar{g}(0)$ by above)

let $\tilde{f}, \tilde{g} : [0,1] \to S^1$ be lifts of $\bar{f}, \bar{g}$, respectively

s.t. $\tilde{f}(0) = \tilde{g}(0)$

now $\deg f = \deg g \Rightarrow \tilde{f}(1) = \tilde{g}(1)$

set $\tilde{H}(s,t) = t\tilde{f}(s) + (1-t)\tilde{g}(s)$

note: $\tilde{H}(0,t) = t\tilde{f}(0) + (1-t)\tilde{g}(0) = \tilde{f}(0)$

$\tilde{H}(1,t) = t\tilde{f}(1) + (1-t)\tilde{g}(1) = \tilde{f}(1)$

so $p \circ \tilde{H}_t : [0,1] \to S^1$ decends to a map $H_t : S^1 \to S^1$ $\forall t$

$H_t$ give homotopy of $f$ to $g$ ▦

exercise: 1) the constant map $f : S^1 \to S^1$ has degree 0

2) $f_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$ is multiplication by $\deg f$

i.e. $\mathbb{Z} \to \mathbb{Z}$

$[\gamma] \longmapsto (\deg f)[\gamma]$

need to homotop $f$ to preserve base pt!

**Corollary 12:**

two maps $f, g : S^1 \to S^1$ are homotopic

$$\Longleftrightarrow$$

$$f_* = g_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0))$$

In particular, $f : S^1 \to S^1$ is homotopically trivial $\Longleftrightarrow$ it induces trivial map on $\pi_1(S^1, (1,0))$

**Proof:** imediate from exercises ∎

**Remark:** so maps on $S^1$ are completely determined by $\pi_1$!

**Lemma 13:**

a map $f : S^1 \to S^1$ extends to a map $F : D^2 \to S^1$

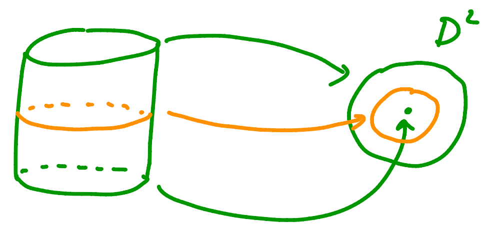
$$\Longleftrightarrow$$

$$\deg f = 0$$

**Proof:** $(\Rightarrow)$ let $P : [0,1] \times S^1 \to D^2$

$$(r, \theta) \longmapsto (r, \theta)$$

polar coords




given $F : D^2 \to S^1$ such that $F|_{\partial D^2} = f$

set $H(s,t) = F \circ P(s,t)$

this is a homotopy from

$$H(s,0) = F \circ P(s,0) = F(0,s) = pt$$

origin

to

$$H(s,1) = F \circ P(s,1) = F|_{\partial D^2} = f(s)$$

so $f \simeq$ constant $\therefore \deg f = 0$ ✓

$(\Leftarrow)$ if $\deg f = 0$, then $\exists$ a homotopy $H : S^1 \times [0,1] \to S^1$

s.t. $H(s,1) = f(s)$ and $H(s,0) = pt$

so we get an induced map $F : D^2 \to S^1$ that extends $f$

$$\begin{tikzcd} S^1 \times [0,1] \arrow[r, "H"] \arrow[d, "\text{quotient map}"'] & S^1 \\ D^2 \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

∎

exercise:

think of $S^1$ as the unit circle in $\mathbb{C}$

let $f_n: S^1 \to S^1 : z \mapsto z^n$

show $\deg(f_n) = n$

$$\begin{tikzcd} {[0,1]} \arrow[r, "t \mapsto nt"] \arrow[d, "q"] & \mathbb{R} \arrow[d, "p"] \\ S^1 \arrow[r, "z^n"] & S^1 \end{tikzcd}$$

Th$^{\underline{m}}$ 14 (Fundamental Th$^{\underline{m}}$ of Algebra):

> any non-constant complex polynomial $P(z)$ has a root
> i.e. $z_0$ such that $P(z_0) = 0$

Remark: Amazing! We are using algebraic topology to prove basic facts about polynomials!

