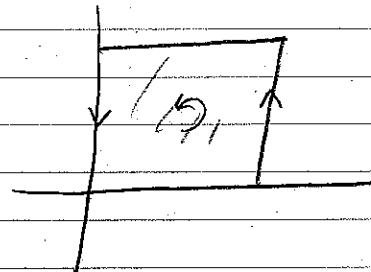


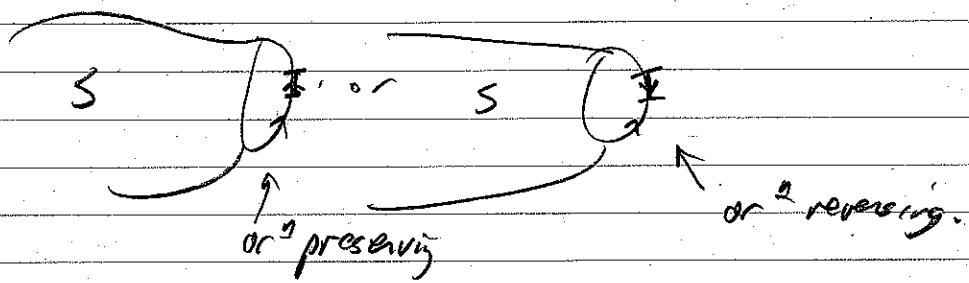
$h' = [0,1] \times [0,1]$ sits in \mathbb{R}^2



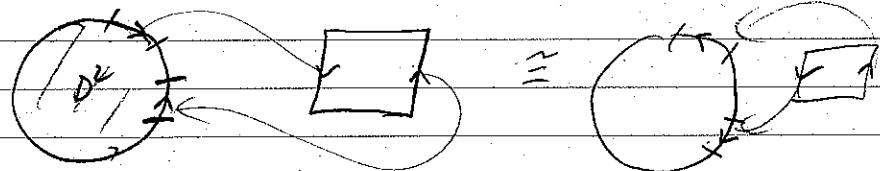
The counterclockwise or 1 or
 h' induces an orientation
on $\partial h'$ and on $\partial^2 h' = [0,1] \times [0,1]$

Lemmas 6, 7, 8 and 9 basically say the only thing that matters when attaching a 1-handle is the relative placement of the "feet" (= components of $\partial h'$) and whether they are glued in an or 1 preserving way or or 2 reversing way

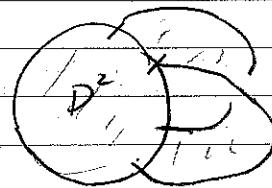
we denote this by



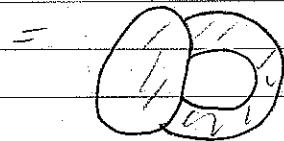
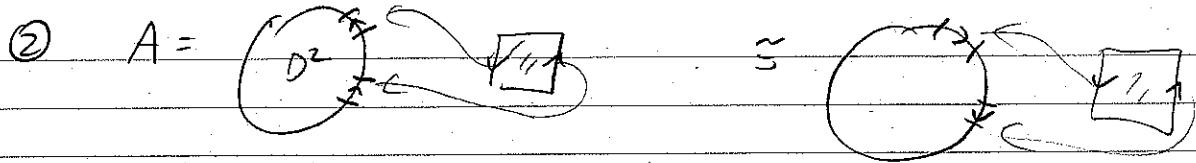
example: ① let $M =$



=



M is called the Möbius band



$\overset{\sim}{S^1 \times [0,1]}$ the annulus
check

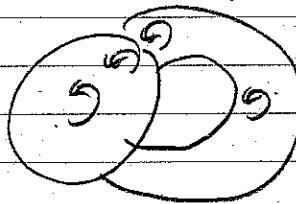
Lemma 10:

M is not homeomorphic to A

exercise: show that if $S \cup h'$ is formed with both feet or¹ reversing or both or² preserving the resulting surfaces are homeomorphic

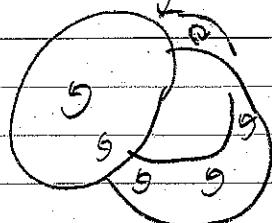
Proof: $\partial M = S^1$ $\partial A = S^1 \cup S^1$

note: \checkmark^A we can pick an or¹ on D^2 and on h' so that when we pass between D^2 & h' the orientations agree



we cannot do this for M .

doesn't agree!



we say A is orientable and M is not orientable

In general we say a surface S is not-orientable

if there is a Möbius band embedded on S

otherwise S is orientable

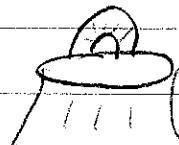
(or equivalently if S has a handle decomposition
and we can choose or's on all the handles
so that they agree then surface is orientable)

from our discussion above it is clear that when attaching a 1-handle the only thing that matters is

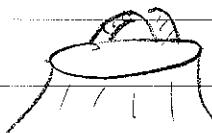
- 1) which boundary components of S the feet of the handle are glued to
- 2) whether or not the α^2 is repeated.

so there are only 4 ways to attach a 1-handle

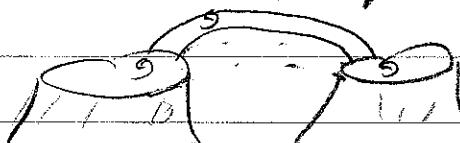
- 1) both feet on 1 boundary cpt with α^2 repeated



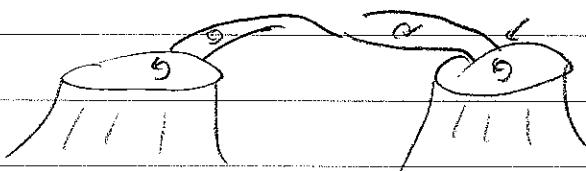
- 2) both feet on 1 boundary cpt with α^2 not repeated



- 3) feet on different boundary components with α^2 repeated



- 4) feet on different boundary components with α^2 not repeated

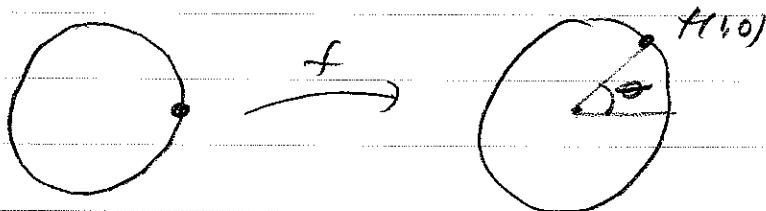


now for 2-handles

Lemma 11: 1) any orientation preserving homeomorphism $f: S^1 \rightarrow S^1$ is isotopic to the identity

2) any orientation reversing homeomorphism $f: S^1 \rightarrow S^1$ is isotopic to $r(x, y) = (x, -y)$

Proof: 1) given f let θ be the angle between $(1,0)$ and $f(1,0)$



$$\text{now let } F_t(x) = R_{-\theta t} f(x)$$

where $R_s = \text{rot}^s$ by s .

