VI Contact structures and symplectic cobordisms

A Contact Structures

a contact structure on a manifold M^{2n+1} is a hyperplane field $\xi^{2n} \subset TM$ such that for each point $p \in M$ there is a 1-form α defined near p such that

$$\xi = \ker \alpha$$

$$\alpha \wedge (d\alpha)^n \neq 0 \quad \text{(means never zero)}$$

note: if α, β both satisfy $\ker \alpha = \xi = \ker \beta$ then $\exists$ a nonzero function f such that $\alpha = f\beta$

so $\quad \alpha \wedge (d\alpha)^n = f^{n+1} \beta \wedge (d\beta)$

thus if n is odd ξ defines an orientation on M! (even if ξ is not orientable!)

if α can be globally defined we call ξ co-orientable

call α a contact form

(TM/ξ is 1-dimensional and α orients this)

when M is orientable, this is equivalent to ξ being orientable

when M is oriented and n odd we call a contact str ξ positive if $\alpha \wedge d\alpha^n$ defines orientation and otherwise negative (it is common for contact str. to mean positive contact structure)

exercise: Show $d\alpha$ defines a symplectic structure on ξ

(note if $\alpha = f\beta$ then $d\alpha|_\xi = f d\beta|_\xi$ so a contact structure has a canonical conformal symplectic structure)

"contact str odd dim'l analog of symplectic str"

examples:

1) given any manifold M

let λ be the Liouville form on T^*M

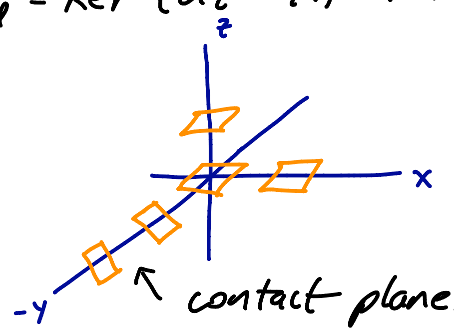
$\alpha = dz - \lambda$ is a 1-form on $T^*M \times \mathbb{R}$ such that

$\xi = \ker(dz - \lambda)$ is a contact structure

Remark: $T^*M \times \mathbb{R}$ is called the 1-jets space of M and denoted $J^1(M)$

2) $\mathbb{R}^{2n+1} = T^*\mathbb{R}^n \times \mathbb{R}$ has the contact structure

$$\xi_{std} = \ker(dz - \lambda) = \ker(dz - \sum y_i dx_i)$$



contact planes tangent to $y = \text{constant}$ and "rotate" in a left handed way

3) If (M^{2n}, ω) a symplectic manifold with boundary and v is a vector field defined near ∂M pointing out of M such that

$$\mathcal{L}_v \omega = \omega \quad \text{(symplectic dilation)}$$

then $\iota_v \omega$ is a contact form for ∂M

indeed $\quad \omega = \mathcal{L}_v \omega = \cancelto{0}{\iota_v d\omega} + d\iota_v \omega$

and
$$\iota_v \omega^n = n(\iota_v \omega) \wedge \omega^{n-1} = n(\iota_v \omega) \wedge (d\iota_v \omega)^{n-1}$$

Remark: So contact structures are a type of str. on the boundaries of some symplectic manifolds if M is compact we call it a symplectic filling of $(\partial M, \ker(\iota_v \omega))$

e.g. let $S^{2n-1} \subset \mathbb{C}^n$ be unit sphere

note $v = \frac{1}{2}\sum x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$ is a symplectic dilation of $\omega = \sum dx_i \wedge dy_i$

so $\alpha = \iota_v \omega|_{S^{2n-1}} = \left(\frac{1}{2}\sum x_i dy_i - y_i dx_i\right)\Big|_{S^{2n-1}}$ is a contact form on S^{2n-1}

and (B^{2n}, ω_{std}) is a symplectic filling of $(S^{2n-1}, \ker \alpha)$

What does the "contact condition" mean

Frobenius Th$^{\underline{m}}$:

$$\boxed{\begin{array}{l}\text{let } M^n \text{ be a manifold} \\ D \subset T(M) \text{ be a } k\text{-plane field} \\ D \text{ is integrable} \Leftrightarrow \begin{array}{l} D \text{ is closed under} \\ \text{Lie bracket}\end{array}\end{array}}$$

