Math 6458 - Spring 2009
Homework 1

Work all the problems, but carefully write up and turn in Problems 2, 3, 4 and 8.

1. Carefully read the section in Lee's book on "Model spaces of Riemannian Geometry", this is pages 33–42, and work all the exercises on those pages. We covered some of this in class, but not all of it and the book has a different perspective and notation.

2. For $n = 1, 2, 3$ and 4. Compute the volume of the n-sphere of radius R in $\mathbb{R}^{n+1}$ with the metric induced from $\mathbb{R}^{n+1}$.

3. Prove the gradient ∇f of a function $f : M \to \mathbb{R}$ at a point p, points in the direction of greatest increase of the function. That is among all unit vectors v, $df(v)$ is largest when pointing in the direction of ∇f. (Recall $df(v)$ can be interpreted as the directional derivative of f in the direction v).

4. Recall that last semester we talked about integrating n-forms over an oriented n-manifold. I also stated but did not prove (we will come back to this later on this semester) Stokes theorem which says that if M is a smooth oriented n-manifold with boundary and ω is an compactly supported $(n-1)$-form on M, then

   $$\int_M d\omega = \int_{\partial M} \omega$$

   if ∂M is given the orientation induced from M (that is if N is an outward pointing vector in T_pM at $p \in \partial N$ then a basis $v_1, \ldots, v_{n-1}$ for $T_p\partial M$ gives the induced orientation on ∂M if $N, v_1, \ldots, v_{n-1}$ is a positively oriented basis for T_pM).

   If (M, g) is a Riemannian n-manifold and we denote its volume form by Ω_g then we say a vector field v is *volume preserving* if the Lie derivative of Ω_g is zero: $L_v\Omega_g = 0$. Recall this means the the flow of v preserves Ω_g. Also recall that

   $$L_v\Omega_g = d\iota_v\Omega_g + \iota_v d\Omega_g$$

   and since Ω_g is an n-form on an n-manifold $d\Omega_g = 0$. Thus $L_v\Omega_g = d\iota_v\Omega$. But we know that $\bigwedge^n M = M \times \mathbb{R}$ so any n-form is $f\,\Omega_g$ for some function $f : M \to \mathbb{R}$. Thus there is some function, called the *divergence of* v, and written div v, such that

   $$L_v\Omega_g = d\iota_v\Omega_g = (\text{div } v)\,\Omega_v.$$

   Now work problems 3-3 and 3-4 in Lee's book.

5. Problem 3-8 in Lee's book.

6. Problems 3-10 and 3-12 in Lee's book.

7. Recall $\mathbb{C}P^1$ is diffeomorphic to S^2. How does the Fubini-Study metric on $\mathbb{C}P^1$ relate to the metric induced on S^2 as the unit sphere in $\mathbb{R}^3$?

8. Show the Lee group $SU(2)$ is diffeomorphic to S^3. How does the metric on $SU(2)$ from class relate to the metric on S^3 induced from $\mathbb{R}^4$?