then $F_t(x) \in \text{fix}_c(1,0)$

$\therefore f$ is isotopic to a map g fixing $(1,0)$

now g induces a map \tilde{g} on $[0, 2\pi]$ by

$$\begin{matrix} [0, 2\pi] & \xrightarrow{\tilde{g}} & [0, 2\pi] \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{g} & S^1 \end{matrix}$$

\tilde{g} is α^4 preserving so by lemma 7 it is isotopic to the identity

this isotopy always sends 0 to 0 & 2π to 2π

so it induces an isotopy of g to identity

2) exercise

Lemma 12: given a 2-handle h^2 and two embeddings

$$f_0: \partial h^2 \rightarrow \partial S \quad \text{and} \quad f_1: \partial h^2 \rightarrow \partial S$$

that map onto the same boundary component of ∂S

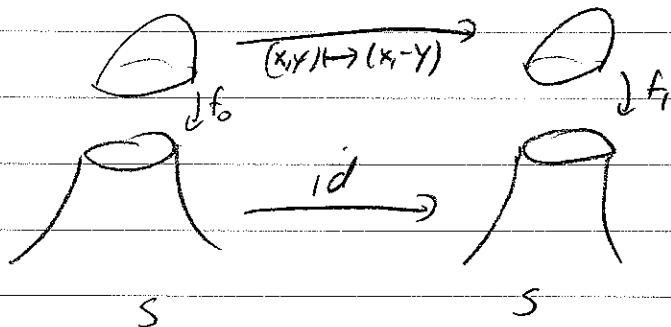
then

$$S^1 f_0 h^2 \cong S^1 f_1 h^2$$

thus when attaching a 2-handle only need to know the boundary comp of S we are attaching to.

Proof: if f_0, f_1 both or² rev or both or² pres then
done by Lemma 12 and 6

if f_0 or² pres & f_1 or² rev then
(after isotopy) can assume $f_i(x, y) = (x, -y)$



This induces a homeomorphism ✓

lets construct some surfaces.

defⁿ: given two surfaces S_1 and S_2 let D_i be a disk
in S_i

let $S_i^\circ = S_i \setminus \text{int } D_i$ intention reversing

let $f: \partial D_1 \rightarrow \partial D_2$ be an homeomorphism

$$\overset{\circ}{\partial S_1} \quad \overset{\circ}{\partial S_2}$$

define the connected sum of S_1 and S_2 to be

$$S_1 \# S_2 = S_1^\circ \cup_f S_2^\circ$$



Th 2B: The connected sum of 2 surfaces is well defined

In other words it does not depend on

- 1) the disks D_1 and D_2 used or
- 2) the bases f .

part 2) is easy since any 2 or " rev. bases are isotopic
for part 1) we need

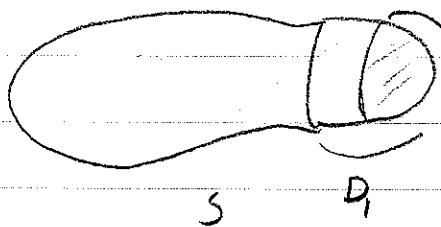
Lemma 4:

If D_1 and D_1' are 2 disks in S then
 $S \setminus \text{int } D_1 \cong S \setminus \text{int } D_1'$

Proof: note if $D_1' \subset D_1$ then this is "easy"

since

$$S \setminus \text{int } D_1' = (S \setminus \text{int } D_1) \cup (D_1 \setminus \text{int } D_1')$$



Claim: $D_1 \setminus \text{int } D_1' \cong S \times [0, 1]$

thus we are done since adding an annulus to ∂S
doesn't change S (lemma 5)

now if $D_1 \cap D_1' \neq \emptyset$ then \exists a pt p and
a disk D around p st. $D \subset D_1 \cap D_1'$

now from above

$$S \setminus \text{int } D_1 \cong S \setminus \text{int } D \cong S \setminus \text{int } D_1'$$

Finally if $D_1 \cap D_1' = \emptyset$ let γ be an ^{embedded} path from a pt on D_1 to a pt on D_2
 a union of γ 's can be assumed to be a disk D
 (you probably checked this on HW2)
 now from above $S^1 \text{ int } D \cong S^1 \text{ int } D \cong S^1 \text{ int } D$

The claim seems obvious and we will not prove it
 but it is actually quite difficult to prove!

$$\text{let } \Sigma_0 = S^2$$

$$\text{let } \Sigma_1 = T^2 \quad \text{②}$$

$$\text{let } \Sigma_2 = \Sigma_1 * \Sigma_1 \quad \text{③} \text{ (a torus with a hole)}$$

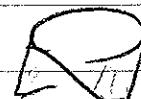
$$\text{in general let } \Sigma_n = \Sigma_{n-1} * \Sigma_1 \quad \text{④} \text{ (a surface with } n \text{ holes)}$$

so we have a bunch of surfaces

are there all compact surfaces w/o boundary?

No these are called closed surfaces

example: let $M = \text{möbius band}$



$$\text{note } \partial M = S^1$$

let $f: \partial D^2 \rightarrow \partial M$ be a homeomorphism

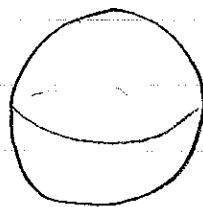
define the projective plane to be

$$P = M \cup_f D^2$$

P is a closed surface (non-orientable)

other descriptions of P

1)



S^2 unit sphere in \mathbb{R}^3

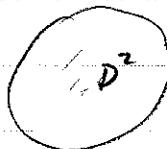
let $r: S^2 \rightarrow S^2: (x, y, z) \mapsto (-x, -y, -z)$

say 2 points in S^2 are equivalent

if $r(p_1) = p_2$ ($\because r(p_2) = p_1$)

exercise: $P \cong S^2/n$

2)



unit disk in \mathbb{R}^2

let $r: S^1 \rightarrow S^1: (xy) \mapsto (-x, -y)$

$\frac{\pi}{2D^2}$ $\frac{\pi}{2D^2}$

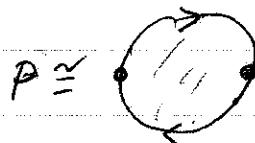
define equivalence rel \sim on S^1 as above

exercise: $P \cong D^2/n$

Hint:

$$\textcircled{1} = \textcircled{X} \textcircled{O}$$

3)

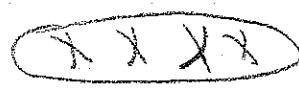


now define $N_1 = P$

$N_2 = P \# P$

~~(X)~~ called "cross cap"
~~(X)~~ we cannot really draw
 P in \mathbb{R}^3 !

