III Cohomology

A Cohomology groups of a chain complex

a sequence of abelian groups C^* and maps

$$\delta^n : C^n \to C^{n+1}$$

is called a co-chain complex if $\delta_{n+1} \circ \delta_n = 0$ for all n

the "homology" of the complex is called the cohomology of (C^*, δ)

$$H^n(C^*, \delta) = \ker \delta_n / \operatorname{im} \delta_{n-1}$$

If (C_*, ∂) is a chain complex and G any abelian group then we get a dual co-chain complex

$$C^n = \operatorname{Hom}(C_n, G) = \{\text{homomorphisms } C_n \to G\}$$

is G omitted, then assumed to be $\mathbb{Z}$

and $\delta_n = \partial^*_{n+1} : C^n \to C^{n+1}$

i.e. $\tau \in C^n$ so $\tau : C_n \to G$

then $\delta(\tau) : C_{n+1} \to G : \sigma \mapsto \tau(\partial_{n+1}\sigma)$

note $[\delta_{n+1} \circ \delta_n(\tau)](\sigma) = [\delta_n(\tau)](\partial_{n+2}\sigma) = \tau(\partial_{n+1} \circ \partial_{n+2}\sigma) = \tau(0) = 0$

so (C^*, δ) is a co-chain complex and

$$H^n(C_*; G) = \ker \delta_n / \operatorname{im} \delta_{n-1}$$

is called the cohomology of (C_*, ∂)

Question: Is there any more information in cohomology

Answer: No ... and Yes

we will see the groups $H^*(C^*, \delta)$ contain same information as groups $H_*(C_*, \partial)$

but the cohomology of a topological space $\bigoplus_n H^n(C_*(X))$ has a ring structure that does give more information about X.

if (A_*,∂) and (B_*,∂') are chain complexes

and $\alpha:(A_*,\partial)\to(B_*,\partial')$ is a chain map

then

$$\alpha^*: \begin{array}{ccc} B^* & \to & A^* \\ \psi & & \psi \\ \beta & \longmapsto & \beta\circ\alpha \end{array} \quad \text{is a co-chain map (i.e. } \delta\circ\alpha^* = \alpha^*\circ\delta'\text{)}$$

and hence induces a map $\alpha^*: H^n(B_*;G)\to H^n(A_*;G)$

exercise: 1) $\alpha:(A_*,\partial)\to(B_*,\partial')$

$\beta:(B_*,\partial)\to(C_*,\partial'')$ chain maps

then $(\beta\circ\alpha)^* = \alpha^*\circ\beta^*$

2) $\mathbb{1}^* = \mathbb{1}$ and $0^* = 0$

As mentioned above $H^*(C_*,\partial)$ is determined by $H_*(C_*,\partial)$

But this is not obvious

example: if (C_*,∂) is

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ \| & & \| & & \| & & \| \\ C_3 & & C_2 & & C_1 & & C_0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccccc} \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\times 2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} \\ \| & & \| & & \| & & \| \\ C^3 & & C^2 & & C^1 & & C^0 \end{array}$$

$\Downarrow$ homology $\qquad\qquad$ $\Downarrow$ cohomology

$$\begin{array}{cccc} H_3 & H_2 & H_1 & H_0 \\ \| & \| & \| & \| \\ \mathbb{Z} & 0 & \mathbb{Z}/2 & \mathbb{Z} \end{array} \quad \Bigg| \quad \begin{array}{cccc} H^3 & H^2 & H^1 & H^0 \\ \| & \| & \| & \| \\ \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z} \end{array}$$

so H^n is not just something like $\mathrm{Hom}(H_n,\mathbb{Z})$

note there is a natural pairing

$$H^n(C_*;G)\times H_n(C_*,\partial)\longrightarrow G$$
$$([\alpha]\ ,\ [\beta]) \longmapsto \alpha(\beta)$$

exercise: Show $\alpha(\beta)$ is independent of representative you take of $[\alpha]$ and $[\beta]$

thus we get a natural map

$$H^n(C_*;G) \xrightarrow{\Phi} \mathrm{Hom}(H_n(C_*,\partial),G)$$
$$[\alpha] \longmapsto \phi_{[\alpha]}: H_n(C_*,\partial) \to G$$
$$[\beta] \longmapsto \alpha(\beta)$$

we want to understand this map better

if A is an abelian group, then $\exists$ free abelian groups F and R and homomorphisms s.t.

$$0 \to R \xrightarrow{f} F \xrightarrow{g} A \to 0$$

is exact

exercise: $\mathrm{Hom}(\cdot,G)$ is left exact

i.e. $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 0$ exact, then

$$0 \to \mathrm{Hom}(G_3,G) \xrightarrow{\beta^*} \mathrm{Hom}(G_2,G) \to \mathrm{Hom}(G_1,G)$$

is exact

(but if $0 \to G_1 \xrightarrow{\alpha} G_2$ too then don't necessarily get α^* surjective)

define: $\mathrm{Ext}(A,G) = \mathrm{Hom}(R,G)/\mathrm{im}\, f^*$ (i.e. $\mathrm{coker} f^*$)

so $0 \to \mathrm{Hom}(A,G) \xrightarrow{g^*} \mathrm{Hom}(F,G) \xrightarrow{f^*} \mathrm{Hom}(R,G) \to \mathrm{Ext}(A,G) \to 0$ is exact.

so $\mathrm{Ext}(A,G)$ measures failure of $\mathrm{Hom}(F,G) \to \mathrm{Hom}(R,G) \to 0$ from being exact

examples:

$\mathbb{Z}$: $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

$$\mathrm{Ext}(\mathbb{Z},G) = 0/\mathrm{im}\, f^* = 0$$

$\mathbb{Z}/n$: $0 \to \mathbb{Z} \xrightarrow{n\times\cdot} \mathbb{Z} \to \mathbb{Z}_n \to 0$

$$\mathrm{Ext}(\mathbb{Z}_n,G) = \mathrm{Hom}(\mathbb{Z},G)/\mathrm{im}\,(n\times\cdot)^* \cong G/nG$$

in particular: $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}_m) \cong \mathbb{Z}_d$ $\quad d = \text{g.c.d.}(m,n)$

exercises:
1) $\mathrm{Ext}(A,G)$ independent of F, R, f, g
2) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
3) $\mathrm{Ext}(H,G) = 0$ if H is free

4) $\mathrm{Ext}(\mathbb{Z}_n;G) \cong G/nG$

5) from the above we can compute $\mathrm{Ext}(H,G)$ for all finitely generated abelian H and G

6) $\mathrm{Ext}(G;\mathbb{Q}) = 0 \quad \forall G$

Th$^{\underline{m}}$ 1 (Universal Coefficients Theorem):

$$0 \to \mathrm{Ext}(H_{n-1}(C_*),G) \to H^n(C_*;G) \to \mathrm{Hom}(H_n(C_*),G) \to 0$$

is exact and splits and is natural with respects to chain maps

being split means the middle group is the direct sum of the other two.

