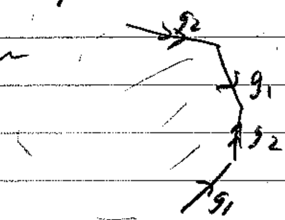
## F. More on Surfaces & the Fundamental Group

**Th^m 32:**

$\boxed{\begin{array}{l} \Sigma_1, \Sigma_2 \text{ two surfaces} \\ \text{given a homomorphism } \phi: \pi_1(\Sigma_1, x_1) \to \pi_1(\Sigma_2, x_2) \\ \exists \text{ a continuous map } f: \Sigma_1 \to \Sigma_2 \text{ s.t. } f_* = \phi \end{array}}$

Proof: let $\langle g_1 \ldots g_{2g} \mid r \rangle$ be a presentation of $\pi_1(\Sigma_1)$ coming from

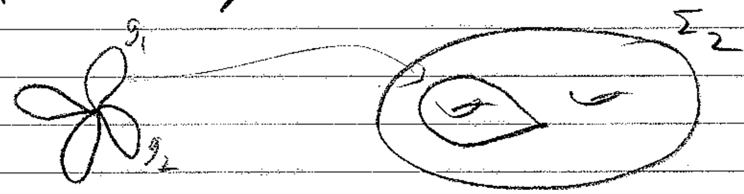



$\phi[g_i] \in \pi_1(\Sigma_2, x_2)$

let $\gamma_i$ be any loop in $\phi[g_i]$

ie. $\gamma_i : [0,1] \to \Sigma_2$ st. $\gamma_i(0) = \gamma_i(1) = x_2$
and $[\gamma_i] = \phi[g_i]$

$\bigcup_{i=1}^{2g} g_i$ = wedge of $2g$ circles




use the $\gamma_i$ to define a function on each $g_i$ call this $f$ =

$f|_{\cup g_i}$

now we want to extend $f$ over $4n$-gon
$f$ is defined on its boundary
and $f(\partial(4n\text{-gon}))$ is a loop $\gamma$ in $\Sigma_2$

but $\partial(4n\text{-gon}) \simeq x_1$ in $\Sigma_1$ $\therefore$ $[\partial(4n\text{-gon})] = e$
so $\phi_*[\partial(4n\text{-gon})] = e$ in $\pi_1(\Sigma_2, x_2)$

$\qquad \| $

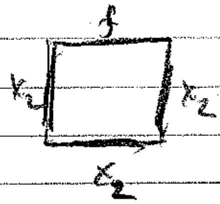
$(f|_{\cup g})_*[\partial(4n\text{-gon})]$

so $f(\partial(4n\text{-gon}))$ is homotopically trivial
or a loop

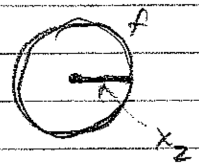
ie $\exists\ H:[0,1]\times[0,1]\to\Sigma_2$
st $H(s,1)=f|_{\partial 4n\text{-gon}}$

$H(s,0)=x_2$ and $H(0,t)=H(1,t)=x_2$

$\therefore$ $H$ decends to a map on the quotient space




$$
\begin{tikzcd}
{[0,1]\times[0,1]} \arrow[r, "H"] \arrow[d, "{(s,t)\mapsto((\cos 2\pi s,\sin 2\pi s),t)}"'] & \Sigma_2 \\
D^2 \arrow[ur, bend right] &
\end{tikzcd}
$$




call this map $f|_{D^2}$

thus we have a map defined on all of $\Sigma_1$

$f:\Sigma_1\to\Sigma_2$

on $U_{g_i}=f|_{U_{g_i}}$

on $\Sigma_1\setminus U_{g_i}=f|_{D^2}$

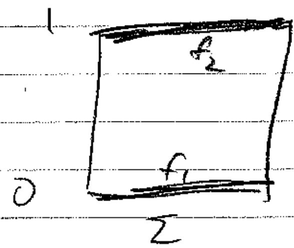
clearly $f_*=\phi$ ✓

<u>Th^m 33:</u>

$\boxed{\text{if } f_1,f_2:\Sigma\to\Sigma' \text{ are two continuous maps st } (f_1)_*=(f_2)_*:\pi_1(\Sigma,x_0)\to\pi_1(\Sigma',f(x_0)) \text{ as homomorphisms then } f_1\simeq f_2 \text{ if } \Sigma'\neq S^2 \text{ (or } P^2)}$