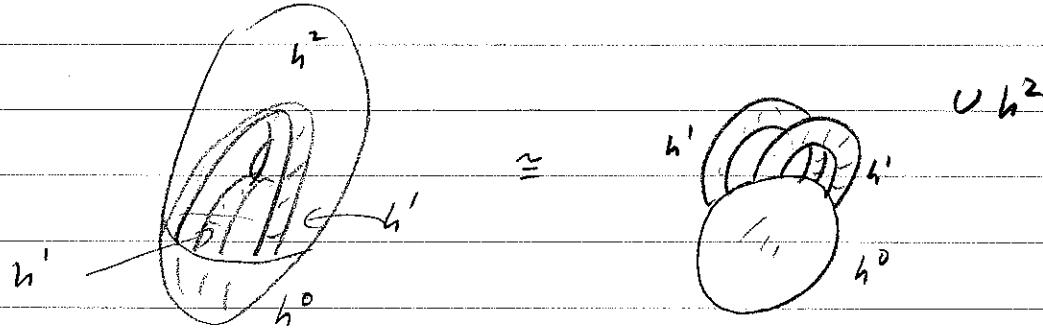
$N_n = N_{n-1} \# P$



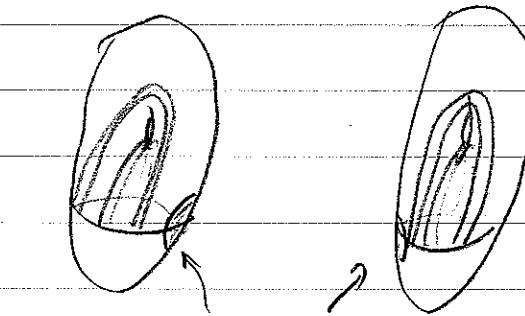
n "cross caps"

handle pictures of Σ_n

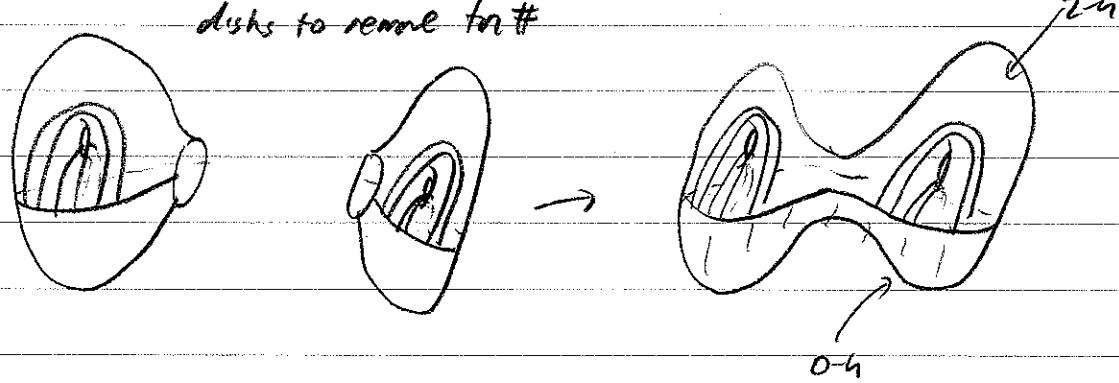
earlier we saw that $\Sigma_1 = T^2$ is



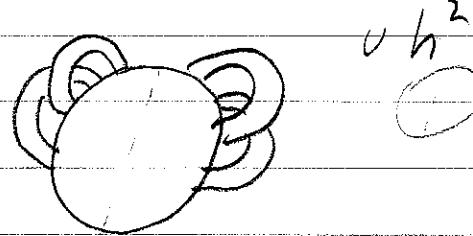
now for $\Sigma_2 = T^2 \# T^2$



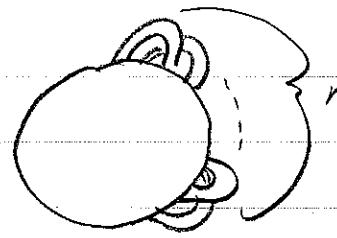
disks to remove for #



\approx

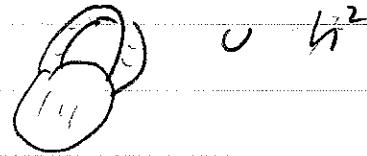


similarly $\Sigma_n \cong$



U^{h^2}

now $N_1 \cong$



U^{h^2}

similarly

$N_n \cong$

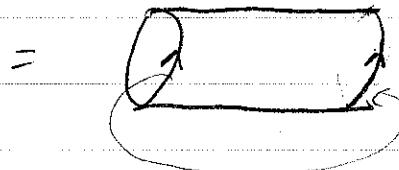
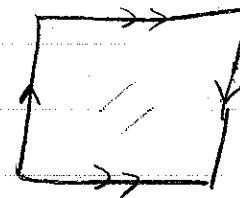


$n U^{h^2}$

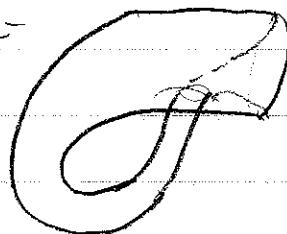
Are these all closed surfaces?

What about:

Klein bottle = $K =$

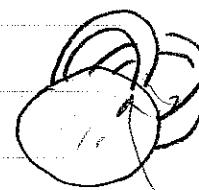
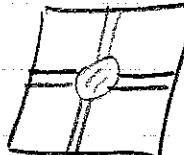


can't draw this in 3D



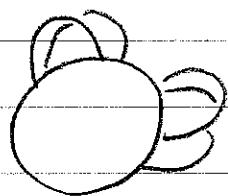
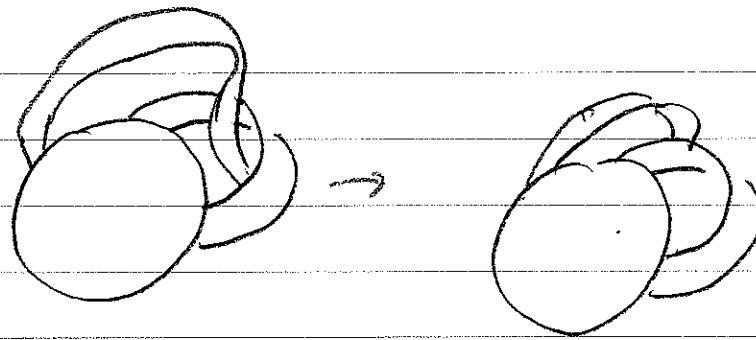
this is a closed surface!

note:



U^{h^2}

slide ears foot!

 $\cup h^2$

$$\cong P^2 \# P^2 = N_2$$

$$\text{so } K \cong N_2$$

so maybe we do have all surfaces!

Th 15:

If S is a closed ^{connected} surface then
there is some $n \geq 0$ st.

$$\begin{aligned} S &\cong \Sigma_n & (\text{if } S \text{ orientable}) \\ \text{or } S &\cong N_n & (\text{if } S \text{ non-orientable}) \end{aligned}$$

we need several preliminary results.

