Knot Groups and Colorings

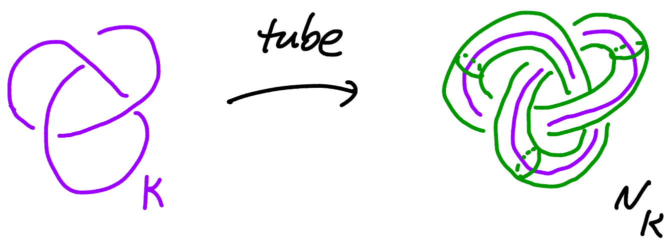
A Knot Groups

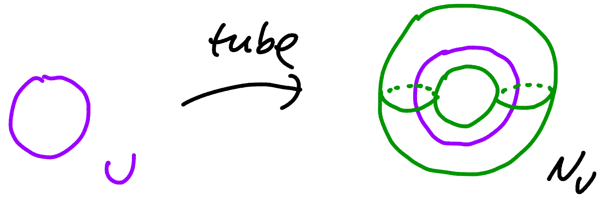
recall a knot K is the image of an embedding

$$f: S^1 \to \mathbb{R}^3$$

(or $S^3 = \mathbb{R}^3 \cup \{\infty\}$, recall stereographic coordinates show $S^3 - \{pt\} \cong \mathbb{R}^3$)

given a knot K we can consider a "tube" about K



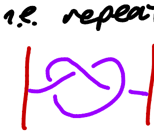


i.e. think of a knot as a piece of string then the tube is a "thickening" of the string

note: $N_K \cong S^1 \times D^2 \; (= K \times D^2)$

Remark: such tubes don't always exist!
but if f is differentiable they do
If tube doesn't exist the knot is called wild

i.e. repeat



shrinking each time

we will not study wild knots
so for us "knot" means "non-wild knot" (tame)

let $X_K = \overline{S^3 - N_K}$ (use S^3 because we like compact things but not important for most of what is below)

exercise:

1) X_K is a compact 3-manifold with boundary
2) $\partial X_K = T^2$

recall we are interested in knots upto isotopy

Fact: For tame knots: K_1 isotopic to K_2

$\Leftrightarrow$

called ambient isotopy $\left\{ \begin{array}{l} \exists \text{ an isotopy } \phi : S^3 \times [0,1] \to S^3 \\ \text{such that } \phi_0 = id_{S^3} \text{ and } \phi_1(K_1) = K_2 \end{array} \right.$

note that given an ambient isotopy ϕ, and a parameterization $\gamma : S^1 \to S^3$ of K_1, then $\phi_t \circ \gamma$ is an isotopy from K_1 to K_2

so $(\Leftarrow)$ is easy

$(\Rightarrow)$ is much more difficult, but true

note: $\phi_1 : (S^3 - K_1) \to (S^3 - K_2)$ is a homeomorphism

lemma 1:

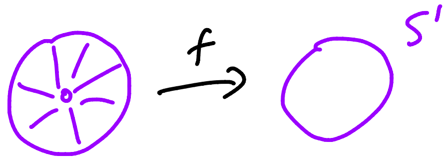
$$\boxed{X_K \simeq S^3 - K}$$

homotopy equivalent

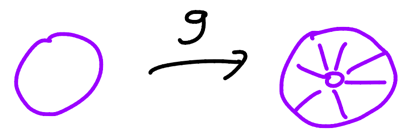
Remark: by above discussion if K_1 is isotopic to K_2 then $X_{K_1} \cong X_{K_2}$

so if we can show $X_{K_1} \not\cong X_{K_2}$ then K_1 and K_2 are different!

