ERRATUM TO: "ON GENERALIZING LUTZ TWISTS"

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Abstract. In this note we point out an error in [2]. We show how to repair the proof in dimension 5. The results are true in general as can easily be seen from recent work of Borman, Eliashberg and Murphy [1].

The proof of Lemma 3.4 in [2] is incorrect. Below we will describe the problem with the proof and then show how it can easily be repaired in dimension 5. We then observe that Lemma 3.4, and thus the main results of the paper, is true in all dimensions based on recent work of Borman, Eliashberg and Murphy [1]. However this approach does not give an explicit construction and hence goes against the sprit of the original paper and in addition all the results of [2] follow directly from [1].
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1. Exact Lagrangians, Liouville flows, and the error in the proof of Lemma 3.4

We begin by recalling the statement of Lemma 3.4 from [2]. To state the lemma we first establish some notation (that is slightly different that what was used in [2]). Consider $T^2 \times [0,1]$ with coordinates (θ, ϕ, r) and the contact structure $\xi_i = \ker \alpha_i$, $i = 1, 2$, where

$$\alpha_i = k_i(r)\, d\theta + l_i(r)\, d\phi.$$

Here we have $k_1(r) = \cos \frac{\pi}{2} r$ and $l(r) = \sin \frac{\pi}{2} r$, and for $i = 2$ we have k_2 and l_2 agreeing with k_1 and l_1 near $r = 0$ and 1, and the curve $(k_2(r), l_2(r))$ in $\mathbb{R}^2$ has $5\pi/2$ winding about the origin. In particular notice that ξ_2 is obtained from ξ_1 by adding Giroux torsion. Lemma 3.4 from [2] now reads as follows.

Lemma 1. *Let W be a manifold with contact form λ, there is a contact structure on $W \times [0,1] \times ([0,1] \times T^2)$ such that the following properties are satisfied:*

1. *near $W \times \{0\} \times [0,1] \times T^2$ and $W \times [0,1] \times \{0,1\} \times T^2$ the contact structure is contactomorphic to $\lambda + e^t \alpha_1$, and*
2. *near $W \times \{1\} \times [0,1] \times T^2$ the contact structure is contactomorphic to $\lambda + e^t \alpha_2$.*

Here t is the coordinate on the first $[0,1]$ factor.

See [2] for details on how the main constructions and theorems of the paper follow from this lemma.

The strategy of the proof in [2] was:

1. To construct a contact structure on $W \times [0,1] \times T^3$ that near $W \times \{0\} \times T^3$ is given by $\lambda + e^t \beta_0$ and near $W \times \{1\} \times T^3$ is given by $\lambda + e^t \times \beta_1$, where β_i is the contact structure on T^3 with Giroux torsion i and we are thinking of T^3 as $S^1 \times T^2$ with the S^1-factors Legendrian curves.
2. Then cut $W \times [0,1] \times T^3$ along $W \times [0,1] \times (\{\theta_0, \theta_1\} \times T^2)$ so that one of the resulting pieces is as described in the lemma.

To try to arrange this let $\beta = p_1\, d\theta_1 + p_1\, d\theta_2$ be the Liouville form on $T^*T^2 = \mathbb{R}^2 \times T^2$ with coordinates $(p_1, p_2, \theta_1, \theta_2)$. Notice that $\alpha = \lambda + \beta$ is a contact form on $W \times T^*T^2$. We will see below that we can arrange the two items above that are needed for our proof if there is a radial vector field v in $\mathbb{R}^2$ centered at a point p whose flow expands $d\beta$ (that is, $L_v d\beta = d\beta$) and a Lagrangian torus T^2 in a small neighborhood of $\{q\} \times T^2 \subset T^*T^2$ that is exact with respect to $\iota_v d\beta$ that is isotopic to $\{q\} \times T^2$ by an isotopy disjoint from $\{p\} \times T^2$. One may easily arrange all of this except for either the last

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requirement of disjointness or the exactness of the Lagrangian torus. In [2] we assumed this could be arranged (though in the presentation there it was not clear these were precisely the conditions necessary), but to the best of our knowledge this cannot be done. More explicitly in [2] we took the Lagrangian torus $T^2 \times \{(0,0)\}$ and the radial vector field v to be centered at a point disjoint from the origin. Notice that the torus is exact with respect to β but not with respect to $\iota_v d\beta$, and thus the construction does not work. In the next section we will see that the condition of having an exact Lagrangian torus can be removed in the 5 dimensional setting. (For simplicity or presentation below we will take $p = (0,0)$.)