Proposition 16:

every surface S has a handle decomposition

for this we need

Hard Theorem (Radó 1925):

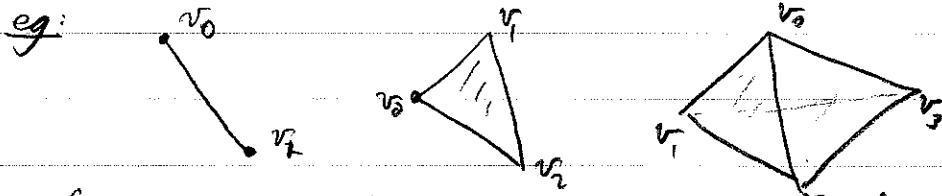
any surface may be triangulated

def[#]: given $k+1$ points v_0, \dots, v_k , called vertices, in \mathbb{R}^N (N large)
in general position" (that is no 3 pts lie on a common line
no 4 on a plane, ...)

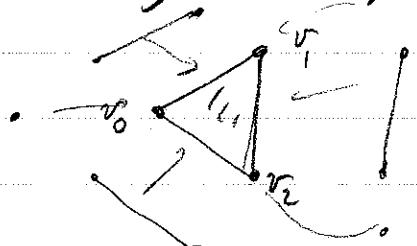
then a k -simplex is the set

$$\Delta_k = \{ \lambda_0 v_0 + \dots + \lambda_k v_k \mid \lambda_0 + \dots + \lambda_k = 1 \}$$

e.g:

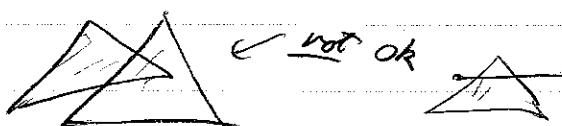
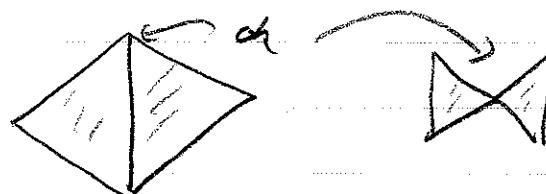


a face of a simplex is a subsimplex formed by discarding some vertices.



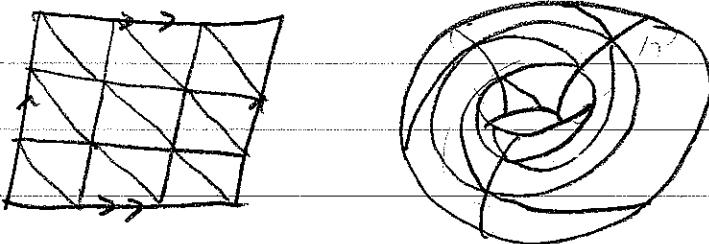
a simplicial complex is a finite collection of simplexes in some \mathbb{R}^N st.

- a) if a simplex is in the collection then so are all its faces
- b) if two simplexes intersect they do so in one common face



a triangulation of a topological space X
 is a simplicial complex K together
 with a homeomorphism $h: K \rightarrow X$

example: T^2



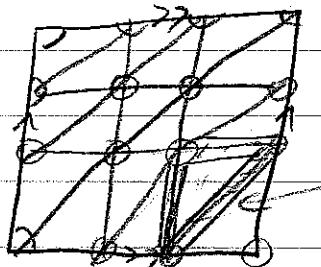
similarly all the surfaces we constructed
 earlier have triangulations, but Rado's thm
 says all surfaces have triangulations.

Proof of Prop 16:

S has a triangulation

let $S_0 = \emptyset$

$$\left. \begin{array}{l} S_1 = \text{nbhd of vertex} \\ S_2 = S_1 \cup \text{nbhd of edge} \end{array} \right\} \text{0-handles}$$



$$S_{k+1} = S_{k+1} \cup \text{nbhd of vertex}$$

$$S_{k+1} = S_{k+1} \cup \text{nbhd of edge}$$

1-handle

$$S_{k+2} = S_{k+1} \cup \text{nbhd of edge}$$

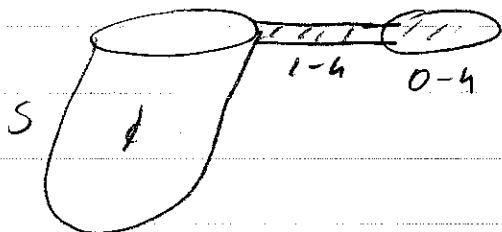
$$\left. \begin{array}{l} S_{k+1} = S_{k+1} \cup \text{face} \\ S_k = S_{k+1} \cup \text{lost face} \end{array} \right\} 2\text{-handles}$$

$$S_k = S_{k+1} \cup \text{lost face}$$

now we need to see how to simplify handle decompositions

lemma 17:

if you attach a 0-handle and 1-handle as follows



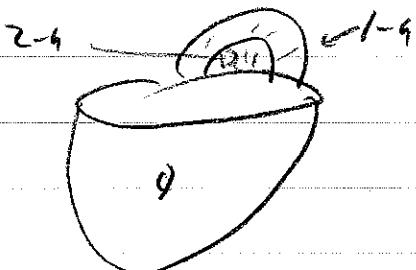
then the resulting surface is homeo to S

Proof: $1-h \cup 0-h \cong \text{disk}$ (by HW2) HW2

and disk is attached to S along an arc in ∂ . \therefore dont change S (HW2)

lemma 18:

if you attach a 1-handle and 2-handle as follows



then the resulting surface is homeo to S

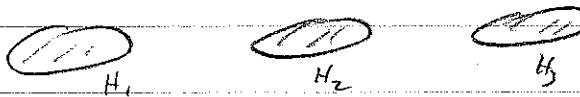
Proof: same as above

Corollary 19

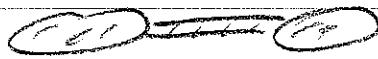
if S is a connected surface then S has a handle decomp with only 1 0-handle

if in addition $\partial S = \emptyset$ then S has a handle decomposition with only 1 0-handle & 1 2-handle

Proof: look at all the 0-handles



Since S connected must have a 1-handle connecting H_1 to H_2 or H_3 (say H_2)

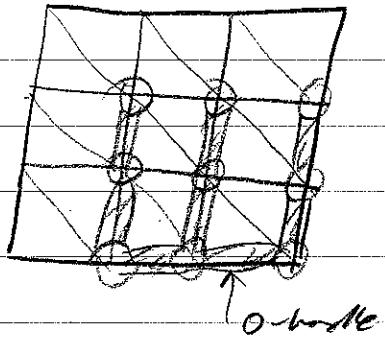


but lemma 17 \Rightarrow this is homeo to H_1 ,
so forget about H_2 & 1-handle



Continuing we get down to 1 0-handle.

