RECOLLECTIONS FROM POINT SET TOPOLOGY
AND
OVERVIEW OF QUOTIENT SPACES

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1. INTRODUCTION

The main idea of point set topology is to (1) understand the minimal structure you need on a set to discuss continuous things (that is things like continuous functions and convergent sequences), (2) understand properties on these spaces that make continuity look more like we think it should (that is properties like Hausdorff and countability properties), and (3) classify topological spaces (that is classify those spaces on which you can talk about continuity). So in some sense this is the starting point, or foundation, of a great deal of mathematics. To do analysis or geometry we need to understand continuous things, so in some very real sense analysis and dynamical systems and geometry are build on top of topology. Of course this is a somewhat biased view of things. In many of these subjects one can restrict to a very simple topological space, like say Euclidean space, and still have a very rich theory. In any event the main focus of this class we will be in (1) developing tools to study topological spaces, (2) seeing what extra "structure" we need to bring analysis and geometry into the picture and (3) study geometry (and some analysis) on these spaces. To get started though we need to assume some background in point set topology. Specifically, you should know

(1) Abstract definition of a topological space, continuous functions and convergent sequences.
(2) Various topological conditions that make spaces nicer. The most important for us are:
 (a) Hausdorff
 (b) Second countable
 (c) Compact
 (d) Connected
(3) Quotient spaces.
(4) Classification of surfaces.

Below, I will briefly recall the main definitions and a few theorems for (1) and (2) (this might seem a little dry as a lot of the fun in point set topology is exploring various exotic topologies and seeing the strange way various properties interact, but this is just supposed to be a quick reminder of some basic fact we will need) and during class we will review (4) (without proof, but you can get by in this class without knowing the proof). Topic (3), quotient spaces, are likely to be the most unfamiliar to most people, but this is an extremely useful way to construct interesting topological spaces so I will give a somewhat thorough

overview of this below. If you have taken an analysis class covering metric space topology you should be fine with the review below, but a good alternate source is Chapters 2-4 in Munkres' book "Topology" together with the statements in Sections 76 and 77 about the classification of surfaces.

2. Topological Spaces

We are all familiar with these ideas using the $\epsilon\delta$-definitions form analysis/calculus. We will see how to recover this form the more abstract definition of a topological space. Let X be a set. A collection of subsets $\mathcal{T}$ of X is called a **topology on** X if

(1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$,
(2) if $A \in \mathcal{T}$ and $B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$, and
(3) the union of sets in any subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$ (*i.e.* if $S \subset \mathcal{T}$ is any collection of sets in $\mathcal{T}$ then $\cup_{A \in S} A \in \mathcal{T}$.

A **topological space** is an ordered pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X. We frequently say X is a topological space without specific reference to $\mathcal{T}$ if the specific topology is not important or implied. Unless specifically given other notation we denote the topology on X by $\mathcal{T}_X$. If U is a subset of X then it is called **open** if $U \in \mathcal{T}$. A set is **closed** if its complement is open.

Given a set X a collection of subsets $\mathcal{B}$ of X is called a **basis for a topology** on X if

(1) X is the union of sets in $\mathcal{B}$ and
(2) if $U, V \in \mathcal{B}$ and $p \in U \cap V$ then there is a set $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$.

Prove the following facts:

(1) If $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}$ is the collection of all unions of sets in $\mathcal{B}$ then $\mathcal{T}$ is a topology on X.
(2) Let X be a set with a metric $d : X \times X \to [0, \infty)$. Let

$$\mathcal{B} = \{B_r(p) | p \in X \text{ and } r \in (0, \infty)\}$$

where $B_r(p) = \{x \in X | d(x, p) < r\}$ is the ball of radius r about p. One may easily check that $\mathcal{B}$ is a basis for a topology on X. The topology $\mathcal{B}$ generates is called the **metric topology** on X induced by d.

There are lots of other interesting topological spaces. I suggest you look at a standard text on topology to see other examples. In particular, you should be familiar with the **subspace topology** induced on a subset of a topological space and the **product topology** on the cartesian product of two topological spaces.