Proof: purely algebraic (not too hard) see Hatcher's book ▦

Cor 2:

if F_n = free part of $H_n(C_*)$
T_n = torsion part of $H_n(C_*)$

then $H^n(C^*;\mathbb{Z}) \cong F_n \oplus T_{n-1}$

Proof: clear from Th$^{\underline{m}}$ 1 and exercises ▦

example: suppose
$$H_n(C_*,\partial) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

then
$$H^n(C_*;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n \text{ odd} \\ \mathbb{Z}/2 & n \text{ even} > 0 \end{cases}$$

$$H^n(C_*;\mathbb{Z}/2) = \begin{cases} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}/2) & = \mathbb{Z}/2 \\ \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/2) \oplus \cancel{\mathrm{Ext}(0,\mathbb{Z}/2)} & = \mathbb{Z}/2 \\ \cancel{\mathrm{Hom}(0,\mathbb{Z}/2)} \oplus \mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2) & = \mathbb{Z}/2 \end{cases}$$

$$= \mathbb{Z}/2$$

Cor 3: if a chain map induces an isomorphism on all homology groups then it induces an isomorphism on all cohomology groups

Proof: if $\alpha : (C_*, \partial) \to (C_*', \partial')$ induces an isomorphism on homology then

$$\begin{array}{ccccccccc} 0 \to & Ext(H_{n-1}(C_*), G) & \longrightarrow & H^n(C_*; G) & \longrightarrow & Hom(H_n(C_*), G) & \longrightarrow 0 \\ & \uparrow (\alpha_*)^* & & \uparrow \alpha^* & & \uparrow (\alpha_*)^* & \\ 0 \to & Ext(H_{n-1}(C_*'), G) & \longrightarrow & H^n(C_*'; G) & \longrightarrow & Hom(H_n(C_*'), G) & \longrightarrow 0 \end{array}$$

two maps on end are isomorphisms so α^* isomorphism (exercise) ∎

B. Cohomology of a space

let X be a topological space

$(C_n(X), \partial)$ be the singular chain complex of X

the cohomology of this complex is the cohomology of X (w/coeff in G)

$$H^n(X; G)$$

similarly for the pair (X, A),

$$H^n(X, A; G)$$

is the cohomology of $(C_n(X,A), \partial)$

from Corollary 3 we know that if X is a CW complex then we get the same cohomology groups if we use $(C_n^{cw}(X), \partial^{cw})$.

if $f : X \to Y$ a continuous map then we get a chain map

$$f_* : C_n(X) \to C_n(Y)$$

and thus a homomorphism

$$f^* : H^n(Y; G) \to H^n(X; G)$$

if $f, g : X \to Y$ are homotopic then f_*, g_* are chain homotopic

i.e. $\exists\ P_n : C_n(X) \to C_{n+1}(Y)$ s.t. $\partial_{n+1} P_n + P_{n-1} \partial_n = f_n - g_n$

dualizing we get

$$P^* \delta + \delta P^* = f^n - g^n$$

<u>exercise</u>: this implies $f^* = g^*$ on $H^n(Y;G)$

Thus cohomology is a contravariant functor from

$\mathcal{H}$ = category of topological spaces and homotopy classes of continuous maps

to

$\mathcal{G}r$ = category of graded abelian groups

Exactly as we did for homology, we can prove

1) <u>Exact sequence of a pair</u>

$$\cdots \to H^n(X,A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \to \cdots$$

$$\begin{array}{ll} i : A \to X & \\ j : (X,\emptyset) \to (X,A) & \text{inclusion maps} \end{array}$$

and if $f : (X,A) \to (Y,B)$ then

$$\begin{tikzcd} H^n(A) \arrow[r, "\delta"] & H^{n+1}(X,A) \\ H^n(B) \arrow[r, "\delta"] \arrow[u, "f^*"] \arrow[ur, phantom, "\circ"] & H^{n+1}(Y,B) \arrow[u, "f^*"'] \end{tikzcd}$$

2) <u>Excision</u>: $Z \subset \bar{Z} \subset \operatorname{int} A \subset A \subset X$ then the inclusion $(X-Z, A-Z) \to (X,A)$ induces an isomorphism

$$H^n(X,A) \to H^n(X-Z, A-Z)$$

3) <u>dimension</u>:

$$H^n(pt;G) \cong \begin{cases} G & n=0 \\ 0 & n \geq 0 \end{cases}$$

4) <u>Mayer-Vietoris</u>: $X = A \cup B$ $\quad A, B$ open sets

$$\cdots \to H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(X) \to \cdots$$

exercise: from above show directly that

$$H^k(D^n;G) \cong \begin{cases} G & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$H^k(S^n;G) \cong H^k(D^n,\partial D^n;) \cong \begin{cases} G & k=0,n \\ 0 & k \neq 0,n \end{cases}$$

C. Products

we will define a

cross product: $H^p(X) \times H^q(Y) \longrightarrow H^{p+q}(X \times Y)$

$$(\alpha,\beta) \longmapsto \alpha \times \beta$$

that is bilinear: $(\alpha_1+\alpha_2)\times\beta = \alpha_1\times\beta + \alpha_2\times\beta$

$$\alpha\times(\beta_1+\beta_2) = \alpha\times\beta_1 + \alpha\times\beta_2$$

and natural: if $f: X' \to X$ and $g: Y' \to Y$ are maps

then $(f^*\alpha)\times(g^*\beta) = (f\times g)^*(\alpha\times\beta)$

cup product: $H^p(X)\times H^q(X) \longrightarrow H^{p+q}(X)$

$$(\alpha,\beta) \longmapsto \alpha\cup\beta$$

that is bilinear: $(\alpha_1+\alpha_2)\cup\beta = \alpha_1\cup\beta + \alpha_2\cup\beta$

$$\alpha\cup(\beta_1+\beta_2) = \alpha\cup\beta_1 + \alpha\cup\beta_2$$

and natural: if $f: X' \to X$ is a map, then

$$f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta)$$

the cup product is more useful and makes cohomology of spaces a stronger invariant of spaces, but the cross product is simpler to define and study

