RECOLLECTIONS FROM POINT SET TOPOLOGY FOR FUNCTIONAL ANALYSIS

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1. Introduction

The main idea of point set topology is to (1) understand the minimal structure you need on a set to discuss continuous things (that is things like continuous functions and convergent sequences), (2) understand properties on these spaces that make continuity look more like we think it should (that is properties like Hausdorff and countability properties), and (3) classify topological spaces (that is classify those spaces on which you can talk about continuity). So in some sense this is the starting point, or foundation, of a great deal of mathematics. You will not need a thorough understanding of topology for this course, but we will use the basic terminology. You probably ran across most of this in an undergraduate analysis course, but here we list the basis notions and some exercises. But in sort you should be familiar with:

(1) Abstract definition of a topological space, continuous functions and convergent sequences.
(2) Various topological conditions that make spaces nicer. The most important for us are:
   (a) Hausdorff
   (b) Second countable
   (c) Compact
   (d) Connected

If you have taken an analysis class covering metric space topology you should be fine with the review below, but a good alternate source is Chapters 2-4 in Munkres' book "Topology".

2. Topological Spaces

We are all familiar with these ideas using the $\epsilon\delta$-definitions form analysis/calculus. We will see how to recover this form the more abstract definition of a topological space. Let X be a set. A collection of subsets $\mathcal{T}$ of X is called a **topology on** X if

(1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$,
(2) if $A \in \mathcal{T}$ and $B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$, and
(3) the union of sets in any subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$ (*i.e.* if $S \subset \mathcal{T}$ is any collection of sets in $\mathcal{T}$ then $\cup_{A\in S} A \in \mathcal{T}$.

A **topological space** is an ordered pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X. We frequently say X is a topological space without specific reference to $\mathcal{T}$ if the specific topology is not important or implied. Unless specifically given other notation we denote the topology on X by $\mathcal{T}_X$. If U is a subset of X then it is called **open** if $U \in \mathcal{T}$. A set is **closed** if its complement is open.

Given a set X a collection of subsets $\mathcal{B}$ of X is called a **basis for a topology** on X if

(1) X is the union of sets in $\mathcal{B}$ and
(2) if $U, V \in \mathcal{B}$ and $p \in U \cap V$ then there is a set $W \in \mathcal{B}$ such that $p \in W \subset U \cap V$.

Prove the following facts:

(1) If $\mathcal{B}$ is a basis for a topology on X and $\mathcal{T}$ is the collection of all unions of sets in $\mathcal{B}$ then $\mathcal{T}$ is a topology on X.
(2) Let X be a set with a metric $d : X \times X \to [0, \infty)$. Let
$$\mathcal{B} = \{B_r(p) | p \in X \text{ and } r \in (0, \infty)\}$$
where $B_r(p) = \{x \in X | d(x, p) < r\}$ is the ball of radius r about p. One may easily check that $\mathcal{B}$ is a basis for a topology on X. The topology $\mathcal{B}$ generates is called the **metric topology** on X induced by d.

There are lots of other interesting topological spaces. I suggest you look at a standard text on topology to see other examples. In particular, you should be familiar with the **subspace topology** induced on a subset of a topological space and the **product topology** on the cartesian product of two topological spaces.

2.1. **Limit points and sequences.** If A is a subset of a topological space $(X, \mathcal{T})$ then $p \in X$ is a **limit point** of A if for each open set U containing p we have
$$(U - \{p\}) \cap A \neq \emptyset.$$
The **closure** of a set A is the set consisting of the point of A and the limit points of A. It is denoted $\overline{A}$.

Prove the following facts:

(1) A set A is closed if it contains all its limit points (that is if $A = \overline{A}$).
(2) If $p \notin A$ then p is a limit point of A if and only if every open set containing p intersects A non-trivially.

Let $(X, \mathcal{T})$ be a topological space. A **sequence** in$X$ is a function from the natural numbers to X
$$p : \mathbb{N} \to X.$$
We denote $p(n)$ by p_n and usually write a sequence $\{p_n\}$ instead of using functional notation. We say a sequence $\{p_n\}$ in X **converges** to a point p, written $p_n \to p$, if for every open set U containing p there is some number N such that $p_n \in U$ for all $n \geq N$.

Prove the following facts:

(1) If X has a metric d then a sequence $\{p_n\}$ converges to p in the metric topology if and only if for all $\epsilon > 0$ there is an N such that $d(p_n, p) < \epsilon$ for all $n \geq N$.
(2) If A is a subset of a topological space, the sequence $\{p_n\} \subset A$ and $p_n \to p$ then $p \in \overline{A}$.