2.1. **Limit points and sequences.** If A is a subset of a topological space $(X, \mathcal{T})$ then $p \in X$ is a **limit point** of A if for each open set U containing p we have

$$(U - \{p\}) \cap A \neq \emptyset.$$

The **closure** of a set A is the set consisting of the point of A and the limit points of A. It is denoted $\overline{A}$.

Prove the following facts:

(1) A set A is closed if it contains all its limit points (that is if $A = \overline{A}$).

(2) If $p \notin A$ then p is a limit point of A if and only if every open set containing p intersects A non-trivially.

Let $(X, \mathcal{T})$ be a topological space. A **sequence** in X is a function from the natural numbers to X

$$p : \mathbb{N} \to X.$$

We denote $p(n)$ by p_n and usually write a sequence $\{p_n\}$ instead of using functional notation. We say a sequence $\{p_n\}$ in X **converges** to a point p, written $p_n \to p$, if for every open set U containing p there is some number N such that $p_n \in U$ for all $n \geq N$.

Prove the following facts:

(1) If X has a metric d then a sequence $\{p_n\}$ converges to p in the metric topology if and only if for all $\epsilon > 0$ there is an N such that $d(p_n, p) < \epsilon$ for all $n \geq N$.
(2) If A is a subset of a topological space, the sequence $\{p_n\} \subset A$ and $p_n \to p$ then $p \in \overline{A}$.

Sequences don't have to behave like we think they should (from our intuition coming form analysis), but if a topological space has extra properties then they do. A topological space $(X, \mathcal{T})$ is called **Hausdorff** if for each pair of distinct points $x, y \in X$ there is a pair of open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

We say a collection of open subset $\mathcal{N}$ of X containing a point $p \in X$ is a **neighborhood basis of a point** p if for all open sets U that contain p there is a set $V \in \mathcal{N}$ such that $V \subset U$. We call $(X, \mathcal{T})$ is **first countable** if every point has a neighborhood basis consisting of a countable number of sets.

Prove the following facts:

(1) If $(X, \mathcal{T})$ is a Hausdorff space and a sequence $\{p_n\}$ converges to p and q then $p = q$.
(2) If $(X, \mathcal{T})$ is a first countable space then a point p is a limit point of a set A if and only if there is a sequence $\{p_n\}$ in A such that $p_n \to p$.
(3) Metric topological spaces are Hausdorff and first countable.
(4) A topological space is called **second countable** if there is a basis for the topology on X that consist of a countable collection of sets. A second countable topological space if first countable. (For our class second countable will be a more important property, but it is nice to know that it implies first countable so the results here and below about first countable spaces also apply the second countable ones too.)

2.2. Continuous Functions.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces. A function

$$f : X \to Y$$

is **continuous** if $f^{-1}(U)$ is open in X for all open sets U in Y.

Prove the following facts:

(1) A function is continuous if and only if the inverse image of all closed sets is closed.
(2) A function is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.
(3) If $(X, \mathcal{T})$ is a first countable space then a function $f : X \to Y$ is continuous if and only if $f(p_n) \to f(p)$ for all $p_n \to p$.
(4) If X and Y are metric spaces with metrics d and d', respectively, then a function $f : X \to Y$ is continuous in the metric topologies if and only if for all $\epsilon > 0$ and

$x \in X$ there is a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon.$$

Recall (or look up) various continuous functions. For example prove the constant function is always continuous, projections to a factor of a product space is always continuous, and restrictions of continuous functions to subspaces are continuous. Also understand the how continuous functions to product spaces relate to continuous functions to the spaces that make up the product. The following is an important tool for building continuous functions.

Lemma 2.1. *Let $(X, \mathcal{T})$ be a topological space and $X = A \cup B$ where A and B are closed sets in X. If*

(1) *$f : A \to Y$ and $g : B \to Y$ are continuous functions and*
(2) *$f(x) = g(x)$ for all $x \in A \cap B$*

then there is a unique continuous function

$$h : X \to Y$$

such that $h(x) = f(x)$ for all $x \in A$ and $h(x) = g(x)$ for all $x \in B$.

A function $f : X \to Y$ between topological spaces is called a **homeomorphism** if f is continuous, one to one, onto and has continuous inverse. We call X and Y **homeomorphic** if there is a homeomorphism between them. You should think of homeomorphic as being indistinguishable form a topological perspective (that is as far as convergent sequences and continuous functions go they seem to be the same space). One of the main question in topology is to try to characterize or classify topological spaces up to homeomorphism.