Proof: $D^2 - \{pt\} \simeq S^1$



$(r,\theta) \longmapsto \theta$



$\theta \longmapsto (1,\theta)$

<u>exercise:</u> $f\circ g = id_{S^1}$ and $g\circ f \simeq id_{D^2-\{pt\}}$

now $N_K - K \cong (D^2-\{pt\})\times S^1 \simeq S^1\times S^1 = T^2$

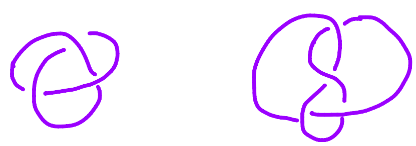
so $S^3 - K = X_K \cup_{T^2} (N_K - K) \simeq X_K \cup_{T^2} (T^2) = X_K$ 

X_K is called the <u>knot complement</u> of K

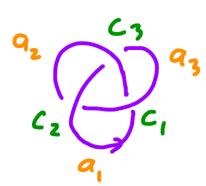
we want to compute the fundamental group of X_K

for this we consider <u>knot diagrams</u>

recall, we discussed these at start of the course. they are projections to xy-plane in $\mathbb{R}^3$ (and remember over and under crossing info.)



<u>note:</u> if the diagram for K has n ($n>0$) crossings, then it also has n arcs $a_1 \dots a_n$ (lable crossings $c_1 \dots c_n$)



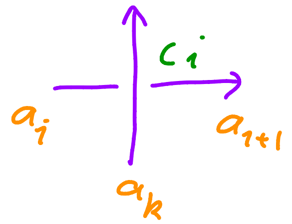
we lable a_i consecutively as we go around K and c_i is tip of a_i

<u>Th$^{\underline{m}}$ 2 (Wirtinger Presentation):</u>

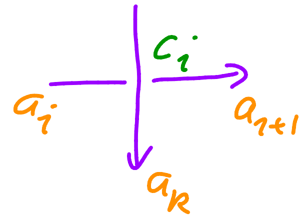
If D_K is a diagram of K with arcs $a_1, \dots, a_n$ and crossings $c_1, \dots, c_n$, then

$$\pi_1(X_K, x_0) \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_{n-1} \rangle$$

where for each crossing c_i we get a relation r_i as follows

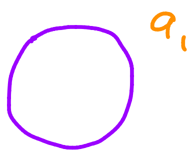


$a_k a_i a_k^{-1} = a_{i+1}$

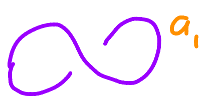


$a_k^{-1} a_i a_k = a_{i+1}$

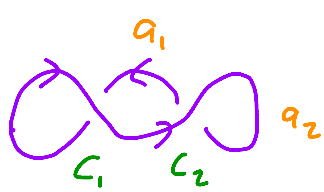
examples:

1) $U =$ 

$\pi_1(X_U) \cong \langle a_1 \mid \ \rangle \cong \mathbb{Z}$

2) $U =$ 

$\pi_1(X_U) \cong \langle a_1 \mid \ \rangle \cong \mathbb{Z}$

3) $U =$ 

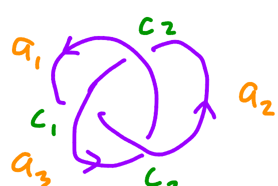
$\pi_1(X_U) \cong \langle a_1, a_2 \mid a_2 a_1 a_2^{-1} = a_2 \rangle$

$$a_2^{-1} a_2 a_1 a_2^{-1} = e$$

$$a_1 a_2^{-1} = e$$

$$a_1 = a_2$$

so $\pi_1(X_U) \cong \langle a_1 \mid \ \rangle \cong \mathbb{Z}$

4) $T =$ 

$\pi_1(X_T) \cong \langle a_1, a_2, a_3 \mid a_3^{-1} a_1 a_3 a_2, \ a_1^{-1} a_2 a_1 a_3^{-1} \rangle$

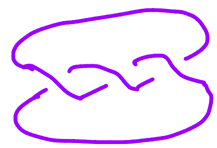
note: $a_2 = a_3^{-1} a_1^{-1} a_3$

so

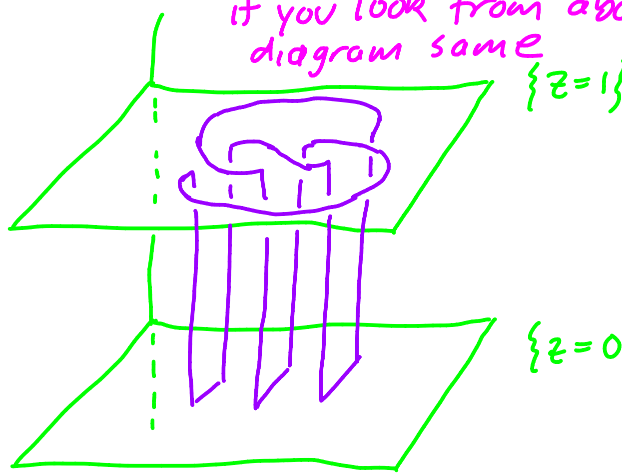
$\pi_1(X_T) \cong \langle a_1, a_3 \mid a_1^{-1} a_3^{-1} a_1^{-1} a_3 a_1 a_3 \rangle$