But the cup and cross products are logically equivalent

to see this let $p_1: X\times Y \to X$ and

$p_2: X\times Y \to Y$ be the projection maps

and $\Delta: X \to X\times X : p \mapsto (p,p)$ be the diagonal map suppose we have a cup product defined with above properties

we define: $\times_\cup : H^p(X)\times H^q(Y) \to H^{p+q}(X\times Y)$

$$(\alpha, \beta) \longmapsto p_1^*\alpha \cup p_2^*\beta$$

exercise: $\times_\cup$ is bilinear and natural

suppose we have a cross product defined with above properties

we define: $\cup_\times : H^p(X)\times H^q(X) \to H^{p+q}(X)$

$$(\alpha, \beta) \longmapsto \Delta^*(\alpha\times\beta)$$

exercise: $\cup_\times$ is bilinear and natural

note: given $\cup$ then $\cup_{\times_\cup} = \cup$

indeed $\alpha \cup_{\times_\cup} \beta = \Delta^*(\alpha \times_\cup \beta) = \Delta^*(p_1^*\alpha \cup p_2^*\beta)$

where $p_i : X\times X \to X$ projection to i^{th} factor

$= \Delta^* p_1^*\alpha \cup \Delta^* p_2^*\beta = (p_1\circ\Delta)^*\alpha \cup (p_2\circ\Delta)^*\beta$

$= \alpha\cup\beta$ (since $p_i\circ\Delta = id_X$)

exercise: Show $\times_{\cup_\times} = \times$

so if we can define either the cup or cross product then we get the other one and

$$\boxed{\begin{aligned} \alpha\cup\beta &= \Delta^*(\alpha\times\beta) \\ \alpha\times\beta &= p_1^*\alpha\cup p_2^*\beta \end{aligned}}$$

note: it is key we are working in cohomology not homology or none of the above would work!

Cross products:

1^{st} we need to "recall" the tensor product of groups (modules, ...)

let G and H be 2 abelian groups

let $F(G\times H)$ be the free abelian group generated by $G\times H$ (i.e. finite formal sums $\sum a_i(g_i, h_i)$)

let S = subgroup generated by

$$((g+g'),h)-(g,h)-(g',h)$$
$$(g,(h+h'))-(g,h)-(g,h')$$

follow from 1st 2 but nice to make explicit $\left\{ \begin{array}{l} (ng,h)-n(g,h) \\ (g,nh)-n(g,h) \end{array} \right.$

$\forall g,g' \in G$, $h,h' \in H$, $n \in \mathbb{Z}$

the tensor product of G and H is the group $G \otimes H = F(G\times H)/S$

the coset of (g,h) is denoted $g\otimes h$

so elements of $G\otimes H$ are $\sum_{i=1}^{k} a_i\, g_i \otimes h_i$ $\quad a_i \in \mathbb{Z}$

and we have
$$(g+g')\otimes h = g\otimes h + g'\otimes h$$
$$g\otimes(h+h') = g\otimes h + g\otimes h'$$
$$ng\otimes h = g\otimes nh = n(g\otimes h)$$

exercises:
1) $G\otimes H \cong H\otimes G$
2) $(\oplus_i G_i)\otimes H \cong \bigoplus_i (G_i \otimes H)$
3) $(G\otimes H)\otimes K \cong G\otimes(H\otimes K)$
4) $\mathbb{Z}\otimes G \cong G$
5) $\mathbb{Z}/n \otimes G \cong G/nG$
6) given homomorphisms $f: G\to G'$ and $g: H \to H'$
   then $f\otimes g : G\otimes H \to G'\otimes H'$ is a homomorphism
   $x\otimes y \mapsto f(x)\otimes g(y)$
7) a bilinear map $\phi: G\times H \to K$ induces a homomorphism
   $G\otimes H \to K$
   $g\otimes h \mapsto \phi(g,h)$

key property! turns bilinear maps into homomorphisms

more generally if R is a commutative ring with unit (think field but without multiplicative inverses) eg. $\mathbb{Z}$

and A and B are R-modules (think vector space over R) eg abelian groups are $\mathbb{Z}$-modules then you can analogously define $A \otimes_R B$

we can also take tensor products of complexes

let $(\mathcal{C}, \partial)$ and $(\mathcal{C}', \partial')$ be two chain complexes

their tensor product is the chain complex

$$(\mathcal{C} \otimes \mathcal{C}')_n = \bigoplus_{i+j=n} (\mathcal{C}_i \otimes \mathcal{C}_j')$$

with boundary maps

$$\partial^{\otimes}(a \otimes b) = (\partial a) \otimes b + (-1)^i a \otimes \partial' b \quad \text{if } a \in \mathcal{C}_i \text{ and } b \in \mathcal{C}_j'$$

exercise: $(\partial^{\otimes})^2 = 0$ so this is a chain complex

we now get an algebraic cross product

$$\times_{alg} : H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') \to H_{p+q}(\mathcal{C} \otimes \mathcal{C}')$$

$$[z] \otimes [w] \longmapsto [z \otimes w]$$

note: if $z = \bar{z} + \partial \tau$, then

$$z \otimes w = \bar{z} \otimes w + \partial \tau \otimes w$$

$$= \bar{z} \otimes w + \partial^{\otimes}(\tau \otimes w) \quad \text{since } \partial' w = 0$$

so $[z \otimes w] = [\bar{z} \otimes w]$

exercise: check $\times_{alg}$ is a well-defined homomorphism that is natural with respect to chain maps

Thm 4 (½ Künneth Sequence)(HA):

$$\boxed{0 \to \bigoplus_{p+q=n} (H_p(\mathcal{C}) \otimes H_q(\mathcal{C}')) \to H_n(\mathcal{C} \otimes \mathcal{C}') \quad \text{is exact}}$$

