F. Lefschetz pencils and fibrations on 4-manifolds

let's start with a Lefschetz fibration

$$\begin{array}{c} \Sigma^2 \to M^4 \\ \quad\ \downarrow \pi \\ \quad\ S \end{array}$$

we assume all critical points have distinct images under π

let c be a critical value and U a closed nbhd of c in S containing no other critical points

we can assume U = 2-disk

note: $\pi^{-1}(\partial U)$ is Σ^2-bundle over S^1

exercise:

1) given any manifold X and a diffeomorphism $\phi: X \to X$, then the mapping torus

$$T_\phi = M \times [0,1] \Big/ (x,1) \sim (\phi(x),0)$$

is an X-bundle over S^1

2) if ϕ and ψ are isotopic, show $T_\phi \cong T_\psi$ as bundles

3) Given any X-bundle over S^1 there is some diffeomorphism $\phi: X \to X$ such that the bundle is T_ϕ, we call ϕ the monodromy of the bundle

let's try to figure out what the monodromy of $\pi^{-1}(\partial U)$ is

let V be a chart about the critical point (and assume U is a chart about c) so that π has the form

$$(z_1, z_2) \mapsto z_1 z_2$$

consider $\pi^{-1}(U) - V$

this will be a $(\Sigma - A)$ bundle over $U = B^2$

where A is an annulus in Σ

(recall $\pi^{-1}(x) \cap V$ is an annulus for $x \neq c$

$\pi^{-1}(c) \cap V$ is 2 disks)

since B^2 contractible, the bundle is trivial

$$\pi^{-1}(U) - V \cong U \times (\Sigma - A)$$

$\therefore$ $\pi^{-1}(\partial U) - V \cong S^1 \times (\Sigma - A)$

i.e. monodromy is trivial away from A (or at least isotopic to identity)

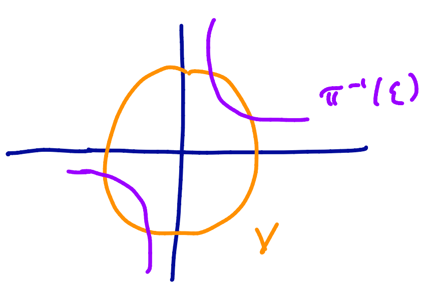
now for $\pi^{-1}(\partial U) \cap V$:

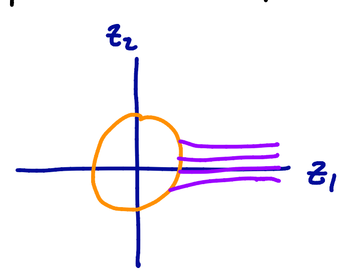
we can assume U is ε-disk

so ∂U parameterized by $t \mapsto \varepsilon e^{i 2\pi t}$ $\quad t \in [0,1]$

recall we need to trivialize $\pi^{-1}(\partial U)$ outside V

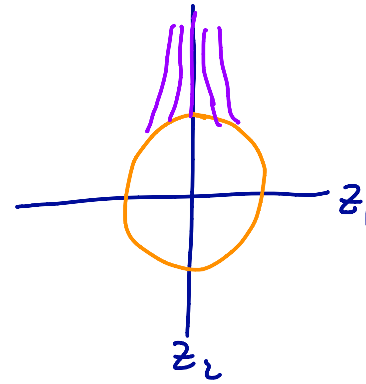
note for $|z_1|$ <u>large</u> $\pi^{-1}(U)$ trivialized by $z_1 \mapsto (z_1, w z_1^{-1})$ $\quad w \in U$





note for $|z_2|$ large $\pi^{-1}(U)$ trivialized

by $z_2 \mapsto (w z_2^{-1}, z_2)$ for $w \in U$



points in $\pi^{-1}(\partial U)$ parameterized by

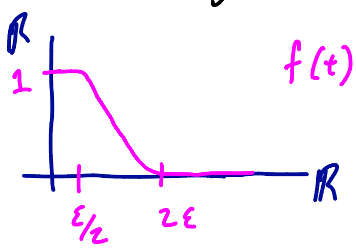
$$z_1 \mapsto (z_1, \varepsilon e^{i2\pi t} z_1^{-1})$$

or

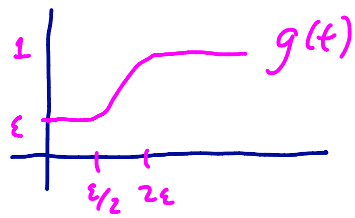
$$z_2 \mapsto (\varepsilon e^{i2\pi t} z_2^{-1}, z_2)$$

but neither agrees with both trivializations!