Recall D integrable means $\forall p \in M$ there is a k-dimensional manifold Σ^k s.t. $p \in \Sigma^k$ and $T_x\Sigma^k = D_x$ $\forall x \in \Sigma^k$

if D is integrable then M has a k-dim'l foliation (that is M is the union of images of injective immersions of k-manifolds s.t. each point has a nbhd $\cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ where k-manifolds go to $\mathbb{R}^k \times \{p\}$)

being closed under Lie bracket means if $v, u \in \Gamma(D) \subset \mathcal{X}(M)$ then $[v,u] \in \Gamma(D)$

if $D^{2n} \subset TM^{2n+1}$ and $D = \ker \alpha$ for some 1-form α

then for $v, u \in \Gamma(D)$ we have

$$d\alpha(v,u) = v \cdot \cancel{\alpha(u)}^{0} - u \cdot \cancel{\alpha(v)}^{0} - \alpha([u,v])$$
$$= -\alpha([v,u])$$

so $[v,u] \in \Gamma(D) \Leftrightarrow d\alpha(v,u) = 0$

thus if D closed under Lie bracket then $d\alpha|_D = 0$

so $\xi = \ker \alpha$ a contact structure is as far from being integrable as possible

some people say a contact structure is a "maximally non-integrable hyperplane field"

exercise:

if $\Sigma^k \subset (M^{2n+1}, \xi)$ is a submanifold and

$T_x\Sigma^k \subset \xi_x \quad \forall\, x \in \Sigma$

then show $T_x\Sigma$ isotropic in $(\xi_x, d\alpha)$

where $\xi = \ker \alpha$

so $k \leq n$ (recall, this means $T_x\Sigma \subset T_x\Sigma^{\perp d\alpha}$)

we call such a Σ an isotropic submanifold

and if $k=n$, then Σ called a Legendrian submanifold

examples:

1) recall for any manifold M, the 1-jet space $J^1(M) = T^*M \times \mathbb{R}$ has a contact structure

$\xi = \ker(dz - \lambda)$ where λ is the Liouville form

given $f: M \to \mathbb{R}$ the 1-jet of f is

$j^1(f): M \to J^1(M): x \mapsto (df_x, f(x))$

note: $j^1(f)^*(dz-\lambda) = df - df^*\lambda = df - df = 0$

so $\Gamma_f = \text{image}(j^1(f))$ is Legendrian!

exercise: a section $\sigma: M \to J^1(M)$ is the 1-jet of some function $\Leftrightarrow$ $\sigma(M)$ Legendrian

Hint: let $\pi_z: J^1(M) \to \mathbb{R}$ be projection to $\mathbb{R}$

consider $f = \pi_z \circ \sigma$

Remark: a 1^{st} order P.D.E. on M is simply a function

$$F: J^1(M) \to \mathbb{R}$$

an a solution is a function $f: M \to \mathbb{R}$ s.t.

$$F \circ j^1(f) = 0$$

or more geometrically a solution is a section $\sigma: M \to J^1(M)$ s.t.

1) $F \circ \sigma = 0$
2) $\text{im}(\sigma)$ Legendrian

Lie used this to solve PDEs

2) let M^n be a manifold

let $P^*M = (T^*M - z)/\mathbb{R}_+$ be the oriented projectivised cotangent bundle

here z is the zero section in T^*M and the positive reals $\mathbb{R}_+$ just acts by mult.

note we still have $\pi: P^*M \to M$

let $\xi = \{ v \in T_{[\alpha]}(P^*M) : \alpha(\pi_* v) = 0 \}$

here $\alpha \in T^*M$ and $[\alpha]$ is equivalence class

exercise:

1) show ξ well-defined hyperplane field

2) fibers of π are S^{n-1} and tangent to ξ

we claim (P^*M, ξ) is a contact manifold

one way to see this is note if g is a Riemannian metric on M (so induces metric on T^*M) and $U^*M = \{\alpha \in T^*M : \|\alpha\|_g = 1\}$ is the unit cotangent bundle then the projection

$$T^*M - z \xrightarrow{\rho} U^*M$$

decends to a diffeomorpism

$$P^*M \xrightarrow{\rho} U^*M$$

now if $v \in T^*M$ is the "radial vector field" in each fiber (i.e. in local coords q_i on M and q_i, p_i on T^*M

$$v = \sum p_i \frac{\partial}{\partial p_i} \quad)$$

then $\mathcal{L}_v \, d\lambda = d\lambda$ Liouville form

v also transverse to U^*M

so from example above

$$\alpha = \iota_v d\lambda\big|_{U^*M} = \lambda\big|_{U^*M}$$

is a contact form on U^*M

exercise: Show $\rho : P^*M \to U^*M$ maps ξ to $\ker\alpha$ $(=: \xi')$

so (U^*M, ξ') depends on g but is contactomorphic to (P^*M, ξ) which doesn't! (diffeomorphic preserving contact structures)

exercise: $U^*\mathbb{R}^n$ is contactomorphic to $J^1(S^{n-1})$

clearly if $f: M \to N$ is a diffeomorphism then the induced map "f^*": $P^*N \to P^*M$ a contactomorphism

the contact analog of an earlier question is

Major Open Question:

$$\boxed{\begin{gathered}\text{are } M, N \text{ diffeomorphic} \\ \Leftrightarrow \\ P^*M \text{ and } P^*N \text{ are contactomorphic}\end{gathered}}$$

Remark: maybe nicer than symplectic analog since P^*M compact (when M is) but T^*M not

Aside: recall the metric g induces a bundle isomorphism

$$TM \to T^*M$$

an ∴ an isomorphism

$$UT \xrightarrow{\phi_g} U^*T$$

where UT unit tangent bundle

exercise: 1) if $(M, \xi = \ker \alpha)$ a contact manifold, $\exists!$ vector field X_α such that

$$\iota_{X_\alpha} d\alpha = 0 \quad \text{and} \quad \alpha(X_\alpha) = 1$$

X_α called a Reeb field of ξ

note: $X_\alpha \pitchfork \xi$ and flow of X_α preserves ξ

2) Show a vector field v on (M, ξ) is a Reeb field (for some α)

$$\Leftrightarrow$$

$$v \pitchfork \xi \text{ and flow of } v \text{ preserves } \xi$$

So $\lambda|_{U^*M}$ gives a natural flow (Reeb flow) on U^*M

Recall, on UM there is also a natural flow, the geodesic flow

$$\Phi_t : UM \to UM$$
$$v \mapsto \gamma_v'(t) \quad \text{where } \gamma_v \text{ unique geodesic with}$$
$$\gamma_v(0) = \pi(v), \quad \gamma_v'(0) = v$$

Fact: $\boxed{\phi_g \text{ intertwines geodesic flow on } UM \text{ with Reeb flow on } U^*M}$

lots of information about g can be seen from geodesic flow, so it can also be seen from the contact geometry of U^*M! Arnold "contact geometry is all geometry"

now if $S \subset M$ a submanifold of dimension k

let $L_S = \{[\alpha] \in P^*M : \alpha(v) = 0 \ \forall v \in T_{\pi^*(\alpha)} S\}$

this is the "unit" conormal bundle of S

exercise:

1) L_S is an S^{n-k-1}-bundle over S

so $\dim L_S = n-1$

2) $TL_S \subset \xi$ so L_S is Legendrian!

3) if S and S' are isotopic in M then L_S and $L_{S'}$ are isotopic through Legendrian submanifolds in P^*M

Remark:

So we can use invariants of Legendrian submanifolds to construct invariants of smooth submanifolds!

e.g. Knots $K \subset \mathbb{R}^3$ are completely determined by L_S in $U^*\mathbb{R}^3 = J^1(S^2)$

and there are invariants of Legendrian submanifolds of $J^1(S^2)$ that gives a complete invariant of knots!

B. Local Theory

Th$^{\underline{m}}$ 1:

- M^{2n+1} a smooth manifold
- $N \subset M$ a compact set with a nbhd U that strongly deformation retracts to N (e.g. N a submanifold)
- ξ_0, ξ_1 be contact structures defined near N

$\xi_0 = \xi_1$ on $TM|_N$

if $n > 1$, then $\exists\, \alpha_2$ s.t. $\xi_2 = \ker \alpha_1$ and

$\alpha_0 = \alpha_1$ and $d\alpha_0 = d\alpha_1$ on $TM|_N$

Then $\exists$ a diffeomorphism $f: M \to M$ fixing N isotopic to the identity on M (rel N) s.t. $f^* \omega_1 = \omega_0$ on some nbhd of N in U.