Proof: let $P(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \qquad n \geq 1$

assume $P(z)$ has no root

let $M = \max\{|a_0|, \ldots, |a_{n-1}|\}$ and choose $k \geq \max\{1, 2nM\}$

note: $P(z) = z^n(1 + \underbrace{a_{n-1}\frac{1}{z} + \ldots + a_1 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n}}_{b(z)})$

so if $|z| = k$, then

$$\begin{aligned} |b(z)| &\leq |a_{n-1}| \frac{1}{|z|} + \ldots + |a_0| \frac{1}{|z|^n} \\ &\leq M\left(\frac{1}{k} + \ldots + \frac{1}{k^n}\right) \leq M \frac{n}{k} \\ &\leq M \frac{n}{2nM} = \frac{1}{2} \end{aligned}$$

let $f: S^1 \to S^1 : z \mapsto \dfrac{P(kz)}{|P(kz)|}$ ← well-defined since never zero by assumption

this extends to

$$F: D^2 \to S^1 : z \mapsto \frac{P(kz)}{|P(kz)|}$$

so $\deg f = 0$ by lemma 14

but let $P_t(z) = z^n(1+tb(z))$

from above $P_t(z) \neq 0$ for $|z|=k$

so $f_t : S^1 \to S^1 : z \mapsto \dfrac{P_t(kz)}{|P_t(kz)|}$

is a homotopy from $f$ to $f_1(z) = \dfrac{(kz)^n}{|kz|^n} = \dfrac{k^n z^n}{k^n \underbrace{|z|^n}_{=1}} = z^n$

$\deg f_1 = n \neq 0$ & $f \simeq f_1$ by Th^m 12

$\therefore$ $P(z)$ has a root! ∎

**lemma 15:**

> If $f: S^1 \to S^1$ is continuous and $f(-x) = -f(x)\ \forall x$
> then $\deg(f)$ is odd

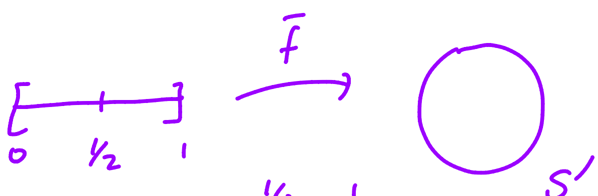
**Proof:** given such an $f: S^1 \to S^1$

let $\bar{f} : [0,1] \to S^1$ be as above (i.e. $f \circ q = \bar{f}$)

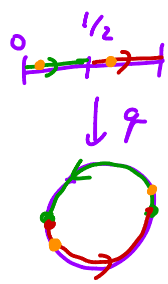
let $a = \bar{f}(0)$ and $p^{-1}(a) = \{\tilde{a}_i\}$ where $p: \mathbb{R} \to S^1$ and $\tilde{a}_i = \tilde{a}_0 + i$

note $\bar{f}(1/2) = f((-1,0)) = -f((1,0)) = -a$ and
$p^{-1}(-a) = \{\tilde{b}_i\}$ where $\tilde{b}_i = \tilde{a}_i + \frac{1}{2}$

let $f_1 = \bar{f}|_{[0,1/2]}$
$f_2 = \bar{f}|_{[1/2,1]}$




since $f(x) = f(-(-x)) = -f(-x)$
and
$q(x - 1/2) = -q(x)$




we have $f_2(x) = \bar{f}(x) = f(q(x))$
$= -f(-q(x)) = -f(q(x-1/2)) = -\bar{f}(x - 1/2) = -f_1(x - 1/2)$

so if $\tilde{f}_1$ is a lift of $f_1$ starting at $\tilde{a}_0$ then $\tilde{f}_1(1/2) = \tilde{b}_i$ some $i$

and $\tilde{f}_1(x - 1/2) + 1/2$ is a lift of $f_2 : [1/2, 1] \to S^1$ starting at $\tilde{a}_0 + 1/2 = \tilde{b}_0$

just like for $q$ above $p(x - 1/2) = -p(x)$

so $\tilde{f}_2(x) = \tilde{f}_1(x - 1/2) + q + 1/2$ is a lift of $f_2$ starting at $\tilde{f}_1(1/2) = \tilde{b}_2$

now $\tilde{f}_2(1) = \tilde{f}_1(1/2) + q + 1/2 = \tilde{b}_2 + q + 1/2 = \tilde{a}_1 + q + 1 = \tilde{a}_0 + 2q + 1$

note $\tilde{f}_1 * \tilde{f}_2$ is a lift of $f$

so $\deg(f) = \tilde{f}_1 * \tilde{f}_2(1) - \tilde{f}_1 * \tilde{f}_2(0) = \tilde{a}_0 + 2q + 1 - \tilde{a}_0 = 2q + 1$ ∎