consider



and



and parameterizations of $\pi^{-1}(\varepsilon e^{i2\pi t})$ given by

$$h_t : z_1 \mapsto \left(g(|z_1|)\, z_1 e^{i2\pi f(|z_1|) t},\ \varepsilon/g(|z_1|)\ e^{i2\pi t} z_1^{-1} e^{-i2\pi f(|z_1|) t} \right)$$

$$\left(g(|z_1|)\, z_1 e^{i2\pi f(|z_1|) t},\ \varepsilon/g(|z_1|)\ z_1^{-1} e^{i2\pi (1-f(|z_1|)) t} \right)$$

for $|z_1| > 2\varepsilon$, get $(z_1, \varepsilon z_1^{-1} e^{i2\pi t})$

for $|z_1| < \varepsilon/2$ (so $|z_2| \gg 0$), get $(\varepsilon e^{i2\pi t} z_1, z_1^{-1})$

i.e. param. by $z_2 \mapsto (\varepsilon e^{i2\pi t} z_2^{-1}, z_2)$

so this param. agrees with both trivializations!

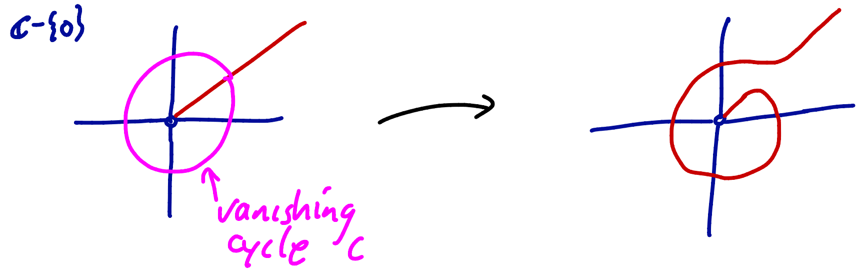
by definition the monodromy is $h_1(z_1) = (g(|z_1|)\, z_1 e^{i2\pi f(|z_1|)}, -)$

compared to $h_0(z_1) = (g(|z_1|) z_1, -)$

identified with $\mathbb{C} - \{0\}$ by projection to 1st factor

so monodromy is multiplication by $z_1 e^{i2\pi f(|z_1|)}$

this is the map

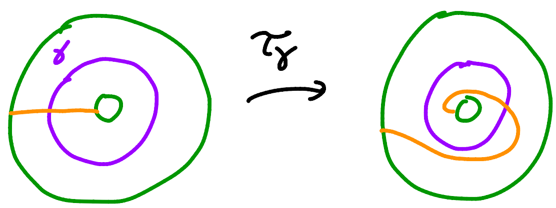


this is called a right handed Dehn twist about c

Remark: In general, if γ a simple closed curve in a surface Σ, the Dehn twist about γ is the diffeomorphism

$$\tau_\gamma : \Sigma \to \Sigma$$

that is the identity outside a nbhd of γ and in that nbhd is

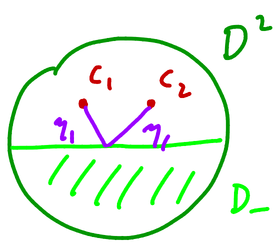


exercise: 1) τ_γ is well-defined upto isotopy (and only depends on γ upto isotopy)

2) given any $\gamma \subset \Sigma$ there is a Lefschetz fibration over the disk with a single critical point, whose vanishing cycle is γ and has monodromy τ_γ (note this is the converse of above discussion)

Hint: Start with nbhd of critical point and glue together with trivial surface bundles

now given $\pi : M \to D^2$ a Lefschetz fibration with more critical points



let D_- be "half" of D^2 w/o crit vals

let η_1 be a path ∂D_- to c_i (intersect D_- in end pt)

let N_1 be small nbhd η_i

$\pi^{-1}(N_1)$ is determined by vanishing cycle γ_1.