Remark: Proof is purely algebraic and we don't really need this so we will skip the proof (see book if you are interested)

now for topological spaces:

if X and Y are CW-complexes, then we get a CW-structure on $X \times Y$ by taking products of cells

i.e. e^i_j an i-cell of X

$\hat{e}^{i'}_{j'}$ an i'-cell of Y

then $e^i_j \times \hat{e}^{i'}_{j'}$ an $(i+i')$-cell of $X \times Y$

and if $a^i_j : \partial e^i_j \to X^{(i-1)}$ attaching map of e^i_j

and $\hat{a}^{i'}_{j'} : \partial \hat{e}^{i'}_{j} \to Y^{(i'-1)}$ attaching map of $\hat{e}^{i'}_{j'}$

then

$$\partial(e^i_j \times \hat{e}^{i'}_{j'}) = (\partial e^i_j) \times \hat{e}^{i'}_j \to e^i_j \times (\partial \hat{e}^{i'}_{j'}) \longrightarrow \overbrace{X^{(i-1)} \times Y^{(i')} \cup X^{(i)} \times Y^{(i'-1)}}^{\subseteq (X \times Y)^{(i+i'-1)}}$$

$$(x,y) \longmapsto (a^i_j(x), y)$$

$$(x,y) \longmapsto (x, \hat{a}^{i'}_{j'}(y))$$

is the attaching map for $e^i_j \times \hat{e}^{i'}_{j'}$

thus if $a \in C^{cw}_i(X)$, $b \in C^{cw}_{i'}(Y)$

$$a = \sum \alpha^k e^i_k$$

$$b = \sum \beta^\ell \hat{e}^{i'}_\ell$$

then $$a \otimes b = \sum \alpha^k \beta^\ell (e^i_k \otimes \hat{e}^{i'}_\ell)$$

and $$\overset{cw}{\partial}(a \times b) = \sum \alpha^k \beta^\ell \left((\partial e^i_k) \times \hat{e}^{i'}_\ell + (-1)^i e^i_k \times \partial \hat{e}^{i'}_\ell \right)$$

think about why sign is here!

$$= (\overset{cw}{\partial} a) \times b + (-1)^i a \times \overset{cw}{\partial} b$$

thus we get a chain map

$$\bigoplus_{p+q=n} C^{cw}_p(X) \otimes C^{cw}_q(Y) \xrightarrow{B} C^{cw}_n(X \times Y)$$

similarly we get a chain map

$$C^{cw}_n(X \times Y) \xrightarrow{A} \bigoplus_{p+q=n} C^{cw}_p(X) \otimes C^{cw}_q(Y)$$

where $A\left(\sum_{p_i+q_j=n} a^{ij} e_i^{p_i} \times e_j^{q_j}\right) = \sum_{p_i+q_j=n} a^{ij} e_i^{p_i} \otimes e_j^{q_j}$

clearly $A \circ B(a) = a$, $\quad B \circ A(a \otimes b) = a \otimes b$

so we get a natural isomorphism

$$H_n(X \times Y) \to H_n(C_*^{cw}(X) \otimes C_*^{cw}(Y)) \quad \forall n$$

Similarly in singular homology we have

$$B : C_p(X) \otimes C_p(Y) \longrightarrow C_{p+q}(X \times Y)$$

$$(\sigma : \Delta^p \to X,\ \tau : \Delta^q \to Y) \longmapsto \sigma \times \tau : \underbrace{\Delta_p \times \Delta_q}_{} \to X \times Y$$

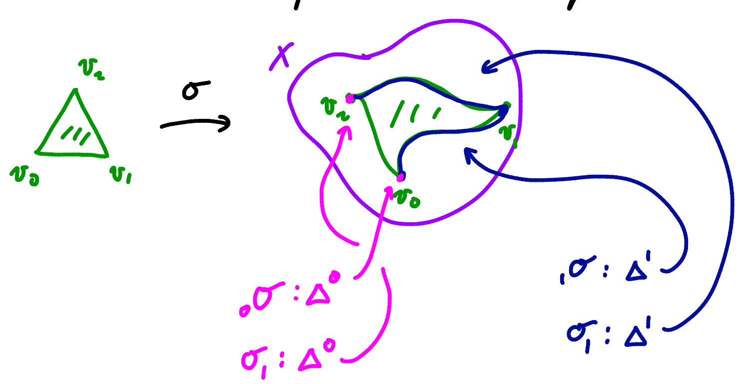
we can break this into union of p+q simplicies

e.g. $\Delta^p =$ $\quad \Delta^q =$

$\Delta^p \times \Delta^q =$ 

given $\sigma : \Delta^n \to X$ set ${}_p\sigma : \Delta^p \to X : (t_0, \dots, t_p) \longmapsto \sigma(t_0, \dots, t_p, 0, \dots, 0)$

$\sigma_q : \Delta^q \to X : (t_0, \dots, t_q) \longmapsto (0, \dots 0, t_0, \dots t_q)$

e.g. 

define $A : C_n(X \times Y) \longrightarrow \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$

$$\sigma \longmapsto \sum_{p+q=n} {}_p(p_X \circ \sigma) \otimes (p_Y \circ \sigma)_q$$

where $p_X : X \times Y \to X$
$p_Y : X \times Y \to Y$ are projections

Th^{m}5 (Eilenberg - Zilber Th^{m}):

$\boxed{\text{A, B induce natural chain maps and induce isomorphisms on homology (they are inverses)}}$

Remark: It is easy to see they are natural chain maps the rest is much more complicated. We will see some of the ideas involved later, but we skip the proof.

the homological cross product is

$$\begin{tikzcd} H_p(X)\otimes H_q(Y) \arrow[r, "\times_{alg}"] \arrow[rr, bend right, "\times"'] & H_{p+q}(C_*(X)\otimes C_*(Y)) \arrow[r, "B"] & H_{p+q}(X\times Y) \end{tikzcd}$$

$Th^{\underline{m}}$ 4 and 5 $\Rightarrow$

$$0 \longrightarrow \bigoplus_{n=p+q} \big(H_p(X)\otimes H_q(Y)\big) \longrightarrow H_n(X\times Y) \quad \text{is exact.}$$

