Section III: Local theory

Th^{m} 1 (Moser-Weinstein):

- M a smooth manifold
- $N \subset M$ a compact set with a nbhd U that strongly deformation retracts to N (e.g. N a submanifold)
- ω_0, ω_1 symplectic forms defined near N, $\omega_0 = \omega_1$ on $TM|_N$

Then $\exists$ a diffeomorphism $f: M \to M$ fixing N isotopic to the identity on M (rel N) s.t. $f^*\omega_1 = \omega_0$ on some nbhd of N in U.

Cor 2 (Darboux Th^{m}):

Given any (M^{2n}, ω) and $p \in M$, there is a neighborhood U of p symplectomorphic to a neighborhood of the origin in $(\mathbb{R}^{2n}, \omega_{std})$

Remark: This says any 2 symplectic manifolds a "locally the same"

Note this is just like for smooth manifolds (all locally Euclidean) but different from Riemannian metrics (curvature obstruction to being same)

so at this level symp. geom. seems closer to topology than Riem. geometry

Proof:

by Theorem I.2 $\exists$ a basis $(v_1 u_1 \ldots v_n u_n)$ for T_pM s.t. in this basis

$$\omega_p = \sum v_i^* \wedge u_i^*$$

let $\phi: \tilde{U} \to \tilde{V}$ be a coordinate chart about p, s.t. $\phi(p) = 0$

i.e. $p \in \tilde{U} \subset M$, $0 \in \tilde{V} \subset \mathbb{R}^{2n}$, ϕ diffeo.

by composing with a linear map $L: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ can assume

$$d\phi(v_i) = \frac{\partial}{\partial x_i}, \quad d\phi(u_j) = \frac{\partial}{\partial y_j}$$

so $\omega_1 = \phi^*\omega_{std}$ and $\omega_0 = \omega$ on M satisfy hypoth. of Th^{m} 1 for $N = \{p\}$

so $\exists$ a diffeo $f: M \to M$ s.t. $f^*\omega_1 = \omega_0$ on some open set U around p

$\therefore (\phi \circ f)^*\omega_{std} = \omega$ on U $\blacksquare$

Cor 3 (Symplectic Nbhd Th$^{\underline{m}}$):

- (M_i, ω_i) symplectic manifolds, $i = 0, 1$
- $\Sigma_i \subset M_i$ compact symplectic submanifolds

if $\exists$ a diffeo. $f : \Sigma_0 \to \Sigma_1$ and bundle isomorphism F

s.t.
$$\begin{tikzcd} TM_0|_{\Sigma_0} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circ"] & TM_1|_{\Sigma_1} \arrow[d] \\ \Sigma_0 \arrow[r, "f"] & \Sigma_1 \end{tikzcd}$$
and $F = df$ on $T\Sigma_0 \subset TM_0$

and $F^* \omega_1 = \omega_0$ on $TM_0|_{\Sigma_0}$

then $\exists$ nbhds N_i of Σ_i and a diffeomorphism

$\phi : N_0 \to N_1$ extending f

s.t. $\phi^* \omega_1 = \omega$ on N_0

Proof: <u>Claim</u>: $\exists$ neighborhoods $\tilde{N}_i$ of Σ_i in M_i and a diffeo. $\tilde{\phi} : \tilde{N}_0 \to \tilde{N}_1$ s.t.

$$\tilde{\phi}|_{\Sigma_0} = f \quad \text{and} \quad d\tilde{\phi} = F \text{ on } \Sigma_0$$

given claim note $\tilde{\phi}^* \omega_1$ and ω_0 satisfy hypoth. of Th$^{\underline{m}}$ 1 with $N = \Sigma_0$

so done as in proof of Cor 2.

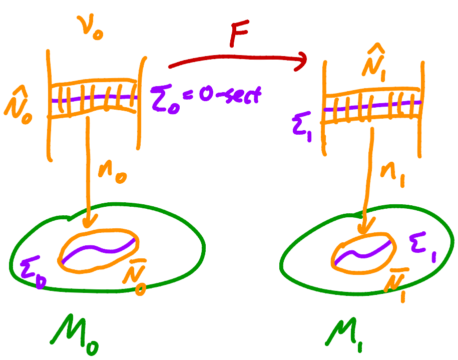
one way to establish claim is as follows:

choose a Riemannian metric g_i on M_i

let $\nu_i = T\Sigma_i^{\perp g_i} \subset TM_i$ be the normal bundle of Σ_i

useing the "normal exponential map"

$$\exp_{\Sigma_i} : \nu_i \to M_i : v = \exp_{\pi_i(v)}(v) \qquad (\pi_i : \nu_i \to \Sigma_i \text{ proj})$$