Remark: Proof very similar to proof of Th$^{\underline{m}}$ III.1 we leave it as an exercise, as well as the following.

Cor 2 (Darboux Th$^{\underline{m}}$):

Given any contact manifold (M^{2n+1}, ξ) and $p \in M$, there is a neighborhood U of p contactomorphic to neighborhood of the origin in $(\mathbb{R}^{2n+1}, \xi_{std})$

So we could have defined a contact structure to be a hyperplane field locally modeled on $(\mathbb{R}^{2n+1}, \xi_{std})$

Cor 3 (Legendrian nbhd theorem):

- (M, ξ) a contact manifold
- $L \subset M$ a compact Legendrian submanifold

Then L has a nbhd in M contactomorphic to a nbhd of 0-section in $(J^1(L), \xi = \ker(dz - \lambda))$

Th$^{\underline{m}}$ 4 (Gray's Th$^{\underline{m}}$):

- M a manifold
- ξ_t, $t \in [0,1]$ a family of contact structures that differ on a compact subset $C \subset M$

Then there is an isotopy $\psi_t : M \to M$ such that

$$\psi_t^* \xi_0 = \xi_t \quad \text{and}$$

$$\psi_t = id \quad \text{off of } C$$

now given a contact manifold (M, ξ) with ξ co-oriented by $v \in \mathfrak{X}(M)$

let $S(M,\xi) = \{\alpha \in T_x^* M : x \in M, \ker \alpha_x = \xi, \alpha_x(v(x)) > 0\}$

then $d\lambda$ is symplectic form on $S(M,\xi)$

$S(M,\xi)$ with form $d\lambda$ is called the symplectization of (M,ξ)

to see $d\lambda$ is symplectic on $S(M,\xi)$ we choose a 1-form α s.t.

$$\xi = \ker \alpha$$

and define
$$\phi : \mathbb{R} \times M \to T^* M$$
$$(z, x) \longmapsto e^z \alpha(x)$$

exercise:
1) ϕ is an embedding
2) $\phi^* d\lambda = d(e^z \alpha)$
3) $d(e^z\alpha)$ is a symplectic form on $\mathbb{R} \times M$

we also sometimes call $(\mathbb{R} \times M, d(e^z\alpha))$ the symplectization of (M,ξ) (but note the original def$^{\underline{n}}$ only depends on (M,ξ))

exercise: If α, β are contact 1-forms for ξ and $\alpha = f\beta$, $f > 0$ then

$$\psi : \mathbb{R}\times M \to \mathbb{R} \times M : (z,x) \mapsto (z + \ln f(x), x)$$

is a symplectomorphism from $(\mathbb{R}\times M, d(e^z\beta))$ to $(\mathbb{R}\times M, d(e^z\alpha))$

Clearly if (M,ξ) is contactomorphic to (M',ξ') then

$S(M,\xi)$ is symplectomorphic to $S(M',\xi')$

Open problem:

are 2 contact structures ξ, ξ' on M contactomorphic

$\Leftrightarrow$

$S(M,\xi)$ and $S(M,\xi')$ are (exact) symplectomorphic

Remark: Sylvain Court showed $\exists$ (M,ξ) and (M',ξ') where M is not diffeomorphic to M' but $S(M,\xi)$ is (exact) symplectomorphic to $S(M',\xi')$

If (M^{2n},ω) a symplectic manifold with boundary and v is a vector field defined near a component C of ∂M such that

$$\mathcal{L}_v \omega = \omega$$

then we say C is convex if v points out of M along C

we say C is concave if v points into M along C

as discussed earlier $\alpha = \iota_v \omega|_C$ is a contact 1-form on C if C is convex and on $-C$ if C is concave

Th^{m} 5:

let C be a convex (concave) boundary component of (M,ω), v the associated vector field, and $\alpha = \iota_v \omega|_C$

then $\exists$ a nbhd of C in M symplectomorphic to

$((-\varepsilon,0]\times C, d(e^{t}\alpha))$ if C is convex

$([0,\varepsilon)\times C, d(e^{t}\alpha))$ if C is concave

exercise: Prove this (almost immediate from Th$^{\underline{m}}$ III.1)