Now for cohomology

C_*, C_*' chain complexes

define $\times_{alg}: C^p(C_*;G_1)\otimes C^q(C_*';G_2) \longrightarrow C^{p+q}(C_*\otimes C_*';G_1\otimes G_2)$

$$\alpha\otimes\beta \longmapsto \alpha\times\beta$$

where $\alpha\times\beta: C_p\otimes C_q' \to G_1\otimes G_2$

$$\sum z_i\otimes w_i \longmapsto \sum \alpha(z_i)\otimes\beta(w_i)$$

note: if $G_1=G_2=$ ring R then $G_1\otimes_R G_2 \cong R$

(e.g. $R=\mathbb{Z}$ then $\times_{alg}$ maps to same coeffs)

easy to check $\times_{alg}$ well-defined and induces homomorphism on homology

to cohomology cross product is

$$\begin{tikzcd} H^p(X;G_1)\otimes H^q(Y;G_2) \arrow[r, "\times_{alg}"] \arrow[rr, bend right, "\times"'] & H^{p+q}(C_*(X)\otimes C_*(Y);G_1\otimes G_2) \arrow[r, "A^*"] & H^{p+q}(X\times Y;G_1\otimes G_2) \end{tikzcd}$$

always use ring coeff. so defined with same coeff

as we mentioned erlier this also gives the cup product

$$H^p(X)\times H^q(X) \to H^{p+q}(X) : (\alpha,\beta) \longmapsto \Delta^*(\alpha\times\beta)$$

Alternate Cup Product definition

given $\alpha \in C^p(X;R)$
$\beta \in C^q(X;R)$

define $\alpha \cup \beta : C_{p+q}(X) \to R$ by

$$\alpha \cup \beta(\sigma) = \alpha({}_p\sigma)\,\beta(\sigma_q)$$ evaluate α on front p-face
evaluate β on back q-face

and on cohomology $[\alpha]\cup[\beta] = [\alpha \cup \beta]$

exercise: check $\cup$ well-defined, bilinear, natural map on cohomology

note: this agrees with above definition

$$\underset{\text{old def}^{n}}{(\alpha \cup \beta)(\sigma)} = \Delta^* A^* (\alpha \times_{alg} \beta)(\sigma) = (\alpha \times_{alg} \beta)\big[(A\circ\Delta)_* \sigma\big]$$
$$= (\alpha \times_{alg} \beta)\Big(\sum_{r+s=p+q} \underbrace{(\rho_1 \circ \Delta(\sigma))}_{\overset{\shortparallel}{\sigma}}{}_r \otimes \underbrace{(\rho_2 \circ \Delta(\sigma))}_{\overset{\shortparallel}{\sigma}}{}_s\Big)$$
$$= \alpha({}_p\sigma)\,\beta(\sigma_q)$$

note: can use this definition to define cross product
$$\alpha \times \beta = (\rho_1^* \alpha) \cup (\rho_2^* \beta)$$

Th^m 6:

1) let $\mathbb{1} \in H^0(X;R)$ be the elemend represented by the cocycle
$$1 : C_0(X) \to R : \sigma \mapsto 1$$ ← unit in R

Then $\mathbb{1} \cup \alpha = \alpha \cup \mathbb{1} = \alpha$

2) $\cup$ makes $C^*(X;R)$ and $H^*(X;R)$ a ring with unit that is natural
(i.e. $\cup$ is bilinear, associative, and has unit)

3) In cohomology $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$
if $\alpha \in H^p(X;R)$ and $\beta \in H^q(X;R)$

So $H^*(X)$ is a skew-commutative graded ring
(note if α has odd grading, then $\alpha \cup \alpha = -\alpha \cup \alpha$
so $\alpha \cup \alpha = 0$ if char $R \neq 2$)

Proof: 1) $1 \cup a : C_p(X) \to R : \sigma \mapsto 1({}_0\sigma)\,a(\sigma_p) = 1 \cdot a(\sigma_p) = a(\sigma)$
so $1 \cup a = a$ (you can check other) ✓

2) $\times$ is bilinear and natural so $\cup$ is too

$\times_{alg}$ clearly associative since $\otimes$ is

exercise: check $\times$, and hence $\cup$, is associative

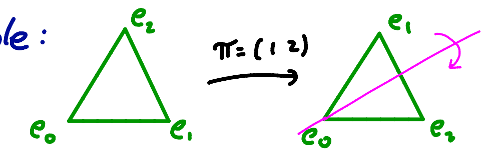
Hint: just need to consider map A^* above.

3) Is more complicated!

given a permutation π of $\{0,\dots,p\}$ we get a linear map

$$\Delta^p \xrightarrow{\pi} \Delta^p$$

that sends e_i to $e_{\pi(i)}$ example:



if σ is a p-simplex we get a new simplex

$\sigma_\pi : \Delta^p \to X$ by composing σ with above map

this defines a homomorphism $C_p(X) \to C_p(X)$

now let $\theta_p(i) = p-i$ be permutation sending $(0,\dots,p)$ to $(p,\dots,0)$

define: $\theta : C_p(X) \to C_p(X)$

$$z \mapsto (-1)^{\frac{1}{2}p(p+1)} z \circ \theta_p \qquad \text{(can do this for all } p)$$

Claim 1: θ is a chain map $\theta \circ \partial = \partial \circ \theta$

Claim 2: θ is chain homotopic to the identity.

we prove these later.

Claim 1 $\Rightarrow$ $\theta^* : C^*(X) \to C^*(X)$ is a cochain map

Claim 2 $\Rightarrow$ $\theta^* = id_{H^*(X)} : H^*(X) \to H^*(X)$

note: $_p(\sigma \circ \theta_{p+q}) = \sigma_p \circ \theta_p$

$(\sigma \circ \theta_{p+q})_q = {}_q\sigma \circ \theta_q$

now if $c \in C^p(X;R)$, $d \in C^q(X;R)$, then

$$\begin{aligned}
(\theta^*(c \cup d))(\sigma) &= (c \cup d)(\theta(\sigma)) = (-1)^{\frac{1}{2}(p+q)(p+q+1)} c\left({}_p(\sigma \circ \theta_{p+q})\right) d\left((\sigma \circ \theta_{p+q})_q\right) \\
&= (-1)^{\frac{1}{2}(p+q)(p+q+1)} c(\sigma_p \circ \theta_p)\, d({}_q\sigma \circ \theta_q) \\
&= (-1)^{\frac{1}{2}(p+q)(p+q+1) + \frac{1}{2}p(p+1) + \frac{1}{2}q(q+1)} c(\theta(\sigma))\, d(\theta(\sigma)) \\
&= (-1)^{\cdots} (\theta^* c)(\sigma_p)(\theta^* d)({}_q\sigma) \\
&= (-1)^{\cdots} (\theta^* d \cup \theta^* c)(\sigma) = (-1)^{\cdots} (\theta^*(d \cup c))(\sigma)
\end{aligned}$$

exponent is $\frac{1}{2}\left[p^2+pq+p+pq+q^2+q+p^2+p+q^2+q\right]$

$$= p^2+q^2+pq+p+q = \underbrace{p(p+1)}_{\text{even}} + \underbrace{q(q+1)}_{\text{even}} + pq$$