Sequences don't have to behave like we think they should (from our intuition coming form analysis), but if a topological space has extra properties then they do. A topological space $(X, \mathcal{T})$ is called **Hausdorff** if for each pair of distinct points $x, y \in X$ there is a pair of open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

We say a collection of open subset $\mathcal{N}$ of X containing a point $p \in X$ is a **neighborhood basis of a point** p if for all open sets U that contain p there is a set $V \in \mathcal{N}$ such that $V \subset U$. We call $(X, \mathcal{T})$ is **first countable** if every point has a neighborhood basis consisting of a countable number of sets.

Prove the following facts:

(1) If $(X, \mathcal{T})$ is a Hausdorff space and a sequence $\{p_n\}$ converges to p and q then $p = q$.
(2) If $(X, \mathcal{T})$ is a first countable space then a point p is a limit point of a set A if and only if there is a sequence $\{p_n\}$ in A such that $p_n \to p$.
(3) Metric topological spaces are Hausdorff and first countable.
(4) A topological space is called **second countable** if there is a basis for the topology on X that consist of a countable collection of sets. A second countable topological space if first countable. (For our class second countable will be a more important property, but it is nice to know that it implies first countable so the results here and below about first countable spaces also apply the second countable ones too.)

2.2. **Continuous Functions.** Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces. A function

$$f : X \to Y$$

is **continuous** if $f^{-1}(U)$ is open in X for all open sets U in Y.

Prove the following facts:

(1) A function is continuous if and only if the inverse image of all closed sets is closed.
(2) A function is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.
(3) If $(X, \mathcal{T})$ is a first countable space then a function $f : X \to Y$ is continuous if and only if $f(p_n) \to f(p)$ for all $p_n \to p$.
(4) If X and Y are metric spaces with metrics d and d', respectively, then a function $f : X \to Y$ is continuous in the metric topologies if and only if for all $\epsilon > 0$ and $x \in X$ there is a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon.$$

Recall (or look up) various continuous functions. For example prove the constant function is always continuous, projections to a factor of a product space is always continuous, and restrictions of continuous functions to subspaces are continuous. Also understand the how continuous functions to product spaces relate to continuous functions to the spaces that make up the product.

A function $f : X \to Y$ between topological spaces is called a **homeomorphism** if f is continuous, one to one, onto and has continuous inverse. We call X and Y **homeomorphic** if there is a homeomorphism between them. You should think of homeomorphic as being indistinguishable form a topological perspective (that is as far as convergent sequences and continuous functions go they seem to be the same space). One of the main question in topology is to try to characterize or classify topological spaces up to homeomorphism.

3. Connectedness and Compactness

A topological space X is **disconnected** if there exists disjoint non-empty open sets U and V in X such that $X = U \cup V$. If no such sets exist the we say X is **connected**.

A topological space X is called **path connected** if every two points in X are connected by a path, that is for all $x, y \in X$ there is a continuous function

$$f : [0, 1] \to X$$

such that $f(0) = x$ and $f(1) = y$.

Prove the following facts:

(1) A topological space X is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.
(2) The real line $\mathbb{R}$ with its metric topology is connected (this is a bit harder than the other facts I have asked you to check). Similarly any open or closed interval is connected.
(3) If $f : X \to Y$ is a continuous surjective function and X is connected then Y is connected. (Said another way, the continuous image of a connected space is connected.)
(4) A path connected space is connected.
(5) Euclidean space $\mathbb{R}^n$ with its metric topology is connected. (Here you can use any metric on $\mathbb{R}^n$ you like.)

A collection of subsets $\{U_\alpha\}_{\alpha \in J}$ of X is a **cover** of X if $X = \cup_{\alpha \in J} U_\alpha$. (Here J is an indexing set, so for example if $J = \{1, 2, \ldots, n\}$ then $\{U_\alpha\}_{\alpha \in J} = \{U_1, U_2, \ldots, U_n\}$.) A topological space X is **compact** if every cover of X by open sets has a finite subcover, in other words if $\{U_\alpha\}_{\alpha \in J}$ is a collection of open sets in X that cover X then there is a subset $I \subset J$ such that I is finite and $X = \cup_{\alpha \in I} U_\alpha$.

Prove the following facts:

(1) A closed subspace of a compact topological space is compact in the subspace topology.
(2) A compact subset of a Hausorff topological space is closed.
(3) In a first countable compact topological space every sequence has a convergent subsequence.
(4) A closed interval $[a, b]$ is compact. (This is also a bit more difficult than many of the previous things I have asked you to do.)
(5) The continuous image of a compact space is compact.
(6) A continuous function $f : C \to \mathbb{R}$ from a compact set to $\mathbb{R}$ always takes on its minima and maxima. That is there are points $a, b \in C$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in C$.