Th$^{\underline{m}}$6:

> let (M_0, ω_0) have convex boundary component C_0
> (M_1, ω_1) have concave boundary component C_1
>
> let ξ_0 be the inducted contact structure on C_0
> ξ_1 " " " " $-C_1$
>
> If $\exists$ a contactomorphism $\phi : (C_0, \xi_0) \to (-C_1, \xi_1)$
> then there is a symplectic structure ω on
> $$M_0 \cup M_1 /_{x \in C_0 \sim \phi(x) \in C_1}$$
> such that $\omega|_{M_0} = \omega_0$ and off of some nbhd of C in M_1, $\omega = k\omega_1$ for some constant k

So we can "glue" symplectic manifold together if the contact structures on their boundaries agrees

Remark: Many of our previous constructions can be proven using Th$^{\underline{m}}$6 (eg. blow-down, blow-up, normal sum)

Proof:

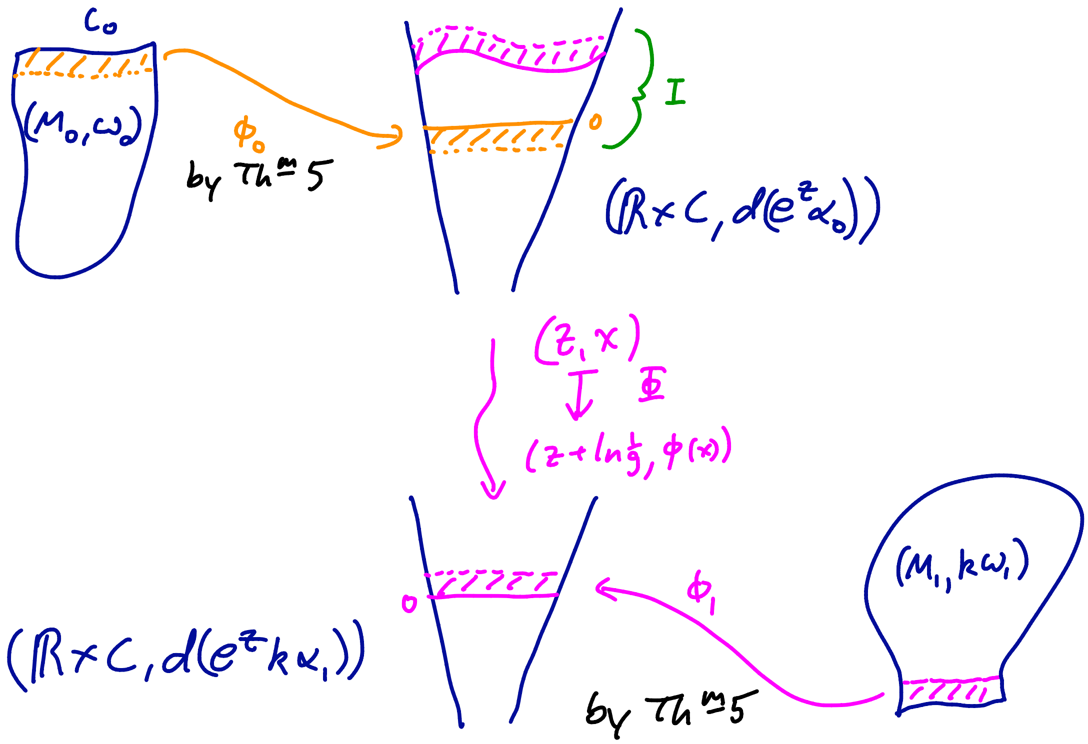
let v_i be vector field from def of convex/concave

and $\alpha_i = \iota_{v_i} \omega_i|_{C_i}$

so $\phi^* \alpha_1 = f \alpha_0$

if we scale ω_1 by a constant then α_1 also scales

so $\exists$ so k st. $\phi^* k\alpha_1 = \frac{1}{g} \alpha_0$ with $\ln \frac{1}{g} \ll 0$



we can now use ϕ_0 to glue (M_0, ω_0) to I and ϕ_1 and Φ to glue I to $(M_1, k\omega_1)$