$$= pq \mod 2$$

$\therefore\ \theta^*(c \cup d - (-1)^{pq} d \cup c) = 0$

θ isomorphism $\Rightarrow$ $c \cup d = (-1)^{pq} d \cup c$ ✓

<u>Proof of Claim 1:</u> σ a p-simplex

$$\partial\theta(\sigma) = (-1)^{\frac{1}{2}p(p+1)}\partial(\sigma\circ\theta_p) = (-1)^{\frac{1}{2}p(p+1)}\sum(-1)^{p-i}\sigma\circ\underbrace{[e_p,\dots\hat{e}_i\dots e_0]}_{\text{recall this is }(p-i)^{th}\text{ face map}}$$

and

$$\theta\partial(\sigma) = \theta\sum(-1)^i\sigma\circ[e_0\dots\hat{e}_i\dots e_p] = (-1)^{\frac{1}{2}(p-1)p}\sum(-1)^i\sigma\circ[e_p\dots\hat{e}_i\dots e_0]$$

but consider exponents:

$$\left(\tfrac{1}{2}p(p+1)+p-i\right) - \left(\tfrac{1}{2}(p-1)p+i\right) = \frac{p^2+3p-p^2+p}{2} - 2i = 2(p-i) \text{ even}$$

so parity of exponents same and $\partial\theta = \theta\partial$ ✓

<u>Proof of Claim 2:</u>

need to construct

$$J_p : C_p(X) \to C_{p+1}(X)$$

s.t. $\quad Id - \theta = \partial_{p+1}\circ J_p + J_{p-1}\circ\partial_p$

we construct J_p by induction on p

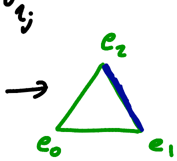
for $p \le 0$, set $J_p = 0$ (note $(Id-\theta)(\sigma^0) = 0$)

to inductively continue we need a little more set up

let $i_0,\dots,i_q$ be $(q+1)$ numbers between 0 and p (don't need to be distinct)

let $(i_0,\dots,i_q) : \Delta^q \to \Delta^p$ be the map

$$\sum_{j=0}^{q} t_j e_j \mapsto \sum_{j=0}^{q} t_j v_{i_j}$$

<u>example:</u> $(1,2) : [0,1] \to$ 

given a p-simplex $\sigma : \Delta^p \to X$

$\sigma(i_0,\dots,i_q)$ will be q-simplex $\sigma\circ(i_0,\dots,i_q)$

let $C(\sigma)_q$ be subgroup (module) of $C_q(X)$ generated by <u>all</u> of the $\sigma(i_0,\dots,i_q)$

<u>note</u>: $\partial(\sigma(i_0,\dots i_q)) = \sum_{i=0}^{q} (-1)^i \sigma(i_0\dots i_q)\big|_{[e_0\dots \hat{e}_i \dots e_q]}$
$= \sum_{i=1}^{q} (-1)^i \sigma(i_0 \dots \hat{i}_i \dots i_q) \in C(\sigma)_{q-1}$

so $(C(\sigma)_*, \partial)$ is a chain complex.

<u>note</u>: $H_q(C(\sigma)_*, \partial) = 0 \quad \forall q > 0$ such a complex is called <u>acyclic</u>

indeed define $B: C(\sigma)_q \to C(\sigma)_{q+1}$ by

$$B(\sigma(i_0 \dots i_q)) = \sigma(0, i_0, \dots i_q)$$

for any $z \in C(\sigma)_q$, $q > 0$

$$\partial(Bz) = z - B(\partial z)$$

so if $\partial z = 0$, then $z = \partial(Bz)$ $\therefore H_q = 0$ ✓

Back to construction of J_p

assume J_k defined for $k<p$ so that

1) $Id - \theta = J\partial + \partial J$ and
2) $\forall \tau \in C_q(X)$, $q<p$, then $J(\tau) \in C(\tau)_{q+1}$

now given σ a p-simplex

$$J\partial\sigma \subset \bigcup_i C(\sigma|_{[e_0 \dots \hat{e}_i \dots e_p]}) = \bigcup_i C(\sigma(0,\dots \hat{i}, \dots p)) \subset C(\sigma)_p$$

also

$$(Id - \theta)(\sigma) \in C(\sigma)_p$$

and

$$\partial[(Id - \theta - J\partial)\sigma] = [\underbrace{Id - \theta - \partial J}_{= J\partial \text{ by induction}}](\partial\sigma) = J\partial\partial\sigma = 0$$

but $C(\sigma)_*$ acyclic so $\exists z \in C(\sigma)_{p+1}$ s.t. $\partial z = (Id - \theta - \partial J)(\sigma)$

so set $J(\sigma) = z$ and we are done ▦

<u>Now for a computation</u>:

recall $H_n(S^n) = H^k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$

and

$$H_k(S^n \times S^m) = H^k(S^n \times S^m) = \begin{cases} \mathbb{Z} & k = 0, n, m, n+m \\ 0 & \text{otherwise} \end{cases}$$

← if $n = m$ get $\mathbb{Z} \oplus \mathbb{Z}$ for $k = n$

can compute using CW-str

e^0, e^n cells for S^n
f^0, f^m cells for S^m

$e^0 \times f^0$, $e^0 \times f^m$, $e^n \times f^0$, $e^n \times e^m$ for $S^n \times S^m$

Künneth gives:

$$0 \to H_n(S^n) \otimes H_m(S^m) \to H_{n+m}(S^n \times S^m)$$

$$\begin{array}{ccccc} \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & \otimes & \mathbb{Z} & & \mathbb{Z} \\ \text{gen } a=[e^n] & & \text{gen } b=[f^m] & & [e^n \times e^m] \end{array}$$

$$\text{gen } a \otimes b \longmapsto a \times b$$

(look back at def of cross prod.)

if $\bar{\alpha}$ dual of a in $\mathrm{Hom}(H_n(S^n);\mathbb{Z}) \cong H^n(S^n)$

$\bar{\beta}$ dual of b in $\mathrm{Hom}(H_m(S^m);\mathbb{Z}) \cong H^m(S^m)$

then $(\bar{\alpha} \times \bar{\beta})(a \times b) = \alpha(a)\beta(b) = 1$ so $\bar{\alpha} \times \bar{\beta}$ gen $H^{n+m}(S^n \times S^m)$

now let $p_1: S^n \times S^m \to S^n$

$p_2: S^n \times S^m \to S^m$ be projections

and $\alpha = p_1^* \bar{\alpha}$, $\beta = p_2^* \bar{\beta}$

then α generates $H^n(S^n \times S^m; \mathbb{Z})$ and β generates $H^m(S^n \times S^m; \mathbb{Z})$