Let $X = D \times T^2$ where D is a disk of radius R about the origin in $\mathbb{R}^2$ and $R > 2$ is some constant. The form β restricts to the contact form β_0 on $T^3 = \partial X$. Now let $X' = D' \times T^2$ where D' is a small disk about the origin in $\mathbb{R}^2$. By noticing that since the radial vector field $v = p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$ is the Liouville field for β on T^*T^2 one easily sees that $\overline{X - X'}$ is a piece of the symplectization of the minimally twisting tight contact structure on T^3.

Let γ_1 be the intersection of D with the ray leaving the origin in $\mathbb{R}^2$ that forms an angle of $\pi/2$ with the positive p_1-axis and similarly let γ_2 be the intersection of D with the ray in $\mathbb{R}^2$ that forms an angle of $-3\pi/4$ with the positive p_1-axis. Setting Y equal to the component of $\overline{X - X'}$ cut along the $\gamma_i \times T^2$ that contains points lying above the negative p_1-axis in $\mathbb{R}^2$, one easily sees that Y is a piece of the symplectization of α_1 (using the notation from above) on $T^2 \times [0, \pi/2]$ (here we have rescaled $[0,1]$ to $[0, \pi/2]$) and the parts of Y lying above γ_1 and γ_2 are the "vertical" or flat boundaries of the symplectization).

If there is an exact Lagrangian torus T^2 in T^*T^2 as discussed above then Lemma 3.6 in [2] shows there is an embedding $\Phi\colon W \times T^2 \to W \times T^*T^2$ so that $\Phi(W \times T^2)$ is contact, disjoint from $Z = W \times \{(0,0)\} \times T^2$ and isotopic to $F_q = W \times \{q\} \times T^2$ in the complement of (a neighborhood of) Z. (For convenience we take q to be a point on the negative p_1-axis in D. If this were not the case we might need to re-choose the γ_i.) We can then let C be the 2-fold cover of $W \times X$ branched over $\Phi(W \times T^2)$. It is well known that C has a contact structure that away from the branched locus is just the lift of the contact structure on $W \times X$, see for example [3, Theorem 7.5.4]. Moreover it is clear that the cover is diffeomorphic to $W \times X$. Notice that the boundary of C is $W \times T^3$ and in a neighborhood of the boundary the contact structure is simply W times a piece of the symplectization of the Giroux torsion 1 contact structure on T^3, so that the boundary of C is W times the convex end of the piece of the symplectization.

Notice that we can take $W \times X'$ to be a neighborhood of Z in $W \times X$ that is disjoint from $\Phi(W \times T^2)$ and the isotopy of $\Phi(W \times T^2)$ to F_q. Clearly $W \times X'$ lifts to two disjoint copies of $W \times X'$ in C. Let N be one of these and set $C' = C \setminus N$. It is clear that $\partial C' - \partial C$ has a neighborhood in C' where the contact structure looks like W times a piece of the symplectization of the standard minimally twisting contact structure on T^3, so that the boundary component is W times the concave end of the piece of the symplectization. Furthermore notice that each $\gamma_i \times T^2$ lifts to two copies in C'. The copies that intersect with N will divide C' into two pieces. Let C'' be the piece that contains the branch locus. Notice that the branched covering map restricted to a neighborhood of $(\partial C'' - ((\partial C) \cap C''))$ in C'' is mapped diffeomorphically (and contactomorphically) to a neighborhood of $(\partial Y - ((\partial X) \cap Y))$ in Y. Moreover the remaining boundary component of C'' can easily be seen to have a neighborhood that is contactomorphic to W times a piece of the symplectization of α_2. From this discussion it should be clear that the contact structure on W'' is the structure described in the lemma.

2. Fixing the error in dimension 5

In this section we show how to fix the proof of Lemma 3.4 from [2] in the 5 dimensional case.

Lemma 2. *There is a contact structure on $S^1 \times [0,1] \times ([0,1] \times T^2)$ such that the following properties are satisfied:*

1. *near $S^1 \times \{0\} \times [0,1] \times T^2$ and $S^1 \times [0,1] \times \{0,1\} \times T^2$ the contact structure is contactomorphic to $d\theta + e^t \alpha_1$, and*
2. *near $S^1 \times \{1\} \times [0,1] \times T^2$ the contact structure is contactomorphic to $d\theta + e^t \alpha_2$,*

where θ is the angular coordinate on S^1 and t is the coordinate on the first $[0,1]$ factor.