the monodromy of $\pi^{-1}(\partial N_1)$ is τ_{γ_i}

exercise:

1) The monodromy of $\pi^{-1}(\partial(D_1 \cup (\cup N_i)))$ is $\tau_{\gamma_k} \circ \dots \circ \tau_{\gamma_1}$

2) Given any curves $\gamma_1 \dots \gamma_k$ in Σ there is a Lefschetz fibration over D^2 with k critical points, vanishing cycles $\gamma_1 \dots \gamma_k$, and monodromy $\tau_{\gamma_k} \circ \dots \circ \tau_{\gamma_1}$

3) let S be a surface of genus g with one boundary component and $\pi: M \to S$ be a Σ-fiber bundle

show $\pi^{-1}(\partial S)$ has monodromy $\prod_{i=1}^{g} [f_i, g_i]$ where $f_i, g_i: \Sigma \to \Sigma$ are any diffeomorphisms and $[f_i, g_i] = f_i \circ g_i \circ f_i^{-1} \circ g_i^{-1}$

4) Show the converse of 3), ie. given any diffeomorphism of the form $\prod_{i=1}^{g} [f_i, g_i]$, there is a Σ-bundle over S with this monodromy on ∂S

Thm 7:

$$\boxed{\begin{minipage}{0.9\textwidth}\end{minipage}}$$

I) if $\pi: M \to S$ is a Lefschetz fibration with fiber Σ over a closed surface of genus g, with k singular points, then there are diffeomorphisms $f_i, g_i: \Sigma \to \Sigma$, $i = 1, \dots g$ such that

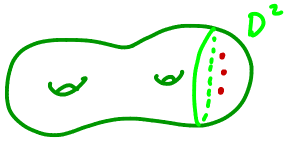
$$\prod \tau_{\gamma_i} \circ \prod [f_i, g_i] \sim \mathbb{1}_\Sigma$$

isotopic

where γ_i are the vanishing cycles

II) Conversely, given such a factorization there is a Lefschetz fibration realizing this data.

Proof: I)



there is a disk $D^2 \subset S$ such that all critical points in D^2

from above $\pi^{-1}(\partial D^2)$ has monodromy $\prod \tau_{\gamma_i}$ and $\pi^{-1}(\partial(\overline{S - D^2}))$ has monodromy $\prod [f_1, g_1]$

these must be glued together by an orientation reversing, fiber preserving diffeomorphism.

exercise: if T_ϕ orientation reversing, fiber preserving diffeomorphic to T_ψ then $\phi \sim h \psi^{-1} h^{-1}$ for some diffeomorphism h

so $\prod \tau_{\gamma_i} \circ \prod [f_1, g_1] \sim \mathbb{1}_\Sigma$ (here f_1, g_1 are possibly conjugates of original f_1, g_1.)

for II) just use above discussions to build fibrations over D^2 and $\overline{S - D^2}$ and glue together ▩

Now, what if $M \xrightarrow{\pi} S$ has a section

$$\begin{array}{c} M \\ \pi \downarrow \ \uparrow \sigma \\ S \end{array}$$

assume $\sigma(S)$ misses critical pts

note: $\sigma(S)$ has a neighborhood N_S that is a D^2 bundle over S and D^2_S's are nbhds of $\sigma(x)$ in Σ_x

let $D^2 \subset S$ contain no critical point, and $S_1 = \overline{S - D^2}$

$N_S \cap \pi^{-1}(S_1) = S_1 \times D^2_f$ (in fiber) so monodromy of $\pi^{-1}(\partial S_1)$ restricted to $D^2_f \subset \Sigma$ is identity!