<u>Proof</u>: to define homotopy need

$$H: \Sigma_1 \times [0,1] \to \Sigma_2$$
s.t. $H(x,0) = f_1(x)$
$H(x,1) = f_2(x)$



now let $g_1 \ldots g_{2g}$ be generators of $\pi_1(\Sigma_1, x_0)$ as in last proof

$(f_1)_*[g_1] = (f_2)_* g_1$ so $f_1 \circ g_1 \simeq f_2 \circ g_1$

let $H_1 : S^1 \times [0,1] \to \Sigma_2$ be the homotopy

now define $H|_{g_1 \times [0,1]} = H_1$

similarly can define $H|_{\cup g_i \times [0,1]}$

<u>note</u>: $\Sigma \times [0,1] \setminus (\cup_i g_i \times [0,1]) = (4n\text{-gon}) \times [0,1] = B^3$

<u>fact</u> any map of $S^2 \to \Sigma'$ (when $\Sigma' \neq S^2$) is homotopic to a constant map

we have map $H$ defined on $\partial B^3 = S^2$
Fact $\Rightarrow$ this is homotopically trivial so
use homotopy to extend $H$ over $B^3$
<u>exercise</u>: write this down!

thus we have our continuous map $H$

**Remark:** 1) Th$^m$s 32–33 $\Rightarrow$ that for surfaces (not $= S^2$ or $P^2$) the homotopy class of a map is determined by its action on $\pi_1$!

(again this is not true in other dimensions)

2) $\{$homotopy classes of maps $S^2$ to $S^2\} \Longleftrightarrow \mathbb{Z}$.

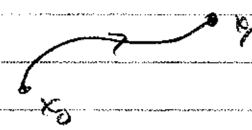
## G. Base Points

**Th$^m$ 34:** $\boxed{\text{Let } x_0, x_1 \text{ be two points in the same path component of a topological space } X. \quad \pi_1(X, x_0) \cong \pi_1(X, x_1)}$

**Proof:** Let $p : [0,1] \to X$ be path $x_0$ to $x_1$

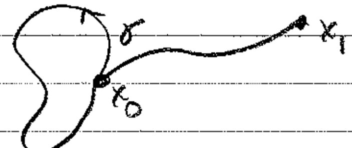
**define:**

$$\phi_p : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

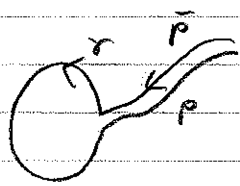


by

given $\gamma \in [\gamma] \in \pi_1(X, x_0)$



let $\gamma^p = p * \gamma * \bar{p} = \begin{cases} p(1-3s) & 0 \le s \le \frac{1}{3} \\ \gamma(3s-1) & \frac{1}{3} \le s \le \frac{2}{3} \\ p(3s-2) & \frac{2}{3} \le s \le 1 \end{cases}$



**exercise:** $\phi_p$ is well defined

**Claim:** $\phi_p$ is onto

given $[g] \in \pi_1(X, x_1)$
let $\gamma = \bar{p} * g * p$

so $\phi_p[\gamma] = \phi_p[\bar{p} * g * p]$
$= [p * \bar{p} * g * p * \bar{p}]$