3. Connectedness and Compactness

A topological space X is **disconnected** if there exists disjoint non-empty open sets U and V in X such that $X = U \cup V$. If no such sets exist the we say X is **connected**.

A topological space X is called **path connected** if every two points in X are connected by a path, that is for all $x, y \in X$ there is a continuous function

$$f : [0, 1] \to X$$

such that $f(0) = x$ and $f(1) = y$.

Prove the following facts:

(1) A topological space X is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.
(2) The real line $\mathbb{R}$ with its metric topology is connected (this is a bit harder than the other facts I have asked you to check). Similarly any open or closed interval is connected.
(3) If $f : X \to Y$ is a continuous surjective function and X is connected then Y is connected. (Said another way, the continuous image of a connected space is connected.)
(4) A path connected space is connected.
(5) Euclidean space $\mathbb{R}^n$ with its metric topology is connected. (Here you can use any metric on $\mathbb{R}^n$ you like.)

A collection of subsets $\{U_\alpha\}_{\alpha \in J}$ of X is a **cover** of X if $X = \cup_{\alpha \in J} U_\alpha$. (Here J is an indexing set, so for example if $J = \{1, 2, \ldots, n\}$ then $\{U_\alpha\}_{\alpha \in J} = \{U_1, U_2, \ldots, U_n\}$.) A topological space X is **compact** if every cover of X by open sets has a finite subcover, in other words if $\{U_\alpha\}_{\alpha \in J}$ is a collection of open sets in X that cover X then there is a subset $I \subset J$ such that I is finite and $X = \cup_{\alpha \in I} U_\alpha$.

Prove the following facts:

(1) A closed subspace of a compact topological space is compact in the subspace topology.
(2) A compact subset of a Hausorff topological space is closed.
(3) In a first countable compact topological space every sequence has a convergent subsequence.
(4) A closed interval $[a, b]$ is compact. (This is also a bit more difficult than many of the previous things I have asked you to do.)
(5) The continuous image of a compact space is compact.

The following lemma is surprisingly useful (try to prove it).

Lemma 3.1. *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff then f is a homeomorphism.*

4. Quotient spaces

Quotient spaces are extremely important. They allow us to construct interesting spaces and maps between them. It initial definition seems a little abstract, but we will quickly see that the notion allows us to "glue simple spaces to create complicated spaces" and define "spaces of things" (ie moduli spaces). Since quotient spaces are so important, and frequently covered lightly (if at all) in a first topology course, we give proofs of the results and lots of examples.

Let

- $(X, \mathcal{T})$ be a topological space,
- Y a set and
- $f : X \to Y$ a surjective function.

Then set

$$\mathcal{T}_f = \{U \subset Y \,|\, f^{-1}(U) \text{ is open in } X\}.$$

We call $\mathcal{T}_f$ the **quotient topology on** Y induced by f. (You should check this is a topology on Y!)

Theorem 4.1. *Let*

- *X and Y be topological spaces, and*
- *$f : X \to Y$ a surjective map.*

Then

The quotient topology on Y agrees with the given topology on Y

$$\Longleftrightarrow$$

$$\underbrace{(U \text{ is open in } Y \Leftrightarrow f^{-1}(U) \text{ is open in } X)}_{(*)}.$$

Proof. The proof is quite simple, but for completeness:

$(\Rightarrow)$ U is open in $Y \Leftrightarrow U \in \mathcal{T}_f \Leftrightarrow f^{-1}(U) \in \mathcal{T}_X \Leftrightarrow U$ is open in X.

$(\Leftarrow)$ $U \in \mathcal{T}_Y \underset{\text{by } (*)}{\Longleftrightarrow} f^{-1}(U)$ open in $X \Leftrightarrow U \in \mathcal{T}_f$. $\square$

Any surjective map f satisfying $(*)$ from the theorem is called a **quotient map**. (Notice a quotient map is clearly continuous.)

