Math 7338 - Fall 2011
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 5, 6, 9, 11, 13, 14.* **Due: In class on October 4.**

1. Let V be a vector space with inner product $\langle \cdot, \cdot \rangle$. We get a map
$$\phi : V \to V^*$$
by defining $\phi(v)$ to be the map $\phi_v : V \to \mathbb{R} : w \to \langle w, v \rangle$. Show that V is a Hilbert space if and only if this map is onto.
2. Prove that $\{1, t^3, t^6, \ldots\}$ span $L^2([0,1])$.
Hint: recall $C^0([0,1])$ is dense in $L^2([0,1])$.
3. Prove that $\{1, t^2, t^4, t^6, \ldots\}$ spans $L^2([0,1])$ but that it does not span $L^2([-1,1])$.
4. Let H be the Hilbert space that comes as the completion of the linear space
$$C^1([0,1]) = \{f : [0,1] \to \mathbb{C} \text{ such that } f \text{ and } f' \text{ are continuous}\}$$
with the inner product
$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx + \int_0^1 f'(x)\overline{g'(x)}\,dx.$$
Show that any $f \in H$ can be represented as a continuous function.
Hint: An element in H is a Cauchy sequence in $C^1([0,1])$. Show such a sequence converges uniformly to a continuous function.
5. With H as in the previous problem show that $\phi(f) = f(0)$ for all $f \in H$ is a bounded linear map. Find $g \in H$ such that $\phi(f) = \langle f, g \rangle$.
Hint: Assume g is twice differentiable and use integration by parts to find equations g must satisfy.
6. Let $T : V_1 \to V_2$ be a map between Hilbert spaces such that $\|Tv\|_2 = \|v\|_1$ (that is T is an isometry). Show that $\langle v, w \rangle_1 = \langle Tv, Tw \rangle_2$.
7. Show that the product of two self-adjoint operators on a Hilbert space is self-adjoint if and only if they commute.
8. Let $T_n : H \to H$ be a sequence of bounded self-adjoint operators on a Hilbert space H. Suppose that $T_n \to T$ (that is $\|T_n - T\| \to 0$, where $\|\cdot\|$ is the norm on the space of bounded linear operators). Show that T is a bounded self-adjoint operator on H.
9. Consider the two subspaces of l^2:
$$W_1 = \{\{x_n\} \in l^2 : \sum (1/n) x_n = 0\}$$
and
$$W_2 = \{\{x_n\} \in l^2 : \sum (1/\sqrt{n}) x_n = 0\}.$$
Which of these spaces is closed in l^2?
10. (Some useful linear algebra facts.)

(a) Let V be a linear space and let $f, f_1, \ldots, f_n \in V^\#$ (recall $V^\#$ is the set of linear maps $V \to \mathbb{R}$ or $V \to \mathbb{C}$ depending on the base field of V). Prove that $f \in \operatorname{span}\{f_1, \ldots, f_n\}$ if and only if $\ker f \supset \cap_{i=1}^n \ker f_i$.

(b) Let W be a linear subspace of V. Prove that the dimension of V/W is $n < \infty$ if and only if there is a linearly independent set $\{f_1, \ldots, f_n\}$ in $V^\#$ such that $W = \cap_{i=1}^n \ker f_i$.

11. Let V be an infinite dimensional Banach space. Show that V does not have a countable basis.
    Hint: If not $V = \cup_{n=1}^\infty (\operatorname{span}\{x_i\}_{i=1}^n)$. Now think about Baire category.
12. Recall $\{v_n\}_{n=1}^\infty$ is a Schauder basis for V if for all $v \in V$ there is a unique sequence $\{a_n\}$ of scalars such that $v = \sum_{n=1}^\infty a_n v_n$. Given such a basis for a Banach space $(V, \|\cdot\|)$ define
$$\left|\sum_{n=1}^\infty a_n v_n\right| = \sup_m \|\sum_{n=1}^m a_n v_n\|.$$
    Show that $(V, |\cdot|)$ and $(V, \|\cdot\|)$ are isomorphic Banach spaces. Use this to show that the following. For any $v \in V$ let $v = \sum a_n v_n$ and for each $i \in \mathbb{N}$ define $f_i(x) = a_i$. Show that $f_i \in V^*$ (that is, f_i is a bounded linear functional).
13. Let V be a Banach space and let W be a closed finite codimensional subspace (that is V/W is finite dimensional). Show that W is complemented in V.
14. Let T be a continuous linear operator from one Banach space to another. Show that T is either onto or its image is of first category.
    Hint: Look at the proof of Lemma III.10.
15. Recall from the last homework you showed that for $p > 1$ we know that $L^p([0,1]) \subset L^1([0,1])$ but that $L^p([0,1]) \neq L^1([0,1])$. Sow that the inclusion map from $(L^p([0,1]), \|\cdot\|_p)$ into $(L^1([0,1]), \|\cdot\|_1)$ is continuous and that its image is of first category.