(to see this let $i_1: S^n \to S^n \times S^m: x \mapsto (x, p_0)$ fixed

$i_1^* \circ (p_1^*(\bar{\alpha})) = (p_1 \circ i_1)^* \bar{\alpha} = id_{S^n}^* \bar{\alpha} = \bar{\alpha}$

so $p_1^* \bar{\alpha}$ is a generator of $H^n(S^n \times S^m)$

(if not α then take use $-\bar{\alpha}$ instead of $\bar{\alpha}$)

done if $n \neq m$

exercise: think about $n=m$ case)

now $\alpha \cup \beta = p_1^* \bar{\alpha} \cup p_2^* \bar{\beta} = \bar{\alpha} \times \bar{\beta}$ generator of $H^{n+m}(S^n \times S^m)$

note, together with $\mathbb{1} \cup g = g$ we know all cup products!

example: $X = S^2 \times S^3$

$Y = S^2 \vee S^3 \vee S^5$

$$H_n(X) \cong H^n(X^n) \cong \begin{cases} \mathbb{Z} & n = 0,2,3,5 \\ 0 & \text{otherwise} \end{cases} \cong H^n(Y) \cong H_n(Y)$$

$\pi_1(X) = \{1\} = \pi_1(Y)$

so all previous invariants same

but if α, β gens in dim $2,3$ in $H^*(X)$ then $\alpha \cup \beta \neq 0$

now consider Y

$$S^5 \xhookrightarrow{i} Y \xrightarrow{\pi} S^5 \quad \text{obvious maps}$$

$$\pi \circ i = id_{S^5}$$

so $i^*: \underset{\mathbb{Z}}{\overset{\cong}{H^5(Y)}} \to \underset{\mathbb{Z}}{\overset{\cong}{H^5(S^5)}}$ is surjective $\therefore$ an isomorphism

$\forall\, x \in H^2(Y)$ any $y \in H^3(Y)$

$$i^*(x \cup y) = i^*(x) \cup i^*(y) = 0$$

$$\therefore \ x \cup y = 0$$

so $S^2 \times S^5$ not homotopy equivalent to $S^2 \vee S^3 \vee S^5$!

Cup products and Relative Cohomology

recall if $A \subset X$, then $C_n(X,A) = C_n(X)/C_n(A)$

so $C^n(X,A;R) = Hom\left(C_n(X)/C_n(A), R\right)$

we note $0 \to C^n(X,A) \xrightarrow{q^*} C^n(X) \xrightarrow{i^*} C^n(A)$ is <u>exact</u>

since $0 \to C_n(A) \xhookrightarrow{i} C_n(X) \xrightarrow{q} C_n(X,A) \longrightarrow 0$ is exact

and $Hom(\cdot, R)$ is left exact

thus $C^n(X,A;R) \cong im\, q^* \cong \ker i^*$

note if $\eta \in C^n(X;R)$ then $i^*(\eta)$ is just η restricted to $C_n(A)$

$$\boxed{\text{so we can think of } C^n(X,A;R) \text{ as homomorphisms from } C_n(X) \text{ that vanish on } C_n(A)}$$

so with the definition of cup product

$$(a \cup b)(\sigma) = a({}_p\sigma)\, b(\sigma_q)$$

for $a \in H^p(X;R)$ and $b \in H^q(X;R)$

we also get products: $H^p(X,A;R) \times H^q(X;R) \to H^{p+q}(X,A;R)$

$H^p(X;R) \times H^q(X,A;R) \to H^{p+q}(X,A;R)$

and
$$H^p(X,A;R) \times H^q(X,A;R) \to H^{p+q}(X,A;R)$$

working a bit more we also get

$$H^p(X,A;R) \times H^q(X,B;R) \to H^{p+q}(X,A\cup B;R)$$

to see this note $\cup$ maps
$$C^p(X,A;R) \times C^q(X,B;R) \to C^{p+q}(X;A+B;R)$$
where $C^{p+q}(X,A+B;R)$ is the cochains that vanish on elements of $C_{p+q}(X)$ of the form $\alpha+\beta$ where $\alpha \in C_{p+q}(A)$ and $\beta \in C_{p+q}(B)$

i.e. $C_{p+q}(X) \big/ C_{p+q}(A)+C_{p+q}(B)$

there is an inclusion map
$$C_{p+q}(X;A+B) \to C_{p+q}(X;A\cup B)$$

similar to our discussion of excision, one can show this induces an isomorphism on homology

$\therefore$ we also get an isomorphism
$$H^{p+q}(X,A\cup B;R) \cong H^{p+q}(X,A+B;R)$$ ✓

lemma 7:

> suppose $X = U\cup V$ with U,V open sets with
> $$\tilde{H}_*(U) = \tilde{H}_*(V) = 0$$
> Then $\alpha\cup\beta = 0$ $\forall \alpha,\beta \in H^*(X)$ of positive degree

Proof: In the long exact sequence of a pair we have

$$H^p(X,U) \xrightarrow{j^*} H^p(X) \xrightarrow{\imath^*} \underset{\substack{\| \\ 0}}{H^p(U)} \qquad p>0$$

so $\forall \alpha \in H^p(X)$, $\imath^*(\alpha) = 0$

$\therefore \exists \bar{\alpha} \in H^p(X,U)$ s.t $j^*\bar{\alpha} = \alpha$

similarly $\forall \beta \in H^q(X)$, $\exists \bar{\beta} \in H^q(X,V)$ s.t. $j^*\bar{\beta} = \beta$ for $q>0$

now $\quad \bar{\alpha} \cup \bar{\beta} \in H^{p+q}(X, U \cup V) = H^{p+q}(X,X) = 0$

but
$$\begin{tikzcd}
H^p(X) \times H^q(X) \arrow[r, "\cup"] & H^{p+q}(X) \\
H^p(X,U) \times H^q(X,V) \arrow[u, "j^* \times j^*"] \arrow[r, "\cup"] \arrow[ur, phantom, "\circ"] & H^{p+q}(X, U \cup V) \arrow[u, "j^*"]
\end{tikzcd}$$

$\therefore \quad \alpha \cup \beta = j^* \bar{\alpha} \cup j^* \bar{\beta} = j^*(\bar{\alpha} \cup \bar{\beta}) = 0$

<u>note</u>: j^* maps not same so think about this to see argument OK.