**Th^m 16 (Borsuk-Ulam I):**

> There does not exist a continuous map
> $$f : S^2 \to S^1$$
> sending antipodal points to antipodal points

**Proof:** If $f : S^2 \to S^1$ is such a map

then let $S^1 \subset S^2$ be the equator

$$f|_{S^1} : S^1 \to S^1 \quad \text{satisfies} \quad f(-x) = -f(x)$$

so $\deg f|_{S^1}$ is odd by lemma 15

but $f|_{S^1}$ extends over northern hemisphere

so $\deg f|_{S^1} = 0$ by lemma 13 ⊗ ∎

**Th^m 17 (Borsuk-Ulam II):**

> Any continuous map $f : S^2 \to \mathbb{R}^2$ must send a pair of antipodal points to the same point

**Proof:** given any continuous $f : S^2 \to \mathbb{R}^2$

assume $f(x) \neq f(-x)$ $\forall x \in S^2$

then consider $g : S^2 \to S^1 : x \longmapsto \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$

exercise: $g$ is continuous

clearly $g(-x) = -g(x)$ ⊗ Th^m 16 ∎

**Remark:** Th^m implies that at any point in time there are antipodal point on the earth with the same temperature and humidity!

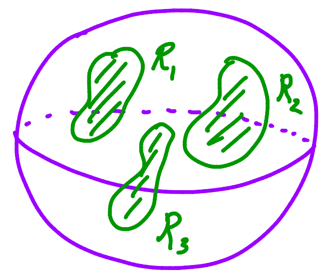
(as ... two ... continuously moving quantities)

## Th$^{m}$ 18 (Ham sandwich th$^{m}$):

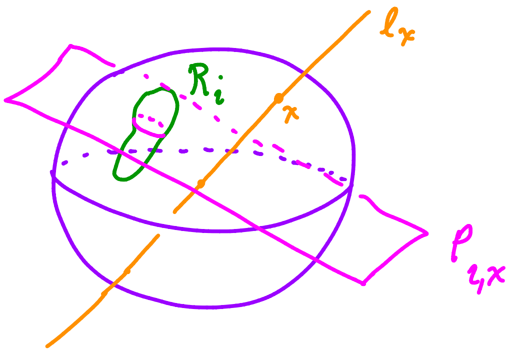
> let $R_1, R_2, R_3$ be three connected open regions in $\mathbb{R}^3$ each of which is bounded and of finite volume
> Then $\exists$ a plane which cuts them in half by volume

**Proof:** let $S^2 \subset \mathbb{R}^3$ be a large sphere about origin containing all $R$

given $x \in S^2$, let $l_x$ be the line through $x$ and origin




for each $i$, $\exists$ plane $P_{i,x}$ perpendicular to $l_x$ that cuts $R_i$ in half




let $d_i(x)$ = distance of $P_{i,x}$ from origin (where $d_i(x) > 0$ if $P_{i,x}$ on same side of origin as $x$)

**exercise:** Show $d_i(x)$ are continuous functions $d_i : S^2 \to \mathbb{R}$

**Hint:** Equation of planes perpendicular to $l_x$ continuously vary with $x$

Volume of regions of $R_i$ cut by plane continuously vary with eq$^{n}$ of plane

clearly $d_i(-x) = -d_i(x)$

consider $f : S^2 \to \mathbb{R}^2 : x \mapsto (d_1(x) - d_2(x),\ d_1(x) - d_3(x))$

Th$^{m}$ 17 $\Rightarrow \exists\, x$ such that $f(x) = f(-x)$

so $d_1(x) - d_2(x) = d_1(-x) - d_2(-x) = -d_1(x) + d_2(x)$

$\therefore\ 2d_1(x) = 2d_2(x) \Rightarrow d_1(x) = d_2(x)$

similarly $d_3(x) = d_1(x) = d_2(x)$

so $\exists$ plane $\perp$ to $l_x$ that cuts $R_1, R_2, R_3$ in half! $\square$