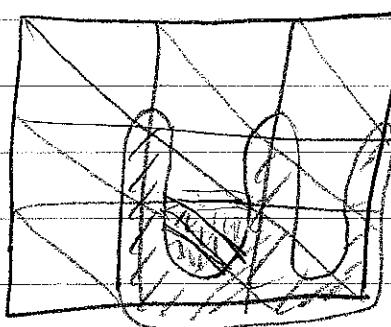
example: T^2



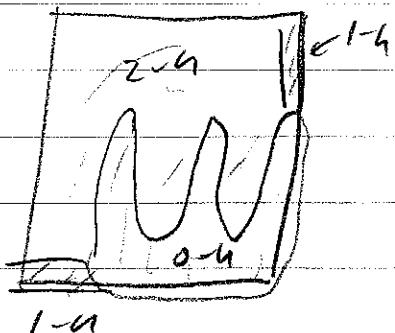
for part 2, if more than one 2-handle
then can find 2 H_1 & H_2 that are
on "opposite sides" of a 1-handle
one of these may be cancelled

e.g.:

T^2



\rightarrow



exercise:) try to make this rigorous

2) note if S is a closed surface

given a handle decomposition we
can "turn it upside down"

that is given $S_0 \dots S_n \cong S$

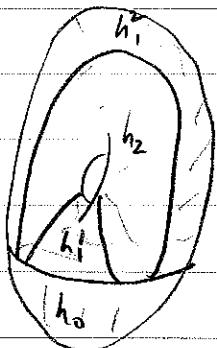
get another decomposition $S'_0 \dots S'_n$

where $S'_0 = \emptyset$ and S'_i is a 0-handle

if S_{n-i} a 2-handle, its a 1-hole

is S_{n-i} a 1-handle and a 2-hole

if S_{n-i} a 0-handle



with h_2 a 0-hole

h_1, h_2 a 1-hole

h_3 a 2-handle

check this and reprove 1) using this

Lemma 20:

given two closed connected surfaces S, \tilde{S}

take a handle decomposition with one 2-handle

for both. Let $S_0 = S \backslash 2\text{-handle}$, $\tilde{S}_0 = \tilde{S} \backslash 2\text{-handle}$

Then

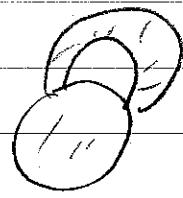
S_0 homeo to \tilde{S}_0 iff S homeo to \tilde{S}

Proof: this is a rephrasing of lemma 14
(+ lemma 12)

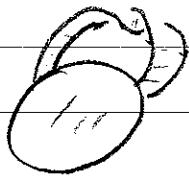
Thus to understand surfaces up to homeomorphism we
just need to consider attaching 1-handles to
a disk!

(44)

before we start "normalizing" our decomposition I want to make an obvious (but sometimes confusing) observation



is homeo to



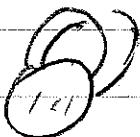
and to



(This is really just lemma 9 & discussion after)

These annuli are sitting in \mathbb{R}^3 differently but we can write down a homeo between them (think about this!)

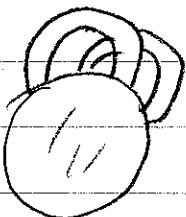
similarly



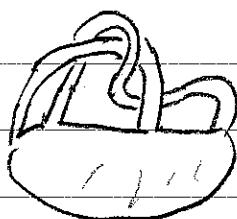
is homeo to



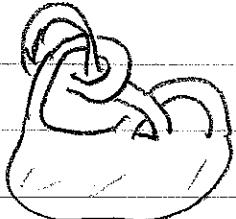
and



is homeo to

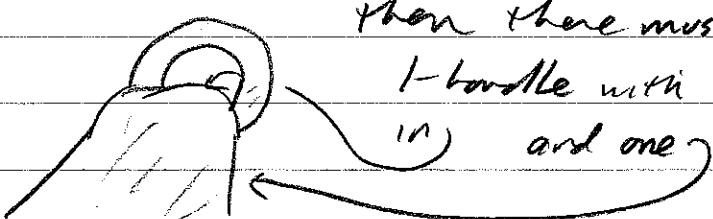


and



OK back to simplifying handle decompositions

if you see

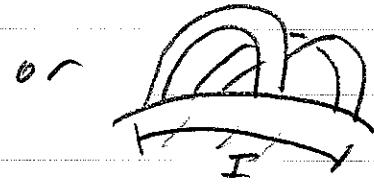
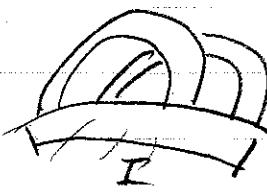


then there must be another 1-handle with one foot

and one

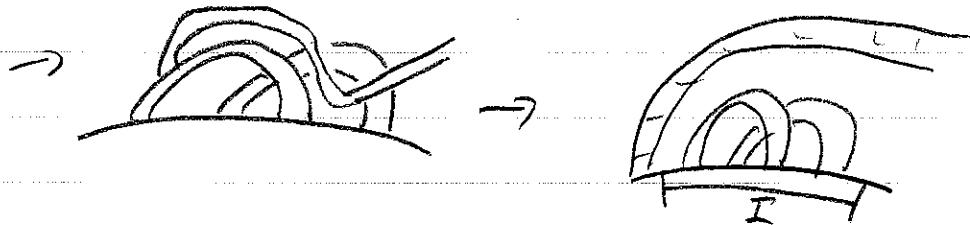
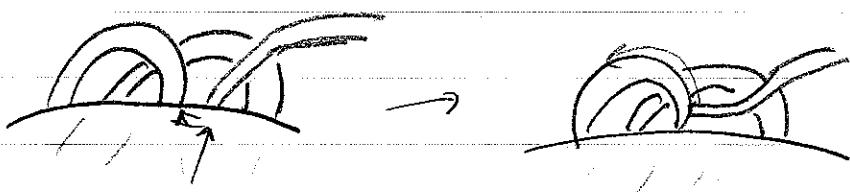
(because after attaching all 1-handles our surface has only one boundary component)

so you will see
either



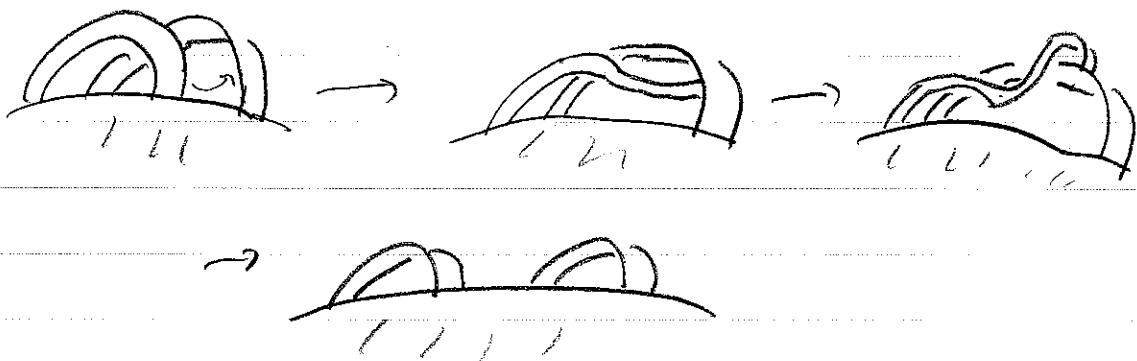
note: if foot of another handle is in region I
we can isotope it away (without, of course,
changing homeomorphism type!)

eg



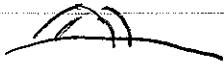
excuse: check other cases.