but $p * \bar{p} \simeq e$ as a loop based at $x_1$

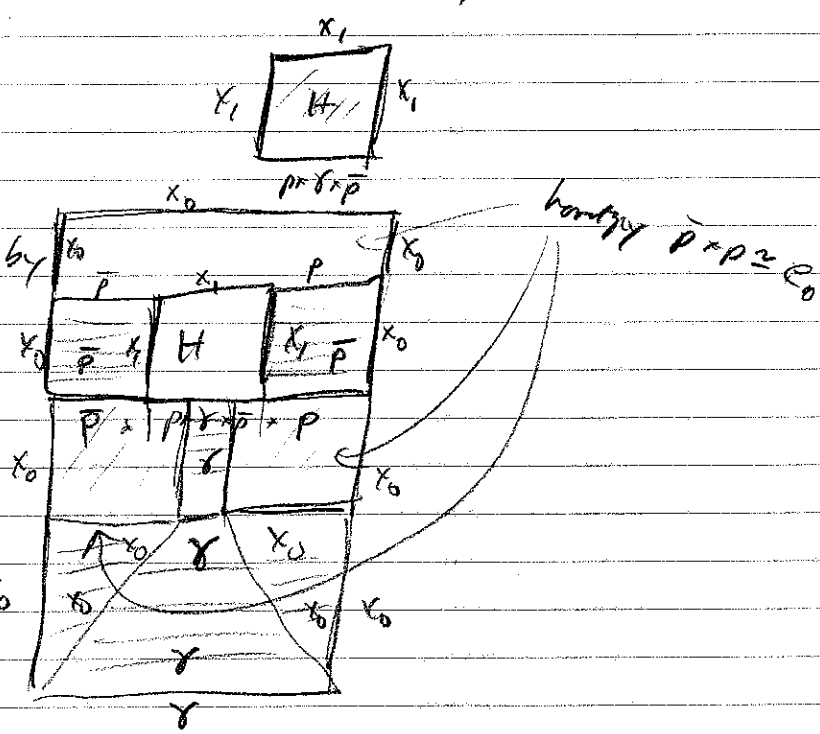
so
$= e * [g] * e$
$= [g]$

<u>Claim</u>: $\phi_p$ is one to one

need to show $\phi_p[\gamma] = e_1 \Rightarrow [\gamma] = e_0$

suppose $p * \gamma * \bar{p} \simeq e_1$ with homotopy $H$

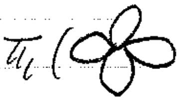
define $\tilde{H}$ by




so $\gamma \simeq e_0$ ✓

## H. Groups and Topology

recall

$\pi_1($  $) \cong F_k$ ← free group on $k$ generators

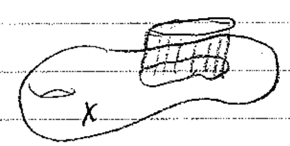
$W_k$ = wedge of $k$-loops

**Lemma 35:**

$$\boxed{\begin{array}{l}\text{let } f : \partial D^2 \to X \text{ be a continuous map} \\ \text{let } r = f_*(1) \in \pi_1(X) \\ \text{let } Y = X \cup_f D \\ \text{then} \\ \qquad \pi_1(Y) = \pi_1(X)/\langle r\rangle \end{array}}$$

Proof: simple application of Seifert-van Kampen.

let $A = X \cup_f S^1 \times [1/2, 1] \simeq X$




let $B$ = disk of radius $\frac{2}{3}$

so $A \cap B = S^1 \times (\frac{1}{2}, \frac{2}{3})$

$\pi_1(B) = \{0\}$
$\pi_1(A) = \pi_1(X)$
$\pi_1(A \cap B) = \mathbb{Z} = \pi_1(\partial D^2)$

so $\pi_1(Y) = \pi_1(A) * \pi_1(B) / \langle f_*(1)e\rangle$

$= \pi_1(X)/\langle r\rangle$ ✓

so if $f: \partial D^2 \to W_k$ then

$$\pi_1(W_k \cup_f D^2) = F_k / \langle f_*(1) \rangle$$
$$= \langle g_1 \ldots g_k \mid f_*(1) \rangle$$

In general we have

Th$^{m}$ 36:

$\boxed{\text{Let } G \text{ be any group with a finite presentation. Then } \exists \text{ a topological space } X \text{ s.t. } \pi_1(X) \cong G}$

Proof: Let $\langle g_1 \ldots g_k \mid r_1 \ldots r_m \rangle$ be a presentation of $G$

Let $f_i: \partial D^2 \to W_k$ be maps s.t.

$$(f_i)_*(1) = r_i$$

Exercise: how do we get such an $f_i$?

Let $X = W_k \cup_{f_i} \left( \bigcup_{i=1}^{m} D^2 \right)$

lemma 35 $\Rightarrow$ $\pi_1(X) = G$ ✓