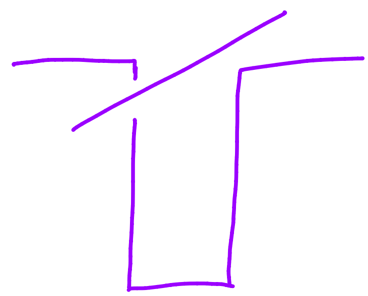
$a_1 a_3 a_1 = a_3 a_1 a_3$

Proof: given a knot diagram in the xy-plane

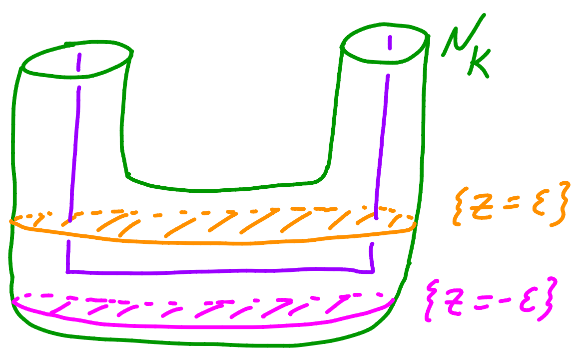


in $\mathbb{R}^3$ we can take almost all of K to be in $\{z=1\}$
with only undercrossings in $\{z=0\}$
(and arcs connecting them)



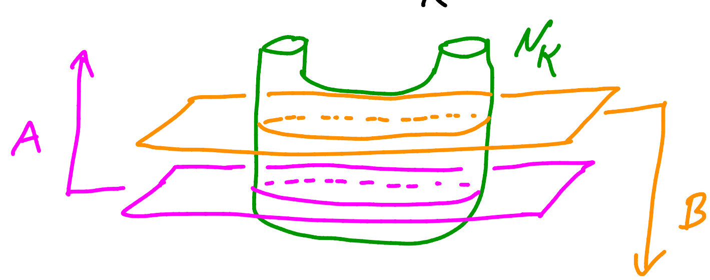


look at N_K near one of the under crossings



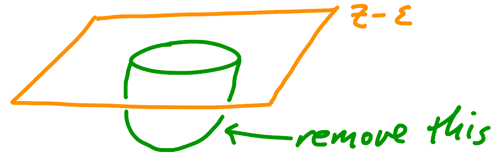
note: $\{z=\pm\varepsilon\}$ intersects
N_K near crossing in
a disk

let $B = \{ (x,y,z) \in \overline{\mathbb{R}^3 - N_K} : z < \varepsilon \}$
$A = \{ (x,y,z) \in \mathbb{R}^3 - N_K : z > -\varepsilon \}$



Identify B: If we did not remove N_K from B we would have an open ball $B^3 = \{z < \varepsilon\}$

for each crossing we remove

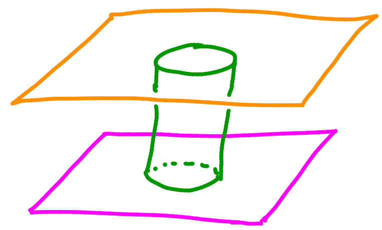


so $B = B^3 - (\cup \text{ balls as above})$

exercise: $B \cong B^3$

so $\pi_1(B) = \{e\}$

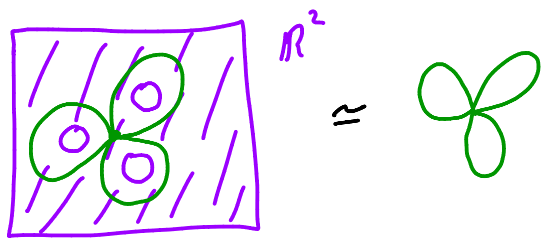
Identify $A \cap B$: in $\mathbb{R}^3$ we see