<u>examples</u>:

1) lemma $\Rightarrow$ $S^n \times S^m$ is <u>not</u> the union of two acyclic sets!

2) a <u>suspension</u> $\Sigma X = X \times [0,1] / X \times \{0\}, X \times \{1\} = X \times [0,1) / X \times \{0\} \cup X \times (1,0] / X \times \{1\}$

<u>always</u> has trivial cup products in pos. degrees!

3) so $S^n \times S^m$ <u>not</u> a suspension

so cup products can tell us interesting things!

<u>exercise</u>: if X the union of n contractible open sets then n fold cup products are trivial.

D. More products

recall we have a map

$$C_p(X;R) \times C^p(X;R) \longrightarrow R$$
$$(\beta, \alpha) \longmapsto \alpha(\beta) \qquad \text{we write this } \langle \alpha, \beta \rangle$$

this pairing is nondegenerate so we can look at the "adjoint" of $\cup$ with respect to this pairing

that is, we define the <u>cap product</u> as the map

$$\cap : C_{p+q}(X;R) \times C^p(X;R) \longrightarrow C_q(X;R)$$

s.t. for $\alpha \in C^p(X;R)$

$\beta \in C_{p+q}(X;R)$

$\beta \frown \alpha$ is the unique element in $C_q(X;R)$ satisfying

$$\langle \beta \frown \alpha, \gamma \rangle = \langle \beta, \alpha \smile \gamma \rangle \quad \forall \gamma \in C^q(X;R)$$

i.e. if we think of $\alpha \smile \cdot$ as a map $C^q(X;R) \to C^{p+q}(X;R)$ then $\cdot \frown \alpha$ is the adjoint with respect to pairing

we can define $\frown$ as follows

$$\beta \frown \alpha = \underbrace{\alpha({}_p\beta)}_{\in R} \underbrace{\beta_q}_{\in C_q(X;R)} \quad \in C_q(X;R)$$

<u>exercise:</u> Check this is the adjoint of $\alpha \smile \cdot$

<u>lemma 8:</u>

$$\boxed{C_*(X;R) \text{ is a unitary } C^*(X;R) \text{ module using } \frown}$$

<u>Proof:</u>

$$\beta \frown (\alpha \smile \gamma) = (\alpha \smile \gamma)({}_{p+q}\beta)\, \beta_r$$
$$= \alpha({}_p\beta)\, \gamma(({}_{p+q}\beta)_q)\, \beta_r$$

and

$$(\beta \frown \alpha) \frown \gamma = [\alpha({}_p\beta)\, \beta_{q+r}] \frown \gamma$$
$$= \alpha({}_p\beta)\, \gamma(\underbrace{{}_q(\beta_{q+r})}_{= ({}_{p+q}\beta)_q})\, \beta_r$$

so $\beta \frown (\alpha \smile \gamma) = (\beta \frown \alpha) \frown \gamma$

<u>exercise:</u> check rest ∎

<u>lemma 9:</u>

$$\boxed{\begin{array}{l} \text{if } \beta \in C_{p+q}(X;R),\ \alpha \in C^p(X;R) \text{ then} \\ \partial(\beta \frown \alpha) = (-1)^p((\partial\beta) \frown \alpha - \beta \frown \delta\alpha) \end{array}}$$

<u>Proof:</u> we need to check each side in equality pairs same with all elements in $C^{q-1}(X;R)$

$$(-1)^p\left(\langle \partial\beta \cap \alpha, \gamma\rangle - \langle \beta \cap \delta\alpha, \gamma\rangle\right)$$
$$= (-1)^p\left(\langle \partial\beta, \alpha \cup \gamma\rangle - \langle \beta, (\delta\alpha)\cup\gamma\rangle\right)$$
$$= (-1)^p\left(\langle \beta, \delta(\alpha\cup\gamma)\rangle - \langle \beta, (\delta\alpha)\cup\gamma\rangle\right)$$
$$= (-1)^p\left(\cancel{\langle \beta, (\delta\alpha)\cup\gamma\rangle} + \langle \beta, (-1)^p \alpha\cup\delta\gamma\rangle - \cancel{\langle \beta, (\delta\alpha)\cup\gamma\rangle}\right)$$
$$= \langle \beta, \alpha\cup\delta\gamma\rangle = \langle \beta\cap\alpha, \delta\gamma\rangle$$
$$= \langle \partial(\beta\cap\alpha), \gamma\rangle$$

from lemma its clear $\cap$ decends to (co)homology

$$\cap : H_{p+q}(X;R) \times H^p(X;R) \to H_q(X;R)$$

exercisé: check well-defined

lemma 10:

$f: X \to Y$ a map

Then $$f_*(\beta \cap f^*\alpha) = f_*(\beta) \cap \alpha$$

for $\beta \in H_{p+q}(X;R)$ and $\alpha \in H^p(Y;R)$

exercise: Prove lemma (just like last lemma)

we also get relative cap products

$$H_{p+q}(X,A;R) \times H^p(X,A;R) \longrightarrow H_q(X;R) \quad \text{and}$$
$$H_{p+q}(X,A;R) \times H^p(X;R) \longrightarrow H_q(X,A;R)$$

let's see the first one is well-defined

$[\beta] \in H_{p+q}(X,A;R)$ so $\beta \in C_{p+q}(X,A;R)$

$[\alpha] \in H^p(X,A;R)$ so $\alpha \in C^p(X,A;R)$ ie. α vanishes on $C_p(A;R)$

$\partial(\beta\cap\alpha) = (-1)^p(\partial\beta\cap\alpha - \beta\cap\delta\alpha)$

since $\partial\beta \in C_{p+q-1}(A;R)$

note: 1) $\langle \partial\beta\cap\alpha, \gamma\rangle = \langle \partial\beta, \alpha\cup\gamma\rangle = 0 \;\; \forall \gamma \;\therefore\; \partial\beta\cap\alpha = 0$

2) $\delta\alpha$ vanishes on $C_{p+1}(A;R) \ni {}_{p+1}\beta$ (since $\partial\beta \subset A$)

$\therefore\; \partial(\beta\cap\alpha) = 0$ not just in $C_{q-1}(A;R)$

so gives eltement in $H_q(X;R)$