Lemma 4.2. *Let $f : X \to Y$ be a continuous surjective map. If f takes closed sets in X to closed sets in Y (that is f is a closed map) or if f takes open sets in X to open sets in Y (that is f is an open map) then f is a quotient map.*

Proof. Assume f is an open map. Note that U open in Y and f continuous implies that $f^{-1}(U)$ is open in X. And if $(f^{-1}(U)$ is open in X then by assumption $U(f(f^{-1}(U))$ is open in Y. Thus f is a quotient map. Similarly if f is a closed map and $f^{-1}(U)$ is open in X then $X - f^{-1}(U)$ is closed so $f(X - f^{-1}(U)) = f(f^{-1}(Y) - f^{-1}(U)) = f(f^{-1}(Y - U)) = Y - U$ is closed. Thus U is open. $\square$

Example 4.3. Think of S^1 as the unit circle in $\mathbb{R}^2$. Let

$$f : [0,1] \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

This map is clearly continuous (since its component functions are continuous by calculus). It is also easy to check that it maps closed sets to closed sets (both the domain and range are compact Hausdorff spaces). Thus f is a quotient map.

We now see how to construct continuous functions on quotient spaces.

Theorem 4.4. *Given a quotient map $f : X \to Y$ and another topological space Z then a function*

$$g : Y \to Z \text{ is continuous}$$
$$\Longleftrightarrow$$
$$g \circ f : X \to Z \text{ is continuous.}$$

Before we prove this theorem them let's think about what it means for our previous example. We have $f : [0,1] \to S^1 : t \to (\cos 2\pi t, \sin 2\pi t)$ is our quotient map. Any function $g : S^1 \to Z$ is continuous if and only if $g \circ f : [0,1] \to Y$ is continuous. Moreover given a continuous function $G : [0,1] \to Z$ such that $G(0) = G(1)$ we clearly have an induced function $g : S^1 \to Z$ such that $G = g \circ f$. Thus since G is continuous so is g. In other words studying continuous functions on S^1 is equivalent to studying continuous functions on $[0,1]$ that map the end points to the same point. Thus studying continuous functions on a "complicated space" (like S^1) is reduced to studying continuous functions on a simpler space (like $[0,1]$). Of course in this case everything is so simple it seems like we are making a big deal out of nothing, but this simple example shows us this powerful principle: that is, if $f : X \to Y$ is a quotient map then we can study continuous functions on Y by studying continuos functions on X that are constant on preimages of points in Y.

To emphasize again, we can understand a lot about S^1 by considering $[0,1]$. In some sense we can think of S^1 as the unit interval with the end points identified! Of course this is a natural thing to do, but this is the rigorous way to think about it (we will make this even more rigorous below). The space S^1 is fairly simple, but the unit interval is even simpler,

so we see the idea that we can use quotient spaces to study more complicated spaces using simpler spaces.

Proof. Again the proof is quite simple, but for completeness:

($\Rightarrow$) We know the composition of continuous functions is continuous.

($\Leftarrow$) If U is open in Z then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. So by the definition of quotient topology, $g^{-1}(U)$ is open in Y thus showing g is continuous. $\square$

We now make more precise the idea of "gluing spaces together from simple pieces". Let X be a topological space. A **decomposition** $\mathcal{D}$ **of** X is a collection of disjoint subset of X whose union is all of X. We define a topology on $\mathcal{D}$ by saying

$$\mathcal{S} \subset \mathcal{D} \text{ is open } \Leftrightarrow \cup_{S \in \mathcal{S}} S \text{ is open in } X.$$

With this topology (you should check it really is a topology on $\mathcal{D}$) $\mathcal{D}$ is called a **decomposition space of** X or a **quotient spaces of** X. Notice that there is a natural surjective map $p : X \to \mathcal{D}$ that takes a point $x \in X$ to the set $S \in \mathcal{D}$ that contains x.

You should think of $\mathcal{D}$ as the space X with the sets in $\mathcal{D}$ crushed to points, that is each $S \in \mathcal{D}$ is a single point in $\mathcal{D}$ while it is a set in X. We will consider many examples to make this idea clear later.

Lemma 4.5. *The topology on $\mathcal{D}$ is the quotient topology on $\mathcal{D}$ coming form the map $p : X \to \mathcal{D}$.*

The proof of this is very easy and left to the reader.