$$A \cap B = (\mathbb{R}^2 \times (-\varepsilon, \varepsilon)) - (\cup (D^2 \times (-\varepsilon, \varepsilon))$$

$$= (\mathbb{R}^2 - \cup D^2) \times (-\varepsilon, \varepsilon) \simeq \mathbb{R}^2 - \cup D^2$$

one for each crossing



wedge of n circles W_n

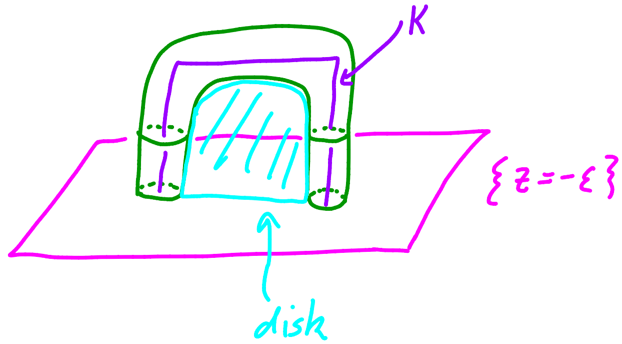
so $\pi_1(A \cap B) \cong \pi_1(W_n) \cong \langle c_1, \dots, c_n \mid \ \rangle$ free group on n generators

Identify A: If we did not remove N_K from A we would have an open ball $B^3 = \{z > -\varepsilon\}$

for each arc a_i in diagram we remove a tube from B^3 (i.e. make a "worm hole")

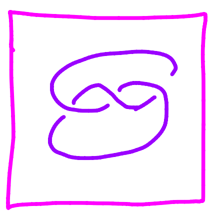
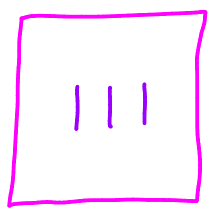
So $A = B^3$ with n worm holes

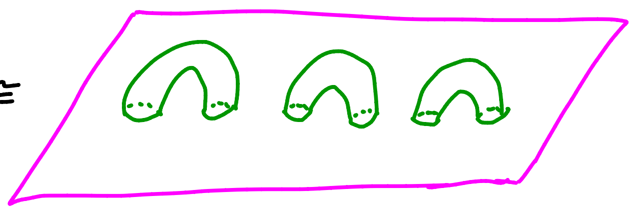
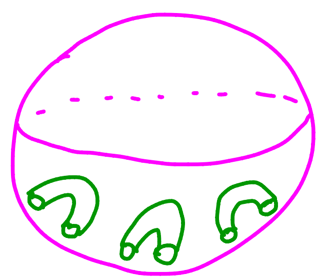
note: each worm hole has a disk under it that is disjoint from other worm holes (and disks)

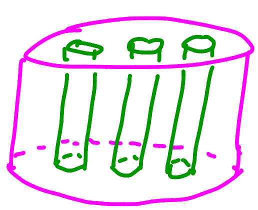


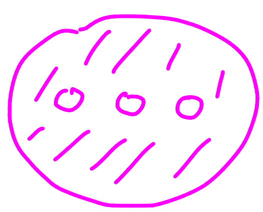
so removing a worm hole is the same as the following: take disjoint arcs on $\{z=-\varepsilon\}$, push interiors into $\{z>-\varepsilon\}$, and removing a nbhd of it

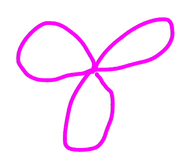
<u>exercise</u>: if you isotop the arcs on $\{z=-\varepsilon\}$ and then push interiors up and remove nbhs then you get homeomorphic spaces

eg  $\cong$ 

so $A \cong$  $\cong$ 

$\cong$  $= \left(D^2 - \coprod_{i=1}^{n} D^2\right) \times (0,1)$

$\simeq D^2 - \coprod_{i=1}^{n} D^2$  $\simeq$ wedge of n-circles

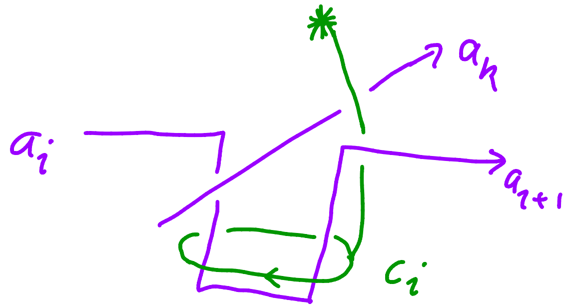
W_n 

so $\pi_1(A) = \langle a_1, \dots, a_n \mid \ \rangle$ free group on n generators to use Seifert - Van Kampen need to see