so the monodromy of $\pi^{-1}(\partial S_1)$ is ϕ, then there is a diffeomorphism

$$\phi_1 : \Sigma_1 \to \Sigma_1$$

where $\Sigma_1 = \overline{\Sigma - D_f^2}$ and $\phi_1|_{\partial\Sigma_1} = id_{\partial\Sigma_1}$

such that ϕ_1 extends over D_f^2 by identity to get ϕ

as above $\phi_1 = \prod \tau_{\gamma_i} \circ \prod [f_i, g_i]$ but

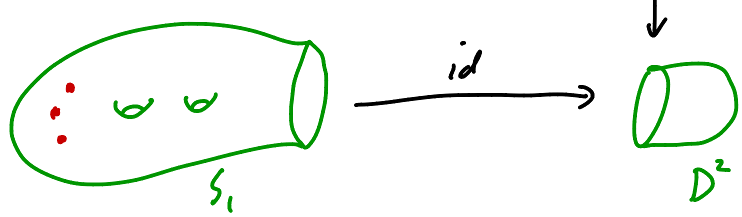
$$\phi_1 \sim \tau_\delta^k$$

some k, where δ is parallel to $\partial\Sigma_1$

Remark: a priori ϕ_1 might be a more general "point push" map but in this case can't extend $S_1 \times D_f^2$ to nbhd of $\sigma(S)$

now we have

$$\begin{array}{ccc}
\partial(\pi^{-1}(S_1)) & & \partial(\pi^{-1}(D^2)) \\
\cong & & \cong \\
S^1 \times \Sigma & & S^1 \times \Sigma \\
\shortparallel & & \\
S^1 \times (\Sigma_1 \cup D_f^2) & \xrightarrow[\text{diffeo}]{\psi} & S^1 \times (\Sigma_1 \cup D_f^2) \\
\downarrow & & \downarrow
\end{array}$$



note $S_1 \times D_f^2$ and $D^2 \times D_f^2$ glue to give N_S the nbhd of $\sigma(S)$

exercise:

1) Show N_S is a D^2-bundle over S with Euler number $-k$

Hint: the isotopy τ_δ^k to id_Σ affects isomorphism $\pi^{-1}(\partial S_1)$ to $S^1 \times \Sigma$

2) Show if we have a factorization of τ_δ^k as above there is a Lefschetz fibration with section of self-intersection $-k$ realizing this data

So we have shown

Thm 8:

I) let $\pi: M \to S$ be a Lefschetz fibration with fiber Σ over a closed surface of genus g, with k singular points, with l disjoint sections having self-intersection $k_1, \dots, k_l$

set $\Sigma_l = \Sigma - (l\text{-disks})$

There are diffeomorphisms $f_i, g_i : \Sigma_l \to \Sigma_l$, $i = 1, \dots g$, s.t.

$$\prod \tau_{\gamma_i} \circ \prod [f_i, g_i] \sim \tau_{\delta_1}^{-k_1} \circ \dots \circ \tau_{\delta_l}^{-k_l}$$

where γ_i are the vanishing cycles and δ_i parallel to i^{th} boundary component.

II) Conversely, given such a factorization there is a Lefschetz fibration with l sections realizing this data.

Open Problem:

Does any Lefschetz fibration over S^2 have a section?
When does any Lefschetz fibration have a section? (multisection?)

note: this is equivalent to: Can a factorization

$$\prod \tau_{\gamma_i} \circ \prod [f_i, g_i] \sim 1_\Sigma$$

be lifted to a similar factorization of τ_δ^k?

(note any diffeo of Σ_1 gives one of Σ, but can you "lift" a diffeo of Σ to Σ_1?)

Finally, note a Lefschetz pencil with base locus having l points then we can blow-up l times to get a Lefschetz fibration with l disjoint sections, all of self-intersection -1

thus we have shown

Th^m 9:

I) let (π, B) be a Lefschetz pencil with B having l points and π having k critical points

let Σ be the fiber and $\Sigma_l = \Sigma - (l \text{ disks})$

Then
$$\prod \tau_{\gamma_i} \sim \prod \tau_{\delta_j}$$
where γ_i are the vanishing cycles and δ_i parallel to i^{th} boundary component.