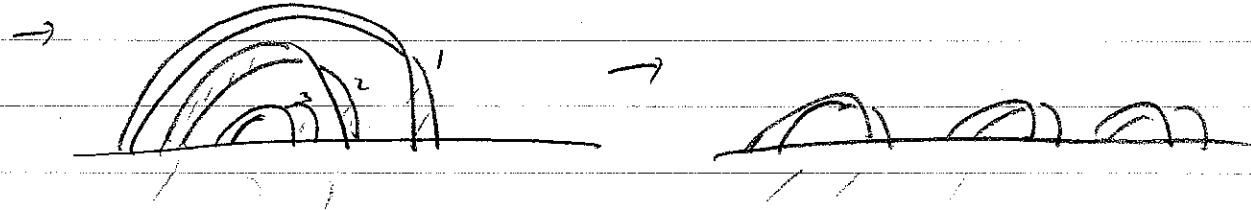
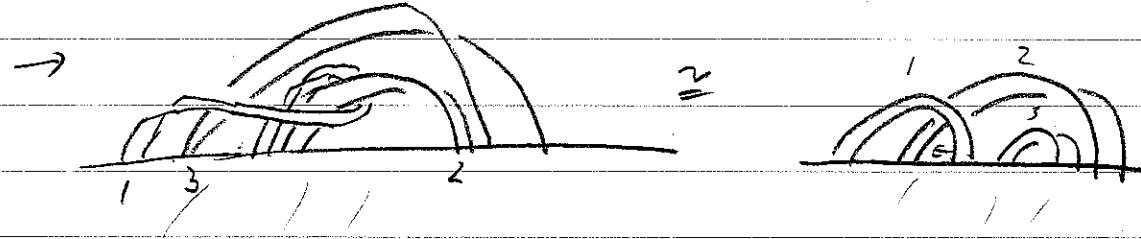
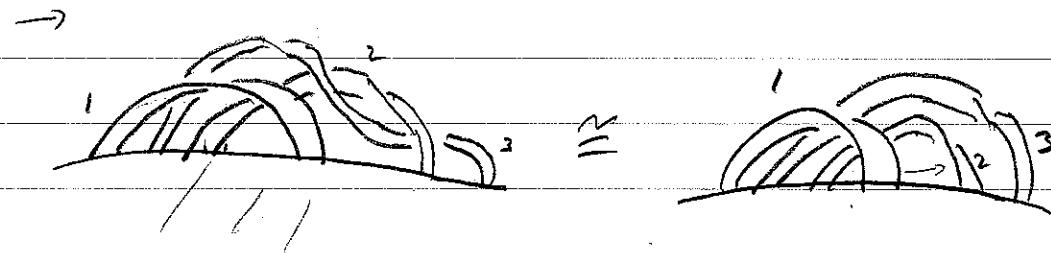
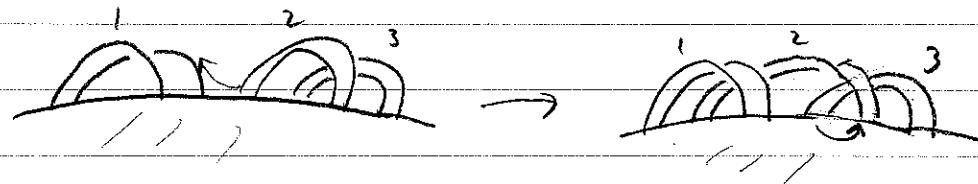
if you are in case 2 above you can do following



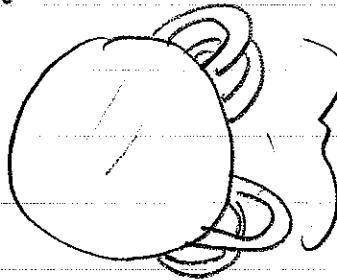
thus we can arrange $S^0 = 5 \backslash 2$ -handle to be



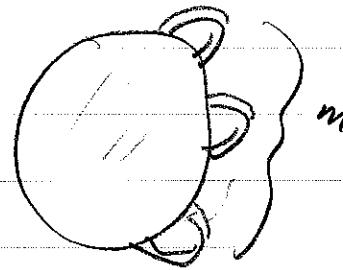
if you have one  then note



thus we have $S^0 = 51$ 2-handle
either



$\{ \}^n$ or



so S^0

either $\Sigma_n = \underbrace{\textcircled{---}}_{n\text{-holes}}$ or $N_m = \underbrace{\textcircled{x-x}}_{m\text{-times}}$

$$= \underbrace{T^2 \# \dots \# T^2}_{n \text{ times}}$$

thus th¹⁵ is proved!

Finally we need to make sure all those surfaces
are different!

C. Distinguishing surfaces - The Euler Characteristic

let K be a simplicial complex (with no n -simplices, for $n \geq k$)
the Euler characteristic of K is

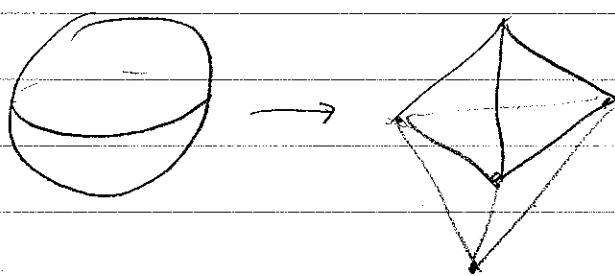
$$\chi(K) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) + \dots + (-1)^k \#(k\text{-cells})$$

$$= \sum_{i=0}^k (-1)^i \#(i\text{-cells})$$

If X is a topological space homeo to K then

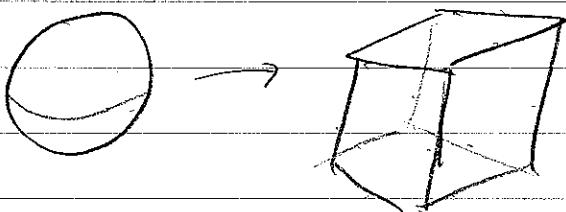
$$\chi(X) = \chi(K)$$

(4P)