Example 4.6. Let $X = [0, 1]$ and set

$$\mathcal{D} = \{\{x\}|x \in (0, 1)\} \cup \{\{0, 1\}.\}$$

In other words we are identifying the points 0 and 1 in $[0, 1]$. We have the map

$$p : X \to \mathcal{D}$$

that induces the quotient topology on $\mathcal{D}$. Not surprisingly we will see $\mathcal{D}$ is (homeomorphic to) S^1.

Theorem 4.7. *Let $f : X \to Z$ be a surjective continuous map. Set*
$$\mathcal{D} = \{f^{-1}(z)|z \in Z\}$$
and give $\mathcal{D}$ the quotient topology. Let
$$p : X \to \mathcal{D}$$
be the natural map discussed above.

The map f induces a continuous bijection $g : \mathcal{D} \to Z$. Moreover
$$g \text{ is a homeomorphism}$$
$$\Longleftrightarrow$$
$$f \text{ is a quotient map.}$$

Example 4.8. Returning to our previous example, if $f : [0, 1] \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ then $\mathcal{D}$ is the decomposition space of $[0, 1]$ induced from f and since we know f is a quotient map, f induces a homeomorphism $g : \mathcal{D} \to S^1$.

We finally have a completely rigorous way to think of S^1 as the interval with endpoints identified!

Proof. Clearly f induces a bijection $g : \mathcal{D} \to Z$. We also knot $g \circ p = f$ is continuous so by Theorem 4.4 we see g is continuous.

($\Rightarrow$) Since g is a homeomorphism we know U is open in $Z \Leftrightarrow g^{-1}(U)$ open in $\mathcal{D} \Leftrightarrow f^{-1}(U) = p^{-1}(g^{-1}(U))$ open in X. Thus f is a quotient map

($\Leftarrow$) Assuming f is a quotient map we see that $\mathcal{S} \subset \mathcal{D}$ is an open set in $\mathcal{D} \Leftrightarrow p^{-1}(\mathcal{S})$ is open in X. But notice that $p^{-1}(\mathcal{S}) = f^{-1}(g(\mathcal{S}))$, so $\mathcal{S}$ is open in $\mathcal{D} \Leftrightarrow g(\mathcal{S})$ is open in Z. Thus $g^{-1} : Z \to \mathcal{D}$ is continuous and g is a homeomorphism. $\square$

Corollary 4.9. *Let $f : X \to Z$ and $\mathcal{D}$ be as in Theorem 4.7. If X is compact and Z is Hausdorff then $g : \mathcal{D} \to Z$ is a homeomorphism.*

Proof. If C is a closed set in X then since X is compact so is C. Thus $f(C)$ is a compact subset of Z and since Z is Hausdorff $f(C)$ is a closed set in Z. Thus f is a closed map and hence a quotient map. $\square$

Example 4.10. Note that $\mathcal{D}$ from the example above is homeomorphic to S^1 follows trivially from this corollary.

Example 4.11. In this example we see that if we take the boundary of a disk and crush it to a point we get S^2. To see this rigorously let
$$D_2^2 = \{(x,y)|x^2 + y^2 \le 4\}$$
and
$$S^2 = \{(x,y,z)|x^2 + y^2 + z^2 = 1\}.$$
We define a continuous map
$$f : D_2^2 \to S^2.$$
To do this we think of D_2^2 as the union of two closed sets
$$D_2^2 = D_1^2 \cup A$$
where D_1^2 is the unit disk in the plane and $A = \{(x,y)|1 \le x^2 + y^2 \le 4\}$. Note $D_1^2 \cap A = S^1$. (Let me encourage you to draw pictures of the various sets and maps as you go along!) Now set
$$f : D_1^2 \to S^1 : (x,y) \mapsto (x,y,-\sqrt{1-x^2-y^2}).$$
In other words f restricted to D_1^2 just sends a point on the unit disk in the xy-plane to the point on the lower hemisphere of S^2 directly below it. We know this map is continuous since each component function is continuous (which we know from, say, calculus). To define f on A we first need to define the map
$$h : A \to D_1^2 : (r,\theta) \mapsto (2-r,\theta)$$
using polar coordinates. Pictorially we are "reflecting" A across the unit circle. Note again that h is continuous. Also note that h sends the the outer boundary of A (the circle of radius 2) to the origin (ie it is crushed to a point) and h is one-to-one elsewhere. Now consider
$$k : D_1^2 \to S^2 : (x,y) \mapsto (x,y,\sqrt{1-x^2-y^2}).$$
This is the map that sends a point in the unit disk in the xy-plane to the point on the upper hemisphere directly above it. Now finally define
$$f : A \to S^2$$