Proof. We will be considering $S^1 \times T^*T^2$ with the contact structure $\alpha = d\theta + p_1\, d\theta_1 + p_2\, d\theta_2$. From the discussion in the previous section we only need to check that there is an embedding $\Phi\colon T^3 \to S^1 \times T^*T^2$ so that $\Phi(T^3)$ is contact, disjoint from $Z = S^1 \times \{(0,0)\} \times T^2$ and isotopic to $F_q = S^1 \times \{q\} \times T^2$ in the complement of (a neighborhood of) Z where $q = (-1-\epsilon, 0)$, for some small $\epsilon > 0$, is a point in $\mathbb{R}^2$.

Using coordinates (ϕ, ϕ_1, ϕ_2) on T^3 we define

$$\Phi(\phi, \phi_1, \phi_2) = (\phi + \phi_2, \sin\phi, -1-\epsilon+\cos\phi, \phi_1, \phi_2).$$

Now we see

$$\begin{aligned}\beta = \Phi^*\alpha &= d\phi + d\phi_2 + (\sin\phi)\, d\phi_1 + (-1-\epsilon)\, d\phi_2 + (\cos\phi)\, d\phi_2 \\ &= d\phi + (\sin\phi)\, d\phi_1 + (\cos\phi)\, d\phi_2 - \epsilon\, d\phi_2.\end{aligned}$$

and

$$d\beta = (\cos\phi)\, d\phi \wedge d\phi_1 - (\sin\phi)\, d\phi \wedge d\phi_2.$$

Thus

$$\begin{aligned}\beta \wedge d\beta &= (\sin^2\phi + \cos^2\phi)\, d\phi \wedge d\phi_1 \wedge d\phi_2 - \epsilon(\cos\phi)\, d\phi \wedge d\phi_1 \wedge d\phi_2 \\ &= (1-\epsilon\cos\phi)\, d\phi \wedge d\phi_1 \wedge d\phi_2.\end{aligned}$$

Since $(1-\epsilon\cos\phi) > 0$ we have a contact embedding. Also note

$$\Phi_\delta(\phi, \phi_1, \phi_2) = (\phi + \phi_2, \delta(\sin\phi), -1-\epsilon+\delta(\cos\phi), \phi_1, \phi_2)$$

is an isotopy from Φ to a map with image the T^3 above $(0, -1-\epsilon)$ in $\mathbb{R}^2$ and the isotopy is disjoint from the T^3 above $(0,0)$. □

3. Overtwisted contact structure approach

In this section we show that Lemma 3.4 from [2], recalled as Lemma 1 above, is indeed true due to Borman, Eliashberg, and Murphy's recent breakthrough [1].

We first note that Lemma 1 explicitly defines a contact structure near the boundary of $W \times [0,1] \times ([0,1] \times T^2)$. It is easy to check that α_2 and α_1 are homotopic, rel boundary, as plane fields, *cf.* [3, Lemma 4.5.3]. Let α_t be the homotopy. Now $e^{-t}\lambda + \alpha_{f(t)}$, for some function $f(t)$, extends the contact form from a neighborhood of the boundary of $W \times [0,1] \times ([0,1] \times T^2)$ to a nonsingular form on the whole manifold. Moreover its kernel splits as $\xi^2 \oplus \xi'$ where ξ^2 is contained in the tangent space of $[0,1] \times T^2$ and ξ' projects isomorphically onto the tangent space of $W \times [0,1]$. Thus ξ' inherits a complex structure from $e^{-t}\lambda$ and ξ^2 inherits one as an oriented plane field. Thus we have constructed an almost contact structure on $W \times [0,1] \times ([0,1] \times T^2)$ that extends our given contact structure. The main result of [1] implies this almost contact structure is homotopic to an actual contact structure by a homotopy that is fixed outside any open neighborhood of the "non-contact" region. The resulting contact structure can be taken to be the one promised by Lemma 3.4 in [2].

References

[1] M. Borman, Y. Eliashberg, and E. Murphy. Existence and classification of overtwisted contact structures in all dimensions. *ArXiv e-prints*, April 2014.

[2] John B. Etnyre and Dishant M. Pancholi. On generalizing Lutz twists. *J. Lond. Math. Soc. (2)*, 84(3):670–688, 2011.

[3] Hansjörg Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

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