II) Conversely, given such a factorization there is a Lefschetz pencil l base points realizing this data

exercise: 1) Compute the homology of a Lefschetz pencil from the above data

2) Compute the intersection form

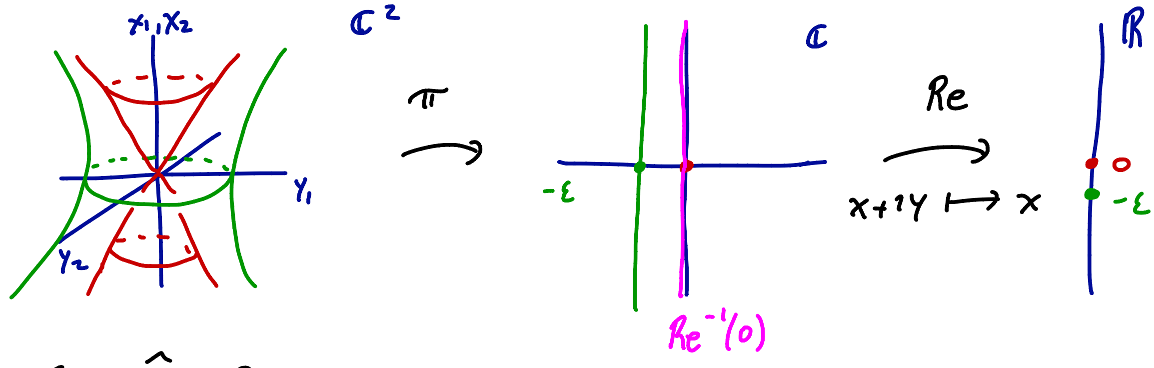
Research Problem:

Can you find an exotic smooth (symplectic) structure on $\mathbb{CP}^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ or $S^2 \times S^2 \ldots$ by finding an appropriate factorization as above?

Or prove they don't exist?

Later we will talk about handlebodies, but here we just observe how to build a neighborhood of a singular fiber by attaching a 2-handle

let U be a disk in S containing a single critical value and V a nbhd of the corresponding critical point such that $\pi(z_1,z_2) = z_1^2 + z_2^2$



Set $\hat{\pi} = Re \circ \pi$

note $\hat{\pi}$ is a Morse function with one critical point at 0 of index 2:

$$\hat{\pi}(x_1+iy_1, x_2+iy_2) = x_1^2 + x_2^2 - y_1^2 - y_2^2$$

so $\hat{\pi}^{-1}((-\infty,\varepsilon))$ is obtained from $\hat{\pi}^{-1}((-\infty,-\varepsilon))$ by "attaching a 2-handle"

the core of the handle is $\{x_1 = x_2 = 0,\ y_1^2 + y_2^2 \leq \varepsilon^2\}$

so the attaching S^1 is $\{x_1 = x_2 = 0,\ y_1^2 + y_2^2 = \varepsilon^2\}$ is the vanishing cycle γ in $\pi^{-1}(-\varepsilon)$

how is the handle attached to $\partial\hat{\pi}^{-1}((-\infty,-\varepsilon])$?

i.e. what is the framing on γ

<u>note:</u> $(i\sqrt{\varepsilon}\cos\theta,\ i\sqrt{\varepsilon}\sin\theta)$ parameterizes γ

so $(-i\sin\theta, i\cos\theta)$ is tangent to γ and $\pi^{-1}(-\varepsilon)$

the fiber $\pi^{-1}(-\varepsilon)$ is holomorphic so

$i(-i\sin\theta, i\cos\theta) = (\sin\theta, -\cos\theta)$ also tangent to $\pi^{-1}(-\varepsilon)$

the tangent space to $\hat{\pi}^{-1}(-\varepsilon)$ along γ is spanned by

$$\{\gamma'(\theta), \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}\}$$

and the normal bundle to the core of the 2-handle is spanned by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$

the framing of the normal bundle to γ in $\hat{\pi}^{-1}(-\varepsilon)$ given by the fiber $\pi^{-1}(-\varepsilon)$

was just computed to be $\sin\theta \frac{\partial}{\partial x_1} - \cos\theta \frac{\partial}{\partial x_2}$
so the fiber framing is <u>one</u> greater than the framing coming from the handle

we have shown

lemma 10:

> If U contains a single critical value of $\pi: M \to S$ then $\pi^{-1}(U)$ is obtained from $U \times \Sigma$ by attaching a 2-handle to the vanishing cycle γ of the critical point with framing one less than the framing induced on γ from the fiber

exercise:

1) Check that we were careful with orientations and got the framing right (i.e. -1 not $+1$).
2) Show this is consistant with the monodromy being a right handed Dehn twist about γ.

Remark: If you know about Kirby calculus then you should now be able to build a handle picture of a Lefschetz fibration given a monodromy presentation.