example: 1) S^2 

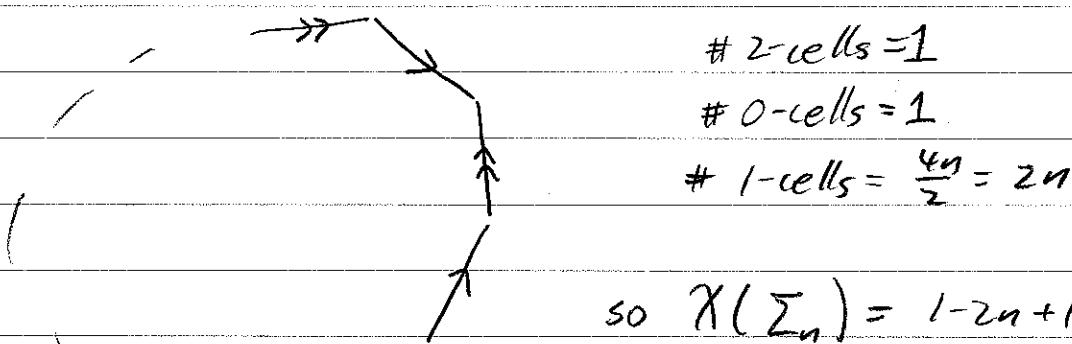
$$\chi(S^2) = 5 - 9 + 6 = 2$$

or



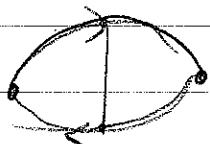
$$\chi(S^2) = 8 - 12 + 6 = 2$$

2) Σ_n can be made by gluing the edges of a $4n$ -gon.



$$\text{so } \chi(\Sigma_n) = 1 - 2n + 1 = 2 - 2n$$

(actually we should cut $4n$ -gon into triangles but get same answer)

3) P^2 

$$\text{eg } \tau^2 \quad \boxed{\text{triangle}} \quad \chi = 1 - 3 + 2 = 0$$

$$\chi(P^2) = 1 - 1 + 1 = 1$$

lemma 20:

X, Y two simplicial complexes
suppose we glue them along a subcomplex Z

$$\chi(X \cup_Z Y) = \chi(X) + \chi(Y) - \chi(Z)$$

Proof: all the simplices in Z are counted
in $\chi(X)$ & in $\chi(Y)$ but they
are only counted one time in $X \cup_Z Y$

example: 1) S'



$$\chi(S') = 3 - 3 = 0$$

2) S_1, S_2 two surfaces

$$S_1 \# S_2 = (S_1 \setminus \text{int } D^2) \cup_{S'} (S_2 \setminus \text{int } D^2)$$

removing a D^2 is like removing a 2-cell

$$\text{so } \chi(S_i \setminus \text{int } D^2) = \chi(S_i) - 1$$

so

$$\begin{aligned} \chi(S_1 \# S_2) &= \chi(S_1 \setminus \text{int } D^2) + \chi(S_2 \setminus \text{int } D^2) - \chi(S') \\ &= \chi(S_1) + \chi(S_2) - 2 \end{aligned}$$

3)

$$\chi(\Sigma_n) = \chi(\Sigma_{n-1}) + \chi(\Sigma_1) - 2 = \chi(\Sigma_{n-1}) - 2$$

$$= \chi(\Sigma_{n-2}) - 2 \cdot 2 = \dots = \chi(\underset{T^2}{\Sigma_{n-(n-1)}}) - 2(n-1)$$

$$= -2n+2$$

$$\begin{aligned} \chi(N_n) &= \chi(N_{n-1}) + \chi(N_1) - 2 = \chi(N_{n-1}) - 1 \\ &= \chi(N_{n-2}) - 2 = \dots = \chi(\underset{1}{N_{n-(n-1)}}) - (n-1) \\ &= 2-n \end{aligned}$$

Note: if $\chi(S)$ does not depend on the triangulation
then

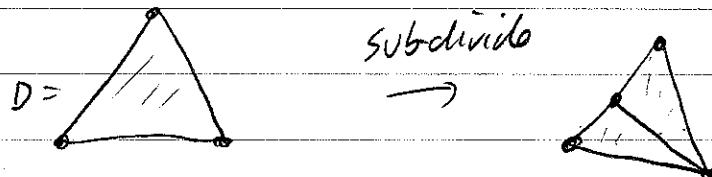
Σ_n is not homeo to Σ_m for $n \neq m$

similarly

$N_n \sim \dots \sim N_m$ for $n \neq m$

It turns out $\chi(S)$ does not depend on triangulation
to see this let's try to understand why we use the
alternating sum in definition of χ .

example:

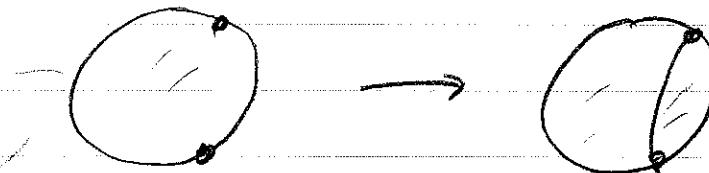


$$\chi(D) = 3 - 3 + 1 = 1$$

$$\chi = 4 - 5 + 2 = 1$$

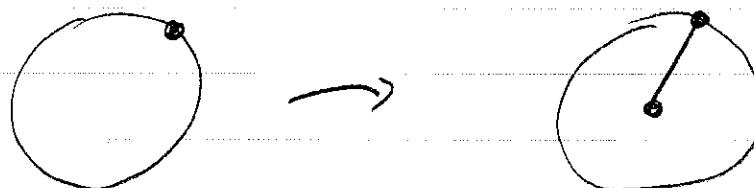
In general the subdivision of a simplicial complex can be done in a sequence of 3-steps

- 1) add an edge between vertices



$$\begin{array}{ccc}
 \# \text{faces} & \xrightarrow{\quad} & f \longrightarrow f+1 \\
 \# \text{edges} & \xrightarrow{\quad} & e \longrightarrow e+1 \\
 \# \text{vertices} & \xrightarrow{\quad} & v \longrightarrow v
 \end{array}
 \text{ so } \chi \text{ is unchanged}$$

- 2) add a vertex in the interior of a 2-simplex and add an edge to a vertex on boundary



$$\begin{array}{ccc}
 f & \longrightarrow & f \\
 e & \longrightarrow & e+1 \\
 v & \longrightarrow & v+1
 \end{array}
 \text{ so } \chi \text{ is unchanged}$$

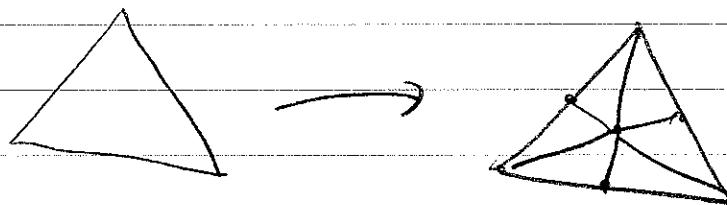
- 3) add a vertex to interior of an edge



$$\begin{array}{ccc}
 f & \longrightarrow & f \\
 e & \longrightarrow & e+1 \\
 v & \longrightarrow & v+1
 \end{array}
 \text{ so } \chi \text{ is unchanged}$$

exercise: convince yourself that any subdivision of a simplicial complex can be done by a sequence of these steps

e.g.



thus we have proved

Lemma 21:

the Euler characteristic is unchanged under subdivision

now we have the theorem

Theorem 22 (Hauptvermutung)

