Math 6457 - Fall 2008
Homework 4

Work all the problems, but carefully write up and turn in Problems 2, 4, 5 and 8.

1. Let M be a smooth manifold with boundary. Let N_1 and N_2 be two components of ∂M. Suppose
$$f : N_1 \to N_2$$
is a diffeomorphism. Show that the quotient space M_f obtained from M by identifying $x \in N_1$ with $f(x) \in N_2$ is a smooth manifold. (If $M = M_1 \cup M_2$ is the union of two disjoint manifolds and $N_i = \partial M_i$ then M_f is usually denoted $M_1 \cup_f M_2$.)

2. Two diffeomorphisms $f_0, f_1 : M \to N$ are called *isotopic* if there is a smooth map
$$H : M \times [0, 1] \to N$$
such that $H(x, 0) = f_0(x), H(x, 1) = f_1(x)$ and for each fixed t, $f_t(x) = H(x, t)$ is a diffeomorphism from M to N. With the notation as in the previous problem show that if f_0 and f_1 are isotopic then M_{f_0} and M_{f_1} are diffeomorphic.
HINT: In class we proved that every neighborhood of ∂M contains a collar neighborhood of ∂M.

3. Recall if $f : M \to M$ is a diffeomorphism then the mapping torus of f, denoted T_f, is the quotient space of $M \times [0, 1]$ obtained by identifying $(x, 0)$ with $(f(x), 1)$. From above we know T_f is a smooth manifold. Show that T_f is a fiber bundle of S^1 with fiber M. Show there are infinitely many different T^2 bundles over S^1.
HINT: For the last part recall that in class we computed $\pi_1(T_f)$. You can take as a fact (or if you are industrious try and show) that any diffeomorphism of T^2 is isotopic to the map induced on T^2 by the matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
with $ad - bc = \pm 1$ and we think of the matrix as a map on $\mathbb{R}^2$. Compute $\pi_1(T_f)$ for various matrices, and maybe abelianize if things are not obvious.

4. Let D_1 and $D_2 = \{(r, \theta) \in \mathbb{R}^2 : |r| \leq 1\}$. Let $f : \partial D_1 \to \partial D_2 : (1, \theta) \to (1, -\theta)$. (You should easily be able to check $D_1 \cup_f D_2$ is diffeomorphic to S^2, but you don't need to do this for the exercise.) Now let $M_i = D_i \times \mathbb{R}^2, i = 1, 2$ and for $n \in \mathbb{Z}$ let
$$F_n : \partial M_1 \to \partial M_2 : (1, \theta, z) \to (1, -\theta, e^{in\theta} z)$$
here we are thinking of $\mathbb{R}^2$ as $\mathbb{C}$. Prove that $E_n = M_1 \cup_{F_n} M_2$ is a vector bundle over S^2.

5. With notation form the previous problem, show that E_n is not isomorphic to E_m as a vector bundle for $|n| \neq |m|$.
HINT: If the where isomorphic as bundles then the complements of their zero sections would be diffeomorphic. Compute π_1 of the complements of their zero sections. To do this notice $\mathbb{R}^2 \setminus \{(0, 0)\}$ retracts onto S^1, so you can reduce this computation to a computation for S^1 bundles over S^2.

6. TS^2 is an vector bundle over S^2. Which E_n is it bundle isomorphic to?
   HINT: Maybe think of stereographic coordinates.

7. Prove that the diffeomorphism group of a connected manifold acts transitively. That is given two point p and q in a manifold there is a diffeomorphism taking p to q.
   HINT: Prove the set of point that p maps to under all diffeomorphism is open and closed. To this end if p and q are in a coordinate chart prove there is a diffeomorphism taking p to q. You don't have to use vector fields and flows for this, but it is probably the easiest way.

8. Let $f : M \to N$ be a smooth map and let v and w be vector fields on M and N respectively. Also assume that $df_x(v(x)) = w(f(x))$ for all $x \in M$. Let $\Psi : M \times \mathbb{R} \to M$ and $\Phi : N \times \mathbb{R} \to N$ be the flows of v and w respectively (we assume the flow exists for all time, this is not essential to do). Show $f(\Psi(x,t)) = \Phi(f(x), t)$.

9. In this exercise we prove: If $f : M \to N$ is a smooth surjective map of compact manifolds and df_x is surjective for all $x \in M$, then $f : M \to N$ is a fiber bundle. Given a point $x_0 \in N$ let (U, V, ϕ) be a coordinate chart around x. Let $\frac{\partial}{\partial x^i}$, $i = 1, \dots, n$, be the coordinate vector fields on $V \subset \mathbb{R}^n$ (we may think of them as coordinate vector field s on U too). Let Ψ_i be the flow of $\frac{\partial}{\partial x^1}$.

   (a) Show the flow commute. That is $\Psi_i(\Psi_j(x,t), s) = \Psi_j(\Psi_i(x,s), t)$. (HINT: just do this in $\mathbb{R}^n$.) Show $\Psi : \mathbb{R}^n \to U$ given by

   $$\Psi(t^1, \dots, t^n) = \Psi_1(\Psi_2(\dots \Psi_{n-1}(\Psi_n(x_0, t^n)t^{n-1}), \dots, t^2), t^1)$$

   is a diffeomorphism near the origin. Actually show this is the identity map if we think of it as a map $\mathbb{R}^n \to V$.

   (b) One can find vector fields e_i on $f^{-1}(U)$ such that $df_x(e_i) = \frac{\partial}{\partial x^i}$ (you don't have to show this, but it is not hard using a partition of unity argument). Given the e_i let Φ_i be their flow. Let $F = f^{-1}(x_0)$ and define

   $$\Phi : F \times \mathbb{R}^n \to M : (x, (t^1, \dots, t^n)) \mapsto \Phi_1(\Phi_2(\dots \Phi_{n-1}(\Phi_n(x, t^n)t^{n-1}), \dots, t^2), t^1).$$

   Show $dF_(x, (0, \dots, 0))$ is an isomorphism for all $x \in F$ and hence conclude that $dF_{(x,(t^1,\dots,t^n))}$ is an isomorphism for all sufficiently small t^i. Conclude that F is a local diffeomorphism for t^i sufficiently small.

   (c) Show F is injective for t^i sufficiently small.

   (d) Conclude F, when restricted to a small enough neighborhood of F is a diffeomorphism onto its image and

   $$f \circ F(x, (t^1, \dots, t^n)) = (t^1, \dots, t^n).$$

   In other words F gives a local trivialization of $f : M \to N$.