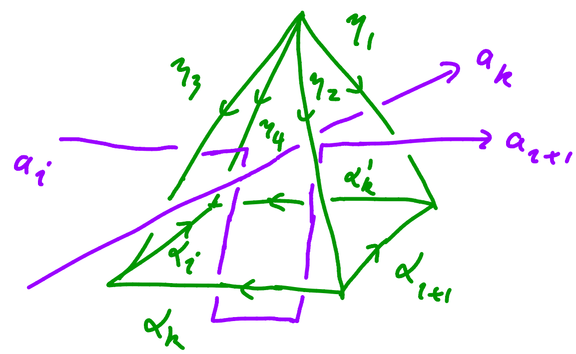
$$\pi_1(A\cap B) \to \pi_1(B) = \{e\} \qquad \text{trivial map}$$
$$\pi_1(A\cap B) \to \pi_1(A)$$

let c_i be one of the generators of $\pi_1(A\cap B)$

c_i in $\pi_1(A)$ is



note this is homotopic to



$$c_i \simeq \eta_1 \bar{\alpha}_{i+1} \alpha_k \alpha_i \bar{\alpha}'_k \bar{\eta}_1$$
$$\simeq (\eta_1 \bar{\alpha}_{i+1} \bar{\eta}_2)(\eta_2 \alpha_k \bar{\eta}_3)(\eta_3 \alpha_i \bar{\eta}_4)(\eta_4 \bar{\alpha}'_k \bar{\eta}_1)$$
$$\simeq a_{i+1}^{-1} \alpha_k \alpha_i \alpha_k^{-1}$$

so $\pi_1(\mathbb{R}^3 - N_K) \cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$

where relations are as above

exercise: Show r_n is a consequence of the other r_i so it is not needed

(you can also do this by taking a different decomposition of $\mathbb{R}^3$)

We applied Seifert - Van Kampen wrong!

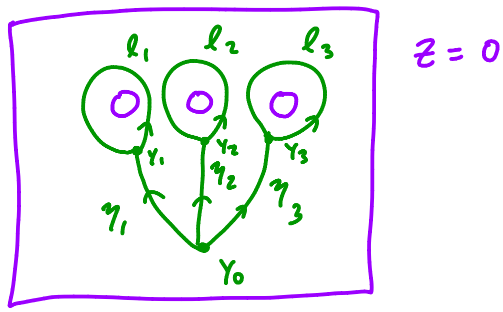
we were not careful with base point need to take base point $y_0 \in A\cap B$ not $x_0 = (0,0,2)$ like we did

(just did this because easier to visualize, and we can fix it!)

let η be a path from x_0 to y_0, then we get isomorphism

$$\Phi_\eta : \pi_1(A, y_0) \to \pi_1(A, x_0)$$

now for generators c_i of $\pi_1(A \cap B, y_0)$ we take



$c_i = \eta_i l_i \bar{\eta}_i$

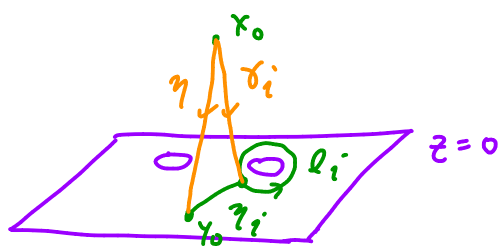
let $y_i = l_i \cap \eta_i$

let γ_i be path x_0 to y_i

note $\gamma_i l_i \bar{\gamma}_i$ are the loops we used above for c_i

in $\pi_1(A, x_0)$ (call them c_i' now)

