ON THE NON-EXISTENCE OF TIGHT CONTACT STRUCTURES

JOHN B. ETNYRE AND KO HONDA

ABSTRACT. We exhibit a 3-manifold which admits no tight contact structure.

1. INTRODUCTION

In 1971, Martinet [16] showed that every 3-manifold admits a contact structure. But in the subsequent twenty years, through the work of Bennequin [2] and Eliashberg [5], it became apparent that not all contact structures are created equal in dimension 3. Specifically, contact structures fall into one of two classes: tight or overtwisted. In this new light, what Martinet actually showed was that every 3-manifold admits an overtwisted contact structure. In [4] Eliashberg classified overtwisted contact structures on closed 3-manifolds by proving the weak homotopy equivalence of the space of overtwisted contact structures up to isotopy and the space of 2-plane fields up to homotopy — hence overtwisted contact structures could now be understood via homotopy theory. On the other hand, it has become apparent that tight contact structures have surprising and deep connections to the topology of 3-manifolds, not limited to homotopy theory. For example, Rudolph [17] and Lisca and Matić [14] found connections with slice knots and slice genus, Kronheimer and Mrowka [12] found connections with Seiberg-Witten theory, and Eliashberg and Thurston [6] found connections with foliation theory. Thus, whether or not every 3-manifold admits a tight contact structure became a central question in 3-dimensional contact topology.

The first candidate for a 3-manifold without a tight contact structure was the Poincaré homology sphere $M = \Sigma(2, 3, 5)$ with reverse orientation. The difficulty of constructing a holomorphically fillable contact structure on M was highlighted in Gompf's paper [8]. Subsequently

Lisca [13], using techniques from Seiberg-Witten theory, proved that M has no symplectically semi-fillable contact structure.

In this paper we prove the following nonexistence result.

Theorem 1. *There exist no positive tight contact structures on the Poincaré homology sphere $\Sigma(2,3,5)$ with reverse orientation.*

This is the first example of a closed 3-manifold which does not carry a positive tight contact structure.

Corollary 2. *Let M be the Poincaré homology sphere with reverse orientation. Then the connect sum $M\#\overline{M}$, where $\overline{M}$ is M with the opposite orientation, does not carry any tight contact structure, positive or negative.*

This follows from Theorem 1, since a tight structures on a reducible manifold may be decomposed into tight structures on its summands [3, 15]

2. CONTACT TOPOLOGY PRELIMINARIES

We assume the reader is familiar with the basics ideas of contact topology in dimension 3 (see for example [1, 5]). A thorough understanding of [9] would be helpful but we include a brief summary of the ideas and terminology. The reader might also find [7, 10] useful for various parts of this section.

In this paper the ambient manifold M will be an oriented 3-manifold, and the contact structure ξ will be *positive*, *i.e.*, given by a 1-form α with $\alpha \wedge d\alpha > 0$. Throughout this section we only consider (M, ξ) *tight*. Also, when we refer to *Legendrian curves* we mean closed curves, in contrast to *Legendrian arcs*.

2.1. **Convexity.** Recall an oriented embedded surface Σ in (M, ξ) is called *convex* if there is a vector field v transverse to Σ whose flow preserves ξ. Perhaps the most important feature of convex surfaces is the *dividing set*. If $\mathcal{F}$ is a singular foliation on Σ then a disjoint union of (properly) embedded curves Γ is said to *divide* $\mathcal{F}$ if Γ divides Σ into two parts $\Sigma^{\pm}$, Γ is transverse to $\mathcal{F}$, and there is a vector field X directing $\mathcal{F}$ and a volume form ω on Σ such that $\pm L_X \omega > 0$ on $\Sigma^{\pm}$ and X points transversely out of Σ^+. If α is a contact 1-form for ξ then the zeros of $\alpha(v)$ provide dividing curves for the characteristic foliation Σ_{ξ}. It is sometimes useful to keep in mind that the dividing curves are where v is tangent to ξ. An isotopy $F : \Sigma \times [0, 1] \to M$ of Σ is called *admissible* if $F(\Sigma \times \{t\})$ is transversal to v for all t. An important result concerning convex surfaces we will (implicitly) be using through out this paper is:

Theorem 3 (Giroux [7]). *Let Γ be the dividing set for Σ_ξ and $\mathcal{F}$ another singular foliation on Σ divided by Γ. Then there is an admissible isotopy F of Σ such that $F(\Sigma \times \{0\}) = \Sigma$, $F(\Sigma \times \{1\})_\xi = \mathcal{F}$ and the isotopy is fixed on Γ.*

Roughly speaking, this says that the dividing set Γ dictates the geometry of Σ, not the precise characteristic foliation.

We will let $\#\Gamma$ denote the number of connected components of Γ. If there is any ambiguity, we will also write Γ_Σ instead of Γ to denote the dividing set of Σ.

2.2. **Edge-rounding.** Let Σ_1 and Σ_2 be compact convex surfaces with Legendrian boundary, which intersect transversely along a common boundary Legendrian curve L. The neighborhood of the common boundary Legendrian is locally isomorphic to the neighborhood $\{x^2 + y^2 \leq \varepsilon\}$ of $M = \mathbf{R}^2 \times (\mathbf{R}/\mathbf{Z})$ with coordinates (x, y, z) and contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$, for some $n \in \mathbf{Z}^+$. Let $A_i \subset \Sigma_i$, $i = 1, 2$, be an annular collar of the boundary component L. We may choose our local model so that $A_1 = \{x = 0, 0 \leq y \leq \varepsilon\}$ and $A_2 = \{y = 0, 0 \leq x \leq \varepsilon\}$ (or the same with A_1 and A_2 switched). Assuming the former, if we join Σ_1 and Σ_2 along $x = y = 0$ and round the common edge, the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on Σ_1 will connect to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, \cdots, 2n - 1$.

2.3. **Bypasses.** We now introduce the main idea on which this work is based. Let L be a Legendrian arc. A half-disk D is called a *bypass for L* if ∂D is Legendrian and consists of two arcs $a_0 = \partial D \cap L$ and a_1 such that, if we orient D, $a_0 \cap a_1$ are both positive (negative) elliptic singular points in D_ξ, there is a negative (positive) elliptic point along a_0, and the singular points along a_1 are all positive (negative) and alternate between elliptic and hyperbolic. See Figure 1. We also allow
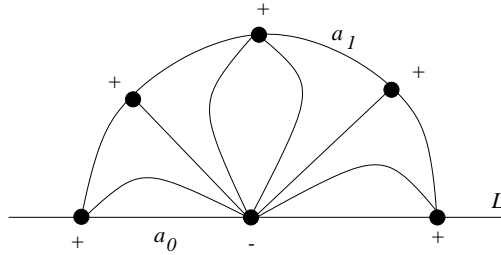

FIGURE 1. A bypass

for the degenerate case when L is a closed curve and the endpoints of

a_0 are the same (*i.e.*, $L = a_0$). We refer to such a bypass as a *bypass of degenerate type*. Once an orientation on D is fixed, the *sign* of the bypass is the sign of the elliptic point on the interior of a_0. The reason for the name 'bypass' is that, instead of traveling along the Legendrian arc L, we may go around and drive through $L' = (L \backslash a_0) \cup a_1$ — this has the effect of increasing the twisting of ξ along the Legendrian curve. (Here we are using the convention that left twists are negative.) One can then show:

Theorem 4 (Honda [9]). *Assume M has a convex boundary Σ and there exists a bypass D along the Legendrian curve $L \subset \Sigma$. Then we can find a neighborhood N of $\Sigma \cup D$ with $\partial N = \Sigma - \Sigma'$, and $\Gamma_{\Sigma'}$ is related to Γ_Σ as shown in Figure 2.*
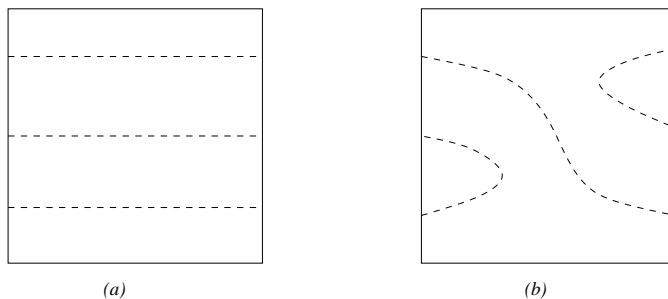

FIGURE 2. (a) Dividing curves (dashed lines) on Σ, (b) Dividing curves on Σ'

Consider a convex torus T. If ξ is tight, then one can show that no dividing curve for T bounds a disk. Thus Γ_T consists of $2n$ parallel dividing curves with $n \in \mathbf{Z}^+$. Assume for now that the dividing curves are all horizontal. Using Theorem 3 we can isotop T so that there is one closed curve of singularities in T_ξ in each region of $T \backslash \Gamma_T$ — these are called *Legendrian divides* and are parallel to the dividing curves. We can further assume all the other leaves of T_ξ form a 1-parameter family of parallel closed curves transverse to the Legendrian divides. These *Legendrian ruling curves* can have any slope not equal to the slope of the dividing curves. A convex torus T is said to be in *standard form* if T_ξ has this nongeneric form, consisting of Legendrian divides and Legendrian ruling curves. Suppose $n > 1$ and we find a bypass for one of the ruling curves for T then we may isotop T across the bypass as in Theorem 4 and reduce the number n by one. If $n = 1$ we have a configuration change as follows:

Theorem 5 (Honda [9]). *Let T be a convex torus with $\#\Gamma_T = 2$ and in some basis for T the slope of the dividing curves is 0. If we find a*

bypass for a ruling curve of slope between $-\frac{1}{m}$ and $-\frac{1}{m+1}$, $m \in \mathbf{Z}$, then after pushing T across the bypass the new torus has two dividing curves of slope $-\frac{1}{m+1}$.

2.4. **Legendrian curves and the twisting number.** Let γ be a Legendrian curve in M. We can always find a neighborhood N of γ whose boundary is a convex torus with two dividing curves. The linking of a dividing curve on ∂N with γ is the Thurston-Bennequin invariant of γ. We call the slope of these dividing curves the *boundary slope* of N. If γ is not null-homologous, then the framing induced on γ relative to some pre-assigned trivialization of the normal bundle of γ will be called the *twisting number* of γ. In contrast to transverse curves, neighborhoods of Legendrian curves have a quantifiable thickness — no matter how small a (nice) neighborhood of a Legendrian curve one takes, the boundary slope of the neighborhood is fixed. If N is the neighborhood of a Legendrian curve with twisting number m (relative to a framing that is already fixed), then the slopes of the dividing curves on ∂N are $\frac{1}{m}$. On the other hand, any tight contact structure on the solid torus $S^1 \times D^2$ with boundary slope $\frac{1}{m}$ is contact isotopic to a neighborhood of a Legendrian curve with twisting number m, see [10].

It is easy to decrease the twisting number inside N by adding 'zigzags' (see below for an explanation of terminology). Hence, increasing the twisting number is one way of thickening N. We now show how to use bypasses to increase the twisting number of γ.

Lemma 6 (Twist Number Lemma: Honda [9])**.** *Let γ be a Legendrian curve in M with a fixed framing. Let N be a standard neighborhood of γ and n the twisting number of γ. If there exists a bypass attached to a Legendrian ruling curve of ∂N of slope r and $\frac{1}{r} \geq n + 1$, then there exists a Legendrian curve with larger twisting number isotopic to γ.*

This lemma is a very useful formulation of the observation that if one has a bypass for a Legendrian knot then one can increase its twisting number.

