Math 7338 - Fall 2011
Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 7, 9, 13, 14, 15.* **Due: In class on September 9.**

1. Recall l^∞ is the set of sequences $s = \{x_n\}$ for which $\sup_{n=1\ldots\infty}\{|x_n|\} < \infty$ and that $\|\|_\infty$ is the function $l^\infty \to \mathbb{R}$ defined by $\|s\|_\infty = \sup_{n=1\ldots\infty}\{|x_n|\}$. Show that $\|\|_\infty$ is a complete norm on l^∞.
2. Show the norm $\|\{x_n\}\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$ is complete on the space $l^p = \{\{x_n\} : \|\{x_n\}\| < \infty\}$.
3. Let $C([a,b])$ be the set of continuous functions from the interval $[a,b]$ to $\mathbb{R}$. For $f \in C([a,b])$ define
$$\|f\|_p = \left(\int_a^b |f(x)|^p\, dx\right)^{1/p}.$$
Show that this is a norm on $C([a,b])$. (Note that this we are only dealing with continuous functions we can use either the Lebesgue or Riemann integral. For this problem you will need to establish the Minkowski inequality for integrals of continuous functions.)
4. Let $C^k([a,b])$ be the set of functions $f : [a,b] \to \mathbb{R}$ that have k continuous derivatives. For any $l < k$ define
$$\|f\|_{C^l([a,b])} = \sum_{k=0}^{l} \sup_{a\le x\le b}\{\left|f^k(x)\right|\},$$
for any $f \in C^k([a,b])$. Show that this gives a norm on $C^k([a,b])$. Would you have to change anything if you considered an open interval?
5. Show that if $l < k$ then $\|f\|_{C^l([a,b])}$ is not a compete norm on $C^k([a,b])$. What is the completion? Show that $\|f\|_{C^l([a,b])}$ is a complete norm on this space. (You don't have to reprove theorems from Analysis. It is OK just to cite them.)
6. Let $(V, \|\|_V)$ and $(W, \|\|_W)$ be two normed vector spaces. Let $\mathcal{L}(V,W)$ be the set of bounded linear maps from V to W. For $f \in \mathcal{L}(V,W)$ define
$$\|f\| = \sup_{v\in V, v\neq 0} \frac{\|f(v)\|_W}{\|v\|_V}.$$
Show this gives a norm on $\mathcal{L}(V,W)$.
7. If $1 \le p < q \le \infty$ then show that $l^p \subset l^q$ but show the sets are not equal.
8. If $1 \le p < q \le \infty$, then show that
$$\|f\|_p \le (b-a)^{1/p-1/q}\|f\|_q.$$
Thus we have $L^q([a,b]) \subset L^p([a,b])$. Hint: Hölder's inequality.
9. Let $\|\|_1$ and $\|\|_2$ be two norms on a vector space V. Show that the norms are equivalent if and only if the topologies associated to the norms are the same.

Let $(V, \|\|)$ be a normed vector space and let W be a closed linear subspace of V. Denote by V/W the *quotient space* of V by W. That is say two vectors v, v' in V are equivalent if $v - v' \in W$. Show that this gives and equivalence relation on V and we denote the set of equivalence classes by V/W. Also denote the equivalence class of v by $[v]$.

10. Show V/W is a vector space.
11. On V/W define

$$\|[v]\| = \inf_{w\in W}\{\|v-w\|\} = \inf_{v'\in[v]}\{\|v'\|\}.$$

    Show that this is a norm on V/W.
12. If $(V, \|\|)$ is a Banach space then show $(V/W, \|\|)$ is a Banach space.

    Let c be the set of all convergent sequences of real numbers. Let c_0 be the set of all sequences of real numbers that converge to 0. Let s^* be the set of sequences of real numbers having only finitely many non-zero terms.
13. Show that $s^* \subset c_0 \subset c \subset l^\infty$ and moreover each is a linear subspace of the spaces it is contained in. The norm $\|\|_\infty$ induces norms on all these spaces. What is the closure of s^* in l^∞? Is $(s^*, \|\|_\infty)$ a Banach space? What is the dimension of s^*? (There are lots of ways to answer this last question, but it is very easy using a result from class.) Is $(c_0, \|\|_\infty)$ or $(c, \|\|_\infty)$ a Banach space?
14. Let e_i be the sequence $\{x_n\}$ with $x_i = 1$ and $x_n = 0$ for $n \neq 1$. Show that $\{e_i\}$ is a Schauder basis for l^p for all p with $1 \leq p < \infty$. Show that it is not a Schauder basis for l^∞. Is $\{e_i\}$ a Hamel basis for any l^p? Show $\{e_i\}$ is a Schauder basis for $(c_0, \|\|_\infty)$.
15. Show l^∞ is not separable. (Thus it cannot have a Schauder basis.)