S a surface

if K_1, K_2 are triangulations of S
(i.e. $K_1 \cong S \cong K_2$)

then there is a triangulation K of S that
is a subdivision of both K_1 and K_2

Corollary 23:

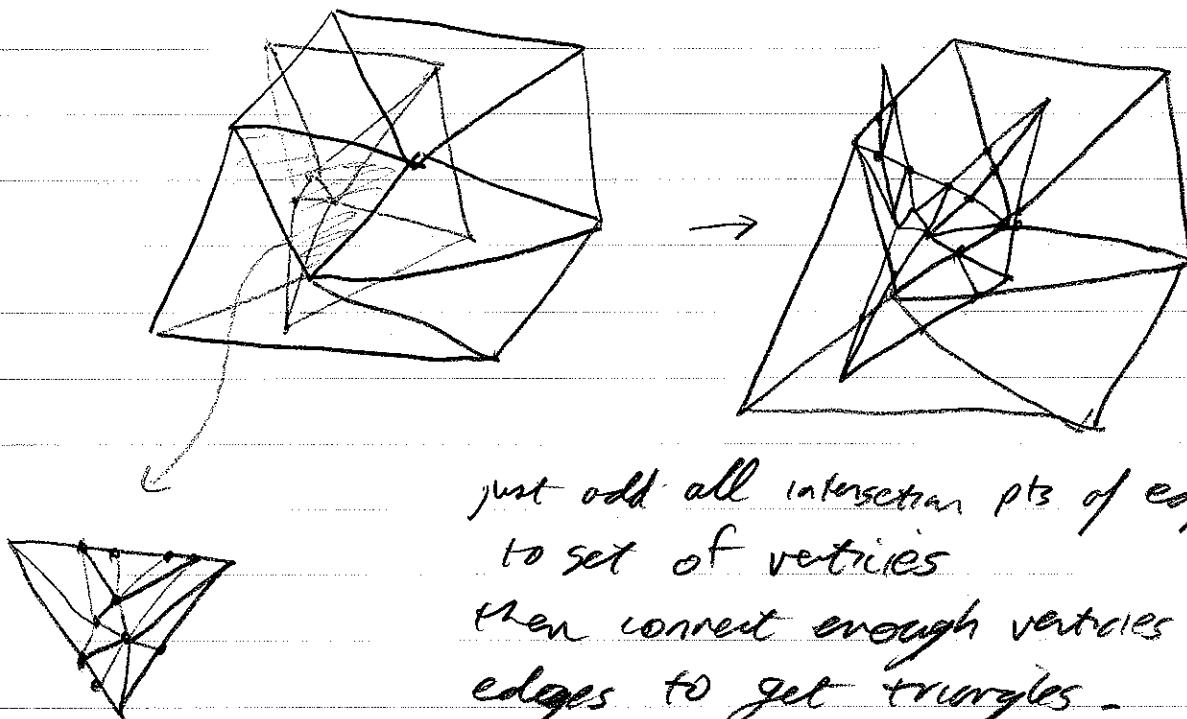
the Euler characteristic of a surface
is a topological invariant

Idea of proof of 2.2:

suppose K_1 and K_2 intersect "nicely"

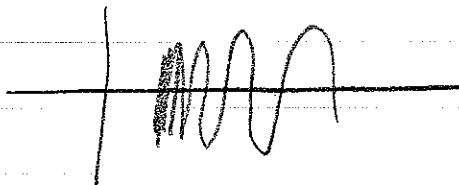
(re. edges do not intersect vertices
and edges intersect other edges
in a finite # of pts)

then



This gives a ^{common} subdivision of K_1 & K_2

Hard part of proof: understanding "nicely"
eg. worry about wild intersections



like sin $\frac{\pi}{7}$ and x-axis

Remark: Hauptvermutung is true in dim 2 and 3

it is false in dim ≥ 5 (Moise book)

X still invt but takes more work

Th^m 24:

Two surfaces are homeomorphic $\Leftrightarrow \chi(S_1) = \chi(S_2)$
and S_1, S_2 are both
orientable or both non-orientable

Moreover, any surface is homeo to

Σ_n or N_n

for some n

This is just a combination of Th^m 15 and Corollary 23

it is useful to observe that any integer $n \leq 2$

is the Euler characteristic of

a unique non-orientable stc if n odd

a non-ori or ori stc if n is even & ≤ 0

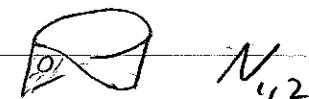
S^2 if $n=2$

given n and m let D_1, \dots, D_m be m disjoint
disks in Σ_n or N_n then let

$$\Sigma_{n,m} = \Sigma_n \setminus \bigcup_{i=1}^m \text{int } D_i \quad \text{with } \Sigma_{1,2}$$

$$N_{n,m} = N_n \setminus \bigcup_{i=1}^m \text{int } D_i$$

denote # of boundary comp by $b(\Sigma)$



Th^m 25:

two compact surfaces S_1, S_2 are homeomorphic

\Leftrightarrow

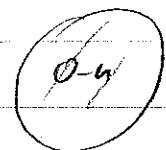
$\chi(S_1) = \chi(S_2)$, $|b(S_1)| = |b(S_2)|$ and S_1, S_2 both
orientable or not orientable

Moreover, any compact stc is homeo to $\Sigma_{n,m}$ or $N_{n,m}$
for some n and m

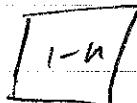
(58)

exercice: draw handle decompo for $N_{n,m}, \Sigma_{n,m}$

Euler characteristic is very useful, we would like to compute it in terms of handles



$$\chi(h^0) = 1$$



$$\chi(h^1) = 1$$



$$\chi(h^2) = 1$$

now if $S'_0 = S$ with a 1-handle attached then

$$\chi(S'_0) = \chi(S) + \chi(h^0) - \chi(\emptyset) = \chi(S) + 1$$

$$\chi(S'_1) = \chi(S) + \chi(h^1) - \chi(\{0,1\} \times [0,1])$$

$$= \chi(S) + 1 - 2 = \chi(S) - 1$$

$$\chi(S'_2) = \chi(S) + \chi(h^2) - \chi(S') = \chi(S) + 1$$

since any S has a handle decompo $\frac{S_0}{\emptyset}, S_1, \dots, S_n \cong S$

$$S_0, S_1, S_2, \dots, S_n$$

$$\chi = 0 \xrightarrow[\text{1-handle}]{} \overset{\text{odd } (-1)^i}{\cdots} \xrightarrow[\text{2-handles}]{} \overset{\text{odd } (-1)^i}{\cdots} \xrightarrow[\text{2-handles}]{} \overset{\text{odd } (-1)^i}{\cdots}$$

$$\text{so } \chi(S) = \#(\text{0-handles}) - \#(\text{1-handles}) + \#(\text{2-handles})$$

exercice: compute $\chi(N_{n,m})$ & $\chi(\Sigma_{n,m})$