gives a diffeo. $n_i : \hat{N}_i \to \bar{N}_i$

where $\hat{N}_i$ nbhd 0-sect. in ν_i

and $\bar{N}_i$ nbhd of Σ_i in M_i

s.t. $n_i =$ "id" on 0-sect. to Σ_i

↑ if γ_v unique geod with $\gamma_v'(0) = \frac{v}{\|v\|}$ and $\gamma_v(0) = \pi_i(v)$

then $\exp_{\pi_i(v)}(v) = \gamma_v(\|v\|)$

now let $\tilde{N}_0 = n_0(\hat{N}_0 \cap F^{-1}(\hat{N}_1))$ and $\tilde{N}_1 = n_1(F(\hat{N}_0) \cap \hat{N}_1)$

and then $n_1 \circ F \circ n_0^{-1} : \tilde{N}_0 \to \tilde{N}_1$ is desired diffeo ▦

Cor 4:

let $f_i : (\Sigma, \omega) \to (M_i, \omega_i)$ be symplectic, codimension 2 embeddings (Σ compact)

If $\exists$ an orientation preserving bundle isomorphism $F : \nu_0 \to \nu_1$ between their normal bundles (F covering $f_1 \circ f_0^{-1}$)

then $f_0(\Sigma_0)$ and $f_1(\Sigma_1)$ have symplectomorphic neighborhoods

Proof:

note $\nu_1 = T(f_1\Sigma_1)^{\perp\omega_1}$ and ν_1 has symplectic 2-dim'l fibers

since F is orientation preserving on fibers of ν_0

it takes $\omega_0|_{\text{fiber } \nu_0}$ to a positive multiple of $\omega_1|_{\text{fiber } \nu_1}$

fiberwise rescaling F results in an isomorphism preserving ω_1 on fibers of ν_1

we can now apply Cor 3 ▦

Cor 5 (Lagrangian nbhd theorem, Weinstein):

- (M,ω) a symplectic manifold
- $L \subset M$ a compact Lagrangian submanifold

Then L has a nbhd in M symplectomorphic to a nbhd of 0-section in $(T^*L, d\lambda)$

Liouville form

Proof: one can show $\exists$ a bundle map $T(T^*L)|_Z \xrightarrow{F} TM|_L$

0-section

such that $F = d\pi$ on TZ

where $\pi : T^*L \to L$ is projection

such that $F^*\omega = d\lambda$ on $T(T^*L)|_Z$

We will prove this later using compatible almost complex structures but given this proof is finished as proof of Cor 5 ▦

Proof of Thm 1:

let $\eta = \omega_1 - \omega_2$ and $\omega_t = \omega_0 + t\eta$

on $TM|_N$ we have $\eta \equiv 0$ and $\omega_t = \omega_0$ (on $TM|_N$)

$\therefore \exists$ a neighborhood of N on which ω_t is symplectic $\forall t$

(non-degeneracy open condition and $[0,1]\times N$ compact)

we want to find an isotopy $\phi_t : M \to M$, $t \in [0,1]$ s.t. $\phi_0 = id$ and

$* \quad \boxed{\phi_t^*\omega_t = \omega_0}$ (on some nbhd of N)

Moser's Trick: find vector field v_t that generates ϕ_t

$$\frac{d}{dt}\phi_t^*\omega_t = \frac{d}{ds}\phi_s^*\omega_t\Big|_{s=t} + \frac{d}{ds}\phi_t^*\omega_s\Big|_{s=t}$$

product rule ↓ by def^n of Lie derivative

$$= \phi_t^*\mathcal{L}_{v_t}\omega_t + \phi_t^*\frac{d\omega_t}{dt}$$

$$= \phi_t^*(\iota_{v_t}\cancel{d\omega_t}^{\,0} + d\iota_{v_t}\omega_t + \eta)$$

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(d\iota_{v_t}\omega_t + \eta)$$

so if we want $\frac{d}{dt}\phi_t^*\omega_t = 0$, then need

$$\boxed{d\iota_{v_t}\omega_t = -\eta} \quad **$$

<u>note:</u> if ϕ_t is the flow of v_t then

ϕ_t satisfies $*$ $\Leftrightarrow$ v_t satisfies $**$

so how to find such v_t? If the d was not in $**$ then easy by non-degen. of ω

so let's remove the d (standard DeRham theory)

let $f_t : U \to U$ be strong deformation retraction of U to N

i.e.
$f_1 = id_U$
$f_0(U) = N$
$f_t(x) = x \quad \forall x \in N$ and $t \in [0,1]$

now for any k-form α set

$$I_f(\alpha) = \int_0^1 \underbrace{\iota_{\frac{\partial}{\partial t}}(f^*\alpha)}\,dt \quad \text{where } f : U \times [0,1] \to U \text{ isotopy}$$