2.5. **Finding bypasses.** Now that we see bypasses are useful let us consider how to find them. Let Σ be a convex surface with Legendrian boundary L. If the Thurston-Bennequin invariant of L (or let us say twisting number with respect to Σ) is negative then we can arrange that all the singularities of Σ_ξ along L are (half)-elliptic. Moreover we can assume the singular foliation Σ_ξ is Morse-Smale. If n is the twisting number of L, then Γ_Σ intersects L at $2n$ points. In this situation we can use the flow of X (the vector field directing Σ_ξ in the definition of the dividing set) to flow any dividing curve to a Legendrian arc on Σ,

all of whose singularities are of the same sign. Now, starting with a dividing curve c with endpoints along L isotopic to the boundary of Σ, we push c 'back into Σ along X' and eventually arrive at a Legendrian arc which cuts a bypass for L off of Σ. Thus to find bypasses we need only look for these ∂-compressible dividing curves.

2.6. **Layering neighborhoods of Legendrian knots.** We now explore the neighborhood of a Legendrian knot in more detail. To this end let N be a standard neighborhood of a Legendrian knot γ with a fixed framing so that $N = S^1 \times D^2$ and γ has twisting number $n \leq -1$ in this framing. Recall this means that ∂N is convex with two dividing curves (and two Legendrian divides) of slope $\frac{1}{n}$. Inside of N we can isotop γ (but not Legendrian isotop) to γ' so as to decrease the twisting number by one. There are actually two different ways to do this. If γ were a knot in $\mathbf{R}^3$ with the standard tight contact structure then this process corresponds to stabilizing the knot (reducing the Thurston-Bennequin invariant) by adding zigzags, and one can do this while increasing or decreasing the rotation number by one.

To see how to detect this difference in N we do the following: first, fix an orientation on N and orient γ so that γ and $\{pt\} \times D^2$ intersect positively. Let N' be a standard neighborhood of γ'. Then consider the layer $U = T^2 \times [0, 1] = N \backslash N'$, where we set $T^2 \times \{0\} = \partial N'$ and $T^2 \times \{1\} = \partial N$. We assume the Legendrian ruling curves are vertical so that $A = S^1 \times [0, 1]$ is an annulus with Legendrian boundary. Note that $S^1 \times \{0\}$ intersects the dividing curves on $\partial N'$, $2n - 2$ times and $S^1 \times \{1\}$ intersects the dividing curves on ∂N, $2n$ times. So there will be a bypass (in fact just one bypass) on A for $S^1 \times \{0\}$. If A is oriented so that the orientation on $S^1 \times \{0\}$ agrees with the one chosen above on γ, then the sign of the bypass is what distinguishes the two possible γ''s. If c is a curve on $S^1 \times \{0\}$ whose slope is not $\frac{1}{n}$ or $\frac{1}{n+1}$ then we can make the Legendrian ruling curves on ∂U parallel to c. Later it will be useful to know what the dividing curves on $c \times [0, 1]$ will look like. The relative Euler class in the next paragraph can be useful for this purpose.

2.7. **Euler class.** Let ξ be any tight structure on a toric annulus $U = T^2 \times [0, 1]$ (not necessarily the same one as in the above paragraphs). Assume the boundary is convex and in standard form. Let v_b be a section of $\xi|_{\partial U}$ which is transverse to and twists (with ξ) along the Legendrian ruling curves. We also take v_b to be tangent to the Legendrian divides. We may now form the relative Euler class $e(\xi, v_b)$ in $H^2(U, \partial U; \mathbf{Z})$. First note that $e(\xi, v_b)$ is unchanged if we perform a C^0-small isotopy of ∂U so as to alter the slopes of the ruling curve.