as $f = k \circ h$. Since the definitions of f agree on the the overlap between A and D_1^2 (and the sets are closed) we know that f is a continuous map $f : D_2^2 \to S^2$. Moreover since D_2^2 is compact and S^2 is Hausdorff we know that the decomposition space of D_2^2 associated to f is homeomorphic to S^2, but of course, this decomposition space is just the one that crushes the boundary of D_2^2 to a point:

$$\mathcal{D} = \{\{p\}|p \in \text{ interior of } D_2^2\} \cup \{\{p|p \in \text{ boundary of } D_2^2\}\}.$$

Example 4.12. Similarly we can think of the unit n-sphere as homeomorphic to the unit n-disk with its boundary crushed to a point. You should check this carefully (note we have already done this for $n = 1$ and 2).

Example 4.13. We can similarly see that S^2 can be thought of as the union of two disks glued together along their boundary. More precisely, let X be two copies of the unit disk in $\mathbb{R}^2$. We can define a map from one of the disks to the lower hemisphere as we did above and similarly a map from the other disk to the upper hemisphere. This will give a quotient map $X \to S^2$ and the induced decomposition space will identify a point on the boundary of one disk to a point on the boundary of the other.

You should carefully work this out! Pick two specific disjoint disk in $\mathbb{R}^2$ to represent X and write down explicit maps that realize the description above.

Similarly show S^n is homeomorphic to the result of gluing two n-disks together along their boundary.

Example 4.14. Let $X = [0, 1] \times [0, 1]$ be the unit square and set

$$\mathcal{D} = \{\{(x,y)\}|0 < x < 1, 0 < y < 1\} \cup \{\{(1,y),(0,y)|0 < y < 1\}\}$$
$$\cup \{\{(x,1),(x,0)\}|0 < x < 1\} \cup \{\{(0,0),(1,0),(0,1),(1,1)\}\}.$$

That is we identify a point on an edge of the unit square with the point on the opposite edge. Pictorially this is given in Figure 1. We claim that $\mathcal{D}$ is homeomorphic to the torus $S^1 \times S^1$. You should prove this. This is actually pretty easy. Just notice that we already have a quotient map $[0, 1] \to S^1$. Now just take the map $[0, 1] \times [0, 1] \to S^1 \times S^1$ to be this map in each factor.

Note we now have a rigorous way to thinking of the "intuitively clear idea" that the torus as a square with edges identified.

Example 4.15. Let Z be the subset of $\mathbb{R}^3$ given in Figure 2. Show that Z is homeomorphic to the indicated quotient space of the octagon X shown in the figure.

Example 4.16. Consider the unit sphere in $\mathbb{R}^4$

$$S^3 = \{(x_1, x_2, x_3, x_4)|x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

More conveniently we will think of S^3 as the unit sphere in complex 2-space $\mathbb{C}^2$

$$S^3 = \{(z_1, z_2)||z_1|^2 + |z_2|^2 = 1\}.$$

Also think of

$$S^1 = \{z \in \mathbb{C}||z| = 1\}.$$

We now say that two points (z_1, z_2) and (w_1, w_2) in S^3 are equivalent if there is some $z \in S^1$ such that
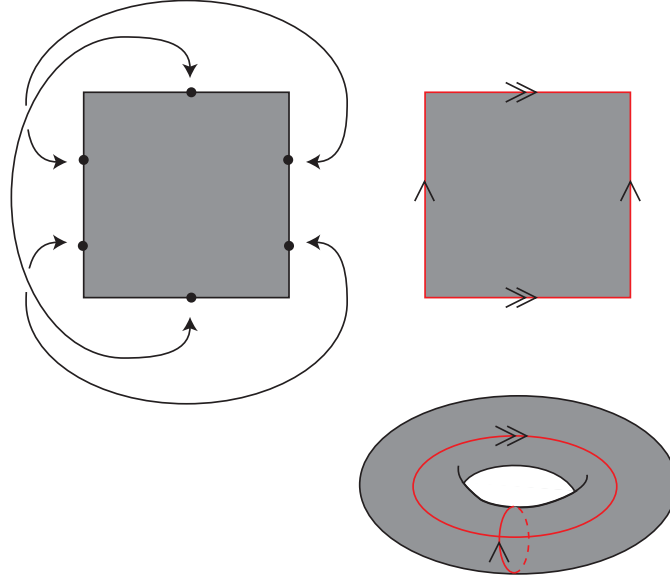
$$(z_1, z_2) = (zw_1, zw_2).$$