this is $(k-1)$-form on $U\times[0,1]$ with <u>no</u> t component so can think of as $(k-1)$-form on U

if $X_t(p) = df_{(p,t)}(\frac{\partial}{\partial t})$, then

$$I_f d\alpha + dI_f(\alpha) = \int_0^1 \left[\iota_{\frac{\partial}{\partial t}} f^* d\alpha + d\iota_{\frac{\partial}{\partial t}} f^*\alpha\right] dt$$

$$= \int_0^1 f^*(\iota_{X_t} d\alpha + d\iota_{X_t}\alpha)\, dt$$

$$= \int_0^1 f^*(\mathcal{L}_{X_t}\alpha)\, dt$$

definition of $\mathcal{L}_{X_t}\alpha$

$$= \int_0^1 \frac{d}{ds} f_s^*\alpha\Big|_{s=t} dt = f_1^*\alpha - f_0^*\alpha$$

Recall: This is what is used to see f^* on DeRham cohom is indep. of homotopy class

for us note $I_f(\eta)$ a 1-form and

$$d\,I_f(\eta) + I_f(\cancel{d\eta}) = f_1^*\eta - f_0^*\eta = \eta|_U$$

$d\eta = 0$ id_U $\eta = 0$ on N

now let v_t be unique vector field satisfying

$$\iota_{v_t}\omega_t = -I_f(\eta)$$

so v_t satisfies ** near N and $v_t = 0$ on N

extend v_t to a vector field on M with compact support

note flow ϕ_t of v_t is well-defined since support of v_t compact

is fixed on N since $v_t = 0$ there

and $\phi_t^*\omega_t$ on some nbhd of N ▦

Th^{m} 6 (Moser's Th^{m}):

- M a closed manifold
- $\{\omega_t \mid 0 \le t \le 1\}$ a smooth family of symplectic forms

If $[\omega_t] \in H^2_{DR}(M)$ is independent of t

then $\exists$ an isotopy $\phi_t : M \to M$ s.t. $\phi_0 = \mathrm{id}_M$ and $f_t^*\omega_t = \omega_0$

In particular, ϕ_1 is a symplectomorphism (M,ω_0) to (M,ω_1)

we call 2 symplectic forms ω_0, ω_1 on M

<u>deformation equivalent</u> if $\exists$ a smooth family of symplectic forms $\{\omega_t\}_{t=0}^{1}$ on M connecting ω_0 and ω_1

happens on bundle level

<u>isotopic</u> if they are deformation equivalent by a family $\{\omega_t\}$ with $[\omega_t]$ independent of t

happens on mfd level

<u>strongly isotopic</u> if $\exists$ an isotopy $\phi_t : M \to M$ such that $\phi_1^*\omega_1 = \omega_0$

clearly   strongly isotopic $\Rightarrow$ isotopic $\Rightarrow$ def. equivalent

$\Leftarrow$ Moser's Th^{m} for <u>compact</u> mfds

not true for non-compact manifolds

Moser's th^{m} says you "can integrate" something on bundle level to something on manifold level

Proof:

note $[\frac{d\omega_t}{dt}] = \frac{d}{dt}[\omega_t] = 0$

so we know for each t, $\exists\ \eta_t$ such that $d\eta_t = \frac{d\omega_t}{dt}$

Claim: can choose η_t to depend smoothly on t

given this $\exists$ a unique vector field v_t s.t. $\iota_{v_t}\omega_t = -\eta_t$

so $d\iota_{v_t}\omega_t = -d\eta_t = \frac{d\omega_t}{dt}$

and discussion in proof of Th^m 1 $\Rightarrow$ flow $\phi_t : M \to M$ of v_t

satisfies $\phi_t^* \omega_t = \omega_0$

the claim can be established by

1) prove in coordinate chart

then use Meyer-Vietoris argument to prove on M

exercise: work this out.

2) Nicer "global" solution: Hodge Theory

given a Riemannian metric on M^{2n} you can define the Hodge $* : \Omega^k(M) \to \Omega^{2n-k}(M)$

and the "adjoint" $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$

of $d : \Omega^{k-1}(M) \to \Omega^k(M)$ exterior derivative

by $\delta = -*d*$ (on odd dim mfds sign is different)

Part of the Hodge theorem says d is an isomorphism from $\delta(\Omega^k)$ to $d(\Omega^{k-1})$

since $\frac{d\omega_t}{dt} \in d(\Omega^1(M))$, $\exists$ smooth family $\eta_t \in \delta(\Omega^2(M))$

s.t. $d\eta_t = \frac{d\omega_t}{dt}$ ▦