Now given an oriented curve c on $T^2 \times \{0\}$ we can assume the annulus $A = c \times [0,1]$ has Legendrian boundary and is also convex. We orient A so that it induces the correct orientation on c. Now we have

$$(1) \qquad e(\xi, v_b)(A) = \chi(A_+) - \chi(A_-),$$

where $A_\pm$ are the positive and negative regions into which the dividing curves cut A. This formula follows from Proposition 6.6 in [11] once one observes that v_b may be homotoped in $\xi|_{\partial U}$ so as to be tangent to and define the correct orientation on ∂A.

2.8. **Twisting.** We end this section by making precise the notion of *twisting* along $T^2 \times [0,1]$.

Let ξ_1 be the kernel of $\alpha_1 = \sin z \, dx + \cos z \, dy$ on $T^3 = T^2 \times S^1 = (\mathbf{R}^2/\mathbf{Z}^2) \times S^1$ with coordinates $((x,y),z)$. This contact structure is tight. The characteristic foliation on $T^2 \times \{p\}$ is by lines of a fixed slope. The slope "decreases" as p moves around S^1 in the positive direction. Let $[a,b]$ be an interval in $[0,\pi]$ so that at one end $T_a = T^2 \times \{a\}$ has slope $\frac{1}{n}$ and at the other end $T_b = T^2 \times \{b\}$ has slope $\frac{1}{n+1}$. We can C^∞-perturb T_a and T_b so that they are convex and each has two dividing curves of slope $\frac{1}{n}$ and $\frac{1}{n+1}$, respectively.

Let $U = T^2 \times [0,1]$ be the layer between N' and N as in 2.6. It is possible to show that $T^2 \times [a,b]$ is contactomorphic to U with the contact structure constructed above (see [9]). The slopes of the characteristic foliations on $T^2 \times \{pt\}$ vary from $\frac{1}{n}$ to $\frac{1}{n+1}$ in $T^2 \times [a,b]$ so they do likewise in U. From this one can conclude that if N is a standard neighborhood of a Legendrian knot γ with twist number n (with respect to some fixed framing), then for any slope in $[\frac{1}{n}, 0)$ one can find a torus around γ whose characteristic foliation has this slope (note if $n > 0$ then $[\frac{1}{n}, 0)$ means $[-\infty, 0) \cup [\frac{1}{n}, \infty]$ where $-\infty$ is identified with ∞).

More generally, if $T^2 \times [0,1]$ has boundary slopes s_i for $T^2 \times \{i\}$, then we may find convex tori parallel to $T^2 \times \{i\}$ with any slope s in $[s_1, s_0]$ (if $s_0 < s_1$ then this means $[s_1, \infty] \cup [-\infty, s_0]$). This follows from the classification of tight contact structures on $T^2 \times I$ — in the proof we layer $T^2 \times I$ into 'thin' toric annuli, each of which is isomorphic to $T^2 \times [a,b]$ above (see [9]).

3. THICKENING THE SINGULAR FIBERS

Consider the Seifert fibered space M with 3 singular fibers over S^2. M is described by the Seifert invariants $(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})$. Let V_i, $i = 1, 2, 3$, be the neighborhoods of the singular fibers F_i, isomorphic to $S^1 \times D^2$ and identify $M \setminus (\cup_i V_i)$ with $S^1 \times \Sigma_0$, where Σ_0 is a

sphere with three punctures. Then $A_i : \partial V_i \to -\partial(M \backslash V_i)$ is given by
$A_i = \begin{pmatrix} \alpha_i & \gamma_i \\ -\beta_i & \delta_i \end{pmatrix} \in SL(2, \mathbf{Z})$. We identify $\partial V_i = \mathbf{R}^2/\mathbf{Z}^2$, by choosing $(1,0)^T$ as the meridional direction, and $(0,1)^T$ as the longitudinal direction with respect to the product structure on V_i. We identify $-\partial(M \setminus V_i) = \mathbf{R}^2/\mathbf{Z}^2$, by letting $(0,1)^T$ be the direction of an S^1-fiber, and $(1,0)^T$ be the direction given by $\partial(M \setminus V_i) \cap (\{pt\} \times \Sigma_0)$.

