Math 7338 - Fall 2011
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 6, 7, 9, 11.* **Due: In class on December 1.**

1. Let V be a topological vector space and $E \subset V$. Show that E is bounded if and only if for every sequence $\{v_n\}$ in E and every sequence of numbers $\{a_n\}$ that converges to 0, we have $a_n v_n \to 0$ as $n \to \infty$.
2. Show that a compact subset of a topological vector space is bounded.
3. Let V be a topological vector space and let $\mathcal{B}$ be a neighborhood basis for 0. We say a sequence $\{v_n\}$ in V is Cauchy if for every $U \in \mathcal{B}$ there is some N such that for all $n, m \geq N$ we have that $v_n - v_m \in U$. Suppose that the topology on V is given by a translation invariant metric d. Then show that the definition of Cauchy just given is equivalent to the normal definition of Cauchy for a metric space.
4. Let S be some subset of a vector space V. The convex hull of S is the set containing all sums $t_1 v_1 + \ldots + t_n v_n$ where $v_i \in S$ and $t_i \geq 1$ satisfy $\sum t_i = 1$ (where n is arbitrary). Show that if V is a topological vector space and S is open then the convex hull of S is also open.
5. Show that for two subsets A and B of a topological vector space $A + B$ is compact if A and B are compact.

   Let $C([0,1])$ be the space of continuous complex valued functions on $[0,1]$. Let

   $$d(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

   This is a metric on $C([0,1])$ and so it defines a topology we denote $\mathcal{S}$. Let $\mathcal{T}$ be the topology of pointwise convergence on $C([0,1])$, that is the topology generated by the semi-norms $\{\|\cdot\|_x\}_{x\in[0,1]}$ where

   $$\|f\|_x = |f(x)|.$$

6. Show that a set in $C([0,1])$ is bounded in the $\mathcal{T}$ topology is also bounded in the $\mathcal{S}$ topology. Conclude that the identity map $id : (C([0,1]), \mathcal{T}) \to (C([0,1]), \mathcal{S})$ is a bounded map (that is takes bounded sets to bounded sets).
7. Show that $id : (C([0,1]), \mathcal{T}) \to (C([0,1]), \mathcal{S})$ is not continuous. Notice that this shows that for topological vector spaces bounded is not equivalent to continuous for linear maps (though continuous does still imply bounded). Also notice by a theorem from class we know that $\mathcal{T}$ does not come from a metric. So this is an example of a topological vector space where the topology is not induced by a metric, norm or inner-products (or even a countable collection of semi-norms).
8. Show that $\mathcal{T}$ does not have a countable neighborhood basis for $0 \in C([0,1])$. (Note this also shows that $\mathcal{T}$ does not come from a metric.)

9. Let V be a Banach space. On V we have the strong topology, induced from the norm, and the weak topology, induced from the bounded linear functionals as in class. Show that a set E in V is strongly bounded if and only if it is weakly bounded.
   Hint: You might need Hahn-Banach and the Uniform Boundedness Principle.
10. Let H be a Hilbert space. Suppose that $\{x_n\}$ is a sequence in H that converges weakly to x. Show that $\{x_n\}$ converges strongly to x if and only if $\{\|x_n\|\}$ converges to $\|x\|$.
11. Let V be a Banach space. Show that the weak* topology on V^* is complete.