so $\Phi_\eta(c_i) = \Phi_\eta(\eta_i l_i \bar{\eta}_i) = \eta \eta_i l_i \bar{\eta}_i \bar{\eta}$



let $\beta_i = \gamma_i \bar{\eta}_i \bar{\eta} \in \pi_1(A, x_0)$

<u>note:</u> $\Phi_\eta(c_i) = (\eta \eta_i \underbrace{\bar{\gamma}_i)(\gamma_i}_{e} l_i \underbrace{\bar{\gamma}_i)(\gamma_i}_{e} \bar{\eta}_i \bar{\eta}) = \bar{\beta}_i c_i' \beta_i$

correct use of Seifert-Van Kampen is

$$\pi_1(X_K, y_0) \cong \Phi_\eta^{-1}(\pi_1(A, x_0)) * \{e\} / \langle c_1, \dots, c_n \rangle$$

$$\cong \pi_1(A, x_0) / \langle \Phi_\eta(c_1), \dots, \Phi_\eta(c_n) \rangle$$

$$= \pi_1(A, x_0) / \langle \bar{\beta}_1 c_1' \beta_1, \dots, \bar{\beta}_n c_n' \beta_n \rangle$$

<u>exercise:</u> $\langle g_1, \dots, g_k \rangle = \langle h_1 g_1 h_1^{-1}, \dots, h_k g_k h_k^{-1} \rangle$

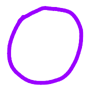
normal subgroups gen by elements

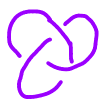
so $\pi_1(X_K, y_0) \cong \pi_1(A, x_0)/\langle c_1', \ldots, c_n' \rangle$

$\cong \langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$

relations given by c_i'

recall

$U =$  has $\pi_1(X_U) \cong \mathbb{Z}$

$T =$  has $\pi_1(X_T) \cong \langle a_1, a_3 \mid a_3 a_1 a_3 a_1^{-1} a_3^{-1} a_1^{-1} \rangle$

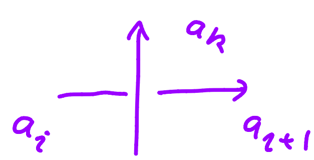
Is $\pi_1(X_U) \cong \pi_1(X_K)$?

as earlier, could try to abelianize (i.e. look at H_1), but

<u>Corollary 3:</u>

$$\boxed{H_1(X_K) \cong \mathbb{Z} \text{ for any knot } K}$$

<u>Proof:</u> each crossing



gives a relation $a_i a_k a_{i+1}^{-1} a_k^{-1} = e$

after we abelianize this is $a_i = a_{i+1}$

so $H_1(X_K)$ has one generator and no relations

so $H_1(K_K) \cong \mathbb{Z}$

next try

<u>Claim:</u> $\pi_1(X_K)$ non-abelian

(so not $\cong \pi_1(X_U)$, so T and U are not isotopic)

to show this, we look for a group G we know is non-abelian and try to find a homomorphism $\phi: \pi_1(X_K) \to G$ onto G.

(since $\exists\, g_1, g_2 \in G$ s.t. $g_1 g_2 \neq g_2 g_1$

and $h_1, h_2 \in \pi_1(X_T)$ s.t. $\phi(h_i) = g_i$

we know $h_1 h_2 \neq h_2 h_1$ and $\pi_1(X_T)$ non-abelian)