FIGURE 1. On the upper left is the square $X = [0,1] \times [0,1]$ and the indication that opposite points on the sides of the square are identified. On the upper right is the same thing, but we are indicating which edges are identified (and the way they are identified) using arrows. On the bottom is the torus $T^2 = S^1 \times S^1$ that is homeomorphic to the decomposition space $\mathcal{D}$ indicated in the figure. Also the red circles correspond the edges of the square after identification.

Show this is an equivalence relation. Let $\mathcal{D}$ be the equivalence classes of points in S^3.

We claim that $\mathcal{D}$ is homeomorphic to S^2. This is far from obvious! Let's carefully work this out. We begin by writing $S^3 = A \cup B$ where

$$A = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1, |z_1|^2 \le \frac{1}{2}\}$$

and

$$B = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 = 1, |z_2|^2 \le \frac{1}{2}\}.$$

We claim A and B are each homeomorphic to the solid torus $D^2 \times S^1$. Indeed consider the map

$$\psi : D^2_{\frac{1}{2}} \times S^1 \to A : (z_1, \theta) \mapsto (z_1, \sqrt{1 - |z_1|^2} e^{i\theta}),$$

where $D^2_{\frac{1}{2}}$ is the disk of radius $\frac{1}{2}$ in $\mathbb{C}$. Clearly ψ is continuous. It is easy to check that it is one-to-one and onto and hence by Lemma 3.1 a homeomorphism.

Note further that

$$A \cap B = \{(z_1, z_2) | |z_1| = |z_2| = \frac{1}{\sqrt{2}}\} = S^1 \times S^1.$$

So the map from the disjoint union of two solid tori to S^3 that sends one solid torus to A and the other to B is a quotient map from which we can see that S^3 is homeomorphic to the
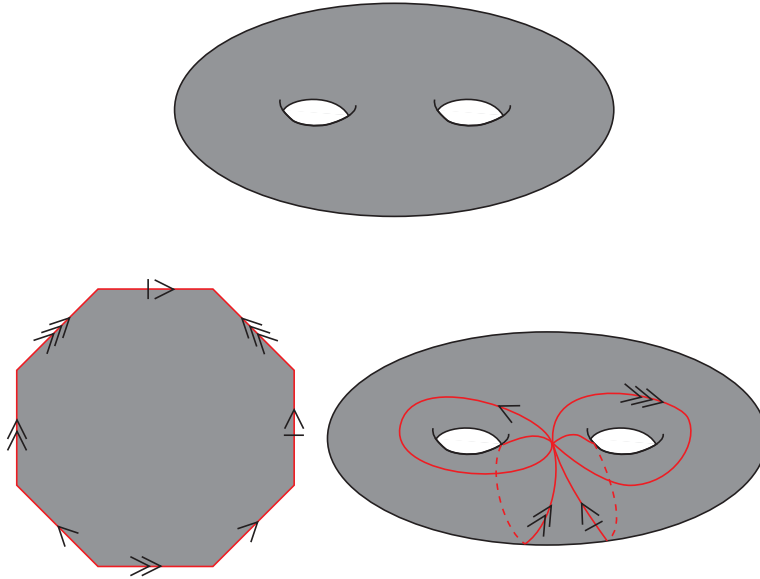
FIGURE 2. The upper figure is the "two holed torus". On the lower right is an octagon in the plane with indications on how to identify the edges. The result of this identification is the two holed torus. The picture on the bottom right should help you see where the edges of the octagon sit after the identifications are made.

result of gluing two solid tori together along their boundary! (You should really convince yourself of this very important fact.)

