Math 7338 - Fall 2011
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 5, 9, 12, 14.* **Due: In class on November 3.**

1. Show that l^1 is not isometric to a quotient of l^2.
2. Let V be a normed linear space and W a linear subspace of it. As in class we define the distance from $v \in V$ to W to be

$$d(v, W) = \inf_{w \in W} \|v - w\|.$$

   Show that the distance also equals

$$\sup |l(v)|$$

   where the supremum is taken over all $l \in V^*$ such that $\|l\| \leq 1$ and $W \in \ker l$.
3. Recall c_0 is the set of sequences that converge to 0 and we use the $\|\cdot\|_\infty$ norm on c_0. Show that there is a sequence $\{x_n\}$ in c_0 that does not converge to 0 but for every continuous linear function $l : c_0 \to \mathbb{R}$ we have $\lim l(x_n) = 0$. (Paraphrasing we say that linear functionals on c_0 cannot detect convergence.)
   Hint: We proved that $(c_0)^* = l^1$.
4. Show that a sequence $\{x_n\}$ in l^1 converges to 0 if and only if for every continuous linear map $l : l^1 \to \mathbb{R}$ we have $\lim l(x_n) = 0$. (Paraphrasing we say that linear functionals on l^1 can detect convergence to zero. Or weak and strong convergence are the same for l^1)
   Warning: The non-trivial implication is fairly hard.
5. From class we know that there is some closed linear subspace V of l^1 such that $c_0 = l^1/V$. Show that V is not complemented in l^1.
6. For a reflexive Banach space V, show that for any functional $l \in V^*$ with $\|l\| = 1$ there is some $v \in V$ with $\|v\| = 1$ and $l(v) = 1$.
7. Show that there is a Banach space V and a functional $l \in V^*$ such that $\|l\| = 1$ but there is no element $v \in V$ with $\|v\| = 1$ and $l(v) = 1$.
   Hint: Try $V = c_0$ and $\{1/n^2(\sum_{k=1}^{\infty} 1/k^2)^{-1}\} \in l^1$.
8. In class we constructed an isometry $l^1 \to (c_0)^*$ and $l^\infty \to (l^1)^*$. We can use these to get an isometry $l^\infty \to (c_0)^{**}$. Show that the natural embedding $c_0 \to (c_0)^{**}$ is just the inclusion of c_0 into l^∞ under the above isometry.
9. Let V be a normed vector space (not necessarily complete). Let $\overline{V}$ be its completion. Show that V^* and $\overline{V}^*$ are isometric.
10. Recall s^* is the set of sequences that have only finitely many non-zero terms. Give s^* the norm $\|\cdot\|_\infty$. Determine the dual space of s^*.
11. Let V be a separable Banach space. Show there is an isometric embedding $T : V^* \to l^\infty$.

12. For $1 < p < \infty$ show that a sequence $\{v_n\}$ in l^p converges weakly to v if and only if $\{\|v_n\|\}$ is bounded and if we write out each v_n as the sequence $\{v_n(k)\}_{k=1}^{\infty}$ and v as $\{v(k)\}_{k=1}^{\infty}$ then for each k we have $v_n(k) \to v(k)$ as $n \to \infty$.
13. Let V be a normed vector space and let $M \subset V^*$ be a total set (recall a set of functions is total if its span is dense in V^*). Show that a sequence $\{v_n\}$ weakly converges to v if and only if the sequence $\{\|v_n\|\}$ is bounded and for every $l \in M$ we have $l(v_n) \to l(v)$ as $n \to \infty$.
14. Show a weakly closed set in a normed vector space is closed.
15. Let $T : V \to W$ be a bounded linear map between two normed linear spaces. If $\{v_n\}$ is a sequence in V that weakly converges to v then show that $\{T(v_n)\}$ weakly converges to $T(v)$ in W.