recall S_3 = group of permutations of $\{1,2,3\}$

$|S_3| = 6$ and S_3 non-abelian

define $\phi: \pi_1(X_T) \to S_3$ by

$$a_1 \longmapsto [2\ 1\ 3]$$
$$a_3 \longmapsto [3\ 2\ 1]$$

recall, this means

$1 \mapsto 2$
$2 \mapsto 1$
$3 \mapsto 3$

this gives a homomorphism since

$$a_3 a_1 a_3 a_1^{-1} a_3^{-1} a_1^{-1} = 1$$

becomes

$$[3\ 2\ 1][2\ 1\ 3][3\ 2\ 1][2\ 1\ 3]^{-1}[3\ 2\ 1]^{-1}[2\ 1\ 3]^{-1}$$

$$= [2\ 3\ 1][3\ 2\ 1][2\ 1\ 3]^{-1}[3\ 2\ 1]^{-1}[2\ 1\ 3]^{-1}$$

$$= [2\ 3\ 1][2\ 3\ 1][2\ 3\ 1]$$

$$= [2\ 3\ 1][3\ 1\ 2] = [1\ 2\ 3] = id$$

image ϕ contains

$$a_1 \longmapsto [2\ 1\ 3]$$
$$a_3 \longmapsto [3\ 2\ 1]$$
$$a_1 a_3 \longmapsto [3\ 1\ 2]$$
$$a_3 a_1 \longmapsto [2\ 3\ 1]$$
$$a_1 a_3 a_1 \longmapsto [1\ 3\ 2]$$
$$e \longmapsto [1\ 2\ 3]$$

so ϕ is onto $\therefore \pi_1(X_T)$ non-abelian

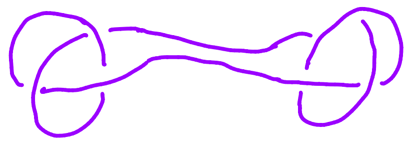
$\therefore \pi_1(X_T) \not\cong \pi_1(X_U)$

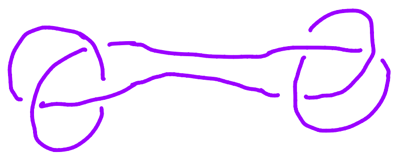
so K and U not isotopic !

How good is $\pi_1(X_K)$ at determining X_K ?

Facts:

1) if $\pi_1(X_K) \cong \mathbb{Z}$, then K is the unknot.

2) if $K_1 =$ 

$K_2 =$ 

then $\pi_1(X_{K_1}) \cong \pi_1(X_{K_2})$

but K_1 is not isotopic to K_2

3) so $\pi_1(X_K)$ is a good invariant of K

but not perfect

but $\pi_1(X_K)$ + tiny bit extra determines K

down side is it can be hard to determine when two group presentations are the same group !

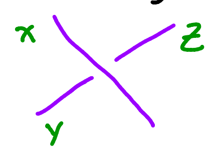
So try to "extract" more "computable" information from $\pi_1(X_K)$

B. Coloring Knots

prime

Recall a p-labeling (or coloring) of a knot diagram is an assignment of an element of $\mathbb{Z}_p$ to each edge of the diagram so that

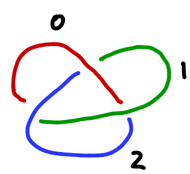
1) at least 2 lables are used and
2) at each crossing



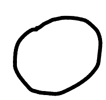
$2x \equiv z+y \mod p$

We saw you can distinguish the unknot, figure 8 knot, and trefoil using 3 and 5 colorings

e.g.



3-colorable



not 3-colorable

What does this have to do with $\pi_1(X_K)$?

Thm 4:

1) Every p-labeling of a diagram of K gives a surjective homomorphism
$$\pi_1(X_K) \longrightarrow D_p$$
2) Every surjective homomorphism $\pi_1(X_K) \longrightarrow D_p$ gives a p-labeling of a diagram of K

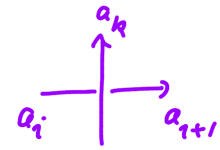
Recall D_p = dihedral group
= symmetries of regular n-gon
$\cong \langle x, y \mid x^n, y^2, xyxy \rangle$

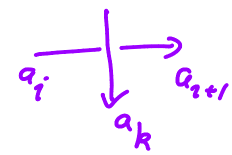
Proof: If a diagram D_K for K has n crossings $c_1, \ldots, c_n$

and n arcs $a_1, \dots, a_n$ (labeled as above)

then Th^{m}2 says

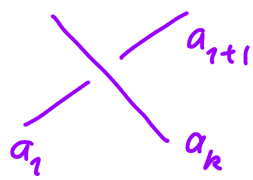
$$\pi_1(X_K) \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_{n-1} \rangle$$