OK, now notice that the equivalence relation above only identifies points of A with other points of A and similarly for B. So we can consider $\mathcal{D}_A$ the equivalence classes in A and similarly for B. It should be clear that the map $A \to D_1^2$ obtained by sending (z_1, z_2) to $\frac{z_1}{z_2}$ induces this decomposition $\mathcal{D}_A$. (Check this!) So $\mathcal{D}_A$ is homeomorphic to D_1^2. Similarly $\mathcal{D}_B$ is homeomorphic to D_1^2. We now have the following diagram

$$
\begin{array}{ccc}
A \coprod B & \xrightarrow{\ h\ } & S^3 \\
{\scriptstyle p_A \cup p_B}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\
\mathcal{D}_A \coprod \mathcal{D}_B & \xrightarrow{\ k\ } & \mathcal{D} \\
{\scriptstyle q}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi} \\
D^2 \coprod D^2 & \xrightarrow{\ l\ } & S^2,
\end{array}
$$

where $\coprod$ means disjoint union, q is a homeomorphism and the horizontal maps are inclusion. Now ϕ is a the map induced on $\mathcal{D}$ from $l \circ q$ and thus by Corollary 4.9 is a homeomorphism.

That was an involved example, but a very important one we now have the map

$$p : S^3 \to S^2.$$

This is called the Hopf map and we will see it reoccur many times and illustrate many important ideas. You should try to think back through the proof and try to understand what $p^{-1}(x)$ for $x \in S^2$ looks like and maybe try $p^{-1}(x) \cup p^{-1}(y)$ for $x, y \in S^2$.

Also S^2 thought of as the quotient of S^3 by the above relation is usually called **complex projective plane** and is denoted $\mathbb{C}P^1$. The complex projective plane is sometimes defined as the set of complex lines in $\mathbb{C}^2$. Try to convince yourself that this is correct.

You can play the same game with S^{2n+1} in $\mathbb{C}^{n+1}$ to get $\mathbb{C}P^n$, called **complex projective n-space**. These are very interesting manifolds that we will see later on and don't have "simple" descriptions for $n \neq 1$ (where $\mathbb{C}P^1 \cong S^2$). So we have described a "complicated" space as a quotient space of a sphere (which is fairly simple). Reemphasizing the philosophy of this section we can study functions on these complicated spaces by studying functions on spheres that are constant on equivalence classes.

We now formalize the idea of "gluing spaces together". Here is the set up

- Y, Z are topological spaces
- $A \subset Y$ and
- $f : A \to Z$ a continuous injective map.

Now let $\mathcal{D}$ be a decomposition space of the disjoint union $Y \coprod Z$ given by

$$\mathcal{D} = \{\{a, f(a)\}\}|a \in A\} \cup \{\{y\}|y \in (Y - A)\} \cup \{\{z\}|z \in (Z - f(A))\}.$$

With the topology induced on $\mathcal{D}$ we say this is the space obtained by **gluing Y to Z along A via f**. We note that we can generalize this idea to allow for f to not be injective map by setting

$$\mathcal{D} = \{\{z\} \cup f^{-1}(z)\}|z \in f(A)\} \cup \{\{y\}|y \in (Y - A)\} \cup \{\{z\}|z \in (Z - f(A))\}.$$

We denote the result of gluing by $Y \cup_f Z$ or if the map f is understood then by $Y \cup_A Z$.

Example 4.17. The example above that S^n is obtained by gluing two n-disks together along there boundary can be expressed

$$S^n = D^n \cup_{\partial D^n} D^n$$

where we understand that we are using the identity map on ∂D^n. Similarly

$$S^3 = (S^1 \times D^2) \cup_{\partial(S^1 \times D^2)} (S^1 \times D^2).$$

It is a very interesting exercise to try to explicitly figure out what map we are using in this gluing.

Example 4.18. Let Y be the unit 4-ball in $\mathbb{C}^2$ and let $A = \partial Y$ be the unit 3-sphere. Set $Z = S^2$ and let $f : A \to S^2$ be the Hopf map from above. We claim that $Y \cup_f Z$ is homeomorphic to $\mathbb{C}P^2$. This is not so easy to see, but using ideas from our proof that $\mathbb{C}P^1 \cong S^2$ you should give this a try.