where r_i is $a_k a_i a_k^{-1} a_{i+1}^{-1}$ if c_i is 

and $a_k^{-1} a_i a_k a_{i+1}^{-1}$ if c_i is 

a p-coloring is a map

$$\{a_1, \dots, a_n\} \xrightarrow{c} \mathbb{Z}_p$$

satisfying 

$$\Rightarrow 2c(a_k) \equiv c(a_i) + c(a_{i+1}) \bmod p$$

given c define

$$\phi_c : \pi_1(X_K) \longrightarrow D_p$$

$$a_i \longmapsto y x^{c(a_i)} \qquad \text{write } c_i = c(a_i)$$

this will give a homomorphism if the relations r_i are respected:

$$a_k^{-1} a_i a_k a_{i+1}^{-1}$$

becomes:

$$(yx^{c_k})^{-1}(yx^{c_i})(yx^{c_k})(yx^{c_{i+1}})^{-1}$$

$xyxy = e$

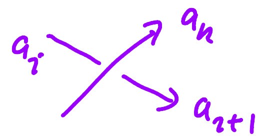
$xy = y^{-1}x^{-1} = yx^{-1}$

$$= x^{-c_k} y^{-1} y x^{c_i} y x^{c_k} x^{-c_{i+1}} y^{-1}$$

$$= x^{c_i - c_k} y x^{c_k - c_{i+1}} y^{-1}$$

$$= x^{c_i + c_{i+1} - 2c_k} y y^{-1} = x^{c_i + c_{i+1} - 2c_k}$$

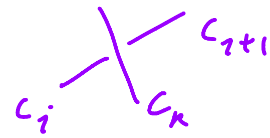
$$= x^{lp} = e$$

similarly for 

so ϕ_c is a homomorphism

<u>Claim:</u> ϕ_c is onto

since at least 2 labels are used there is a crossing s.t. $c_i \not\equiv c_{i+1} \bmod p$



$2c_k \equiv c_i + c_{i+1} \bmod p$

$$c_i \not\equiv c_{i+1} \bmod p \Rightarrow c_{i+1} - c_i \not\equiv 0 \bmod p$$

so $c_{i+1} - c_i$ is represented by an integer between 1 and $p-1$

so $(c_{i+1} - c_i)$ is relatively prime to p (since p prime)

<u>Algebra Fact:</u> $\exists$ integers m, m' such that

$$m(c_{i+1} - c_i) + m'p = 1$$

i.e. $m(c_{i+1} - c_i) \equiv 1 \bmod p$

now $\phi_c((a_i a_{i+1})^m) = (yx^{c_i} y x^{c_{i+1}})^m = (y^2 x^{c_{i+1} - c_i})^m$

$$= x^{m(c_{i+1} - c_i)} = x^1 = x$$

and $\phi_c(a_i((a_i a_{i+1})^m)^{-c_i}) = y x^{c_i} x^{-c_i} = y$

so ϕ_c onto ✓

Now given $\phi : \pi_1(X_K) \to D_p$ surjective

then for a diagram D_K let the arcs be $a_1, \dots, a_n$

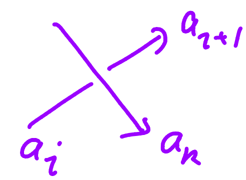
<u>note:</u> $\phi(a_i) = x^{b_1} y x^{b_2} y \dots y x^{b_{k_i}} = y^{\varepsilon_i} x^{c_i}$

where $\varepsilon = 0$ or 1 and $c_i \in \{0, \dots, p-1\}$

Claim: $\varepsilon_i = 1$ for all i

if not, then for some i we have $\varepsilon_i = 0$

now conside



$$\phi(a_k^{-1} a_i a_k a_{i+1}^{-1}) = x^{-c_k} y^{\varepsilon_k} x^{c_i} y^{\varepsilon_k} x^{c_k} x^{-c_{i+1}} y^{\varepsilon_{i+1}}$$

$$= y^{2\varepsilon_k + \varepsilon_{i+1}} x^{?} = y^{\varepsilon_{i+1}} x^{?}$$

since this must be e, we must have $\varepsilon_{i+1} = 0$

inducting we see all $\varepsilon_k = 0$

thus y is not in the image of ϕ $\not\!\!\bigcirc$

thus we see $\phi(a_i) = y x^{c_i}$ $\forall i$

define $c : \{a_1, \dots, a_n\} \longrightarrow \mathbb{Z}_p : a_i \longmapsto c_i$

exercise: check this is a p-labeling