Let M be the Poincaré homology sphere $\Sigma(2,3,5)$ with reverse orientation. It is a Seifert fibered space over S^2 with Seifert invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$. In the case $V = V_1$, we choose $A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$. Notice there we has some freedom to choose γ_1, δ_1, since we could have postmultiplied by $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ if we changed our framing for V_1. Similarly, let $A_2 = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, and $A_3 = \begin{pmatrix} 5 & 1 \\ -1 & 0 \end{pmatrix}$.

Now let ξ be a positive contact structure on M. Assume ξ is tight. The goal of the paper is to obtain a contradiction by finding overtwisted disks inside M.

In the first stage of the proof, we will try to thicken neighborhoods V_i of the singular fibers F_i (we may assume the singular fibers are Legendrian after isotopy). This is done by maximizing the twisting number m_i among Legendrian curves isotopic to F_i, subject to the condition that all three Legendrian curves be simultaneously isotopic to (F_1, F_2, F_3). Let V_i be a standard tubular neighborhood of F_i with minimal convex boundary and boundary slope $\frac{1}{m_i}$ (note this is negative).

It is useful to note how the dividing curves map under A_i. $A_1 : (m_1, 1)^T \mapsto (2m_1 - 1, m_1)^T$, $A_2 : (m_2, 1)^T \mapsto (3m_2 + 1, -m_2)^T$, and $A_3 : (m_3, 1)^T \mapsto (5m_3 + 1, -m_3)^T$. Therefore, the boundary slopes are $\frac{m_1}{2m_1-1}$, $-\frac{m_2}{3m_2+1}$ and $-\frac{m_3}{5m_3+1}$, when viewed on $-\partial(M \setminus V_i)$.

Warning: We will often call the same surface by different names, such as ∂V_i and $-\partial(M \setminus V_i)$. Although the surfaces themselves are the same, their *identifications* with $\mathbf{R}^2/\mathbf{Z}^2$ are not. Therefore, when we refer to slopes on ∂V_i, we implicitly invoke the identification of ∂V_i with $\mathbf{R}^2/\mathbf{Z}^2$ given in the first paragraph of this section.

Lemma 7. *We can increase m_i so that $m_1 = 0$, $m_2 = m_3 = -1$, and thicken V_i to V_i' so that the slopes of $\partial(M \setminus V_i')$ are all infinite.*

Proof. We may modify the Legendrian rulings on both $\partial(M \setminus V_2)$ and $\partial(M \setminus V_3)$ so that they are vertical. Take a vertical annulus $S^1 \times I$

spanning from a vertical Legendrian ruling curve on $\partial(M\backslash V_2)$ to a vertical Legendrian ruling curve on $\partial(M\backslash V_3)$. Here 'vertical' means 'in the direction of the S^1-fibers'. Note $S^1 \times I$ intersects the dividing curves on ∂V_2 and ∂V_3, $3m_2 + 1$ and $5m_3 + 1$ times respectively.

Assume $m_2, m_3 \leq -1$. If $3m_2 + 1 \neq 5m_3 + 1$, then there exists a bypass (due to the imbalance), attached along a vertical Legendrian curve of $\partial(M\backslash V_2)$ or $\partial(M\backslash V_3)$. We transform $(0, 1)^T$ via A_i^{-1} to use the Twist Number Lemma. $A_2^{-1} : (0, 1)^T \mapsto (-1, 3)^T$, and $A_3^{-1} : (0, 1)^T \mapsto (-1, 5)^T$. The Legendrian rulings will have slope -3 and -5, and, therefore, we may increase the twisting number by 1 if the twisting number is < -1.

Next, assume $3m_2 + 1 = 5m_3 + 1$ and there are no bypasses on the vertical annulus. Then we may take $S^1 \times I$ to be standard, with vertical rulings, and parallel Legendrian divides spanning from ∂V_2 to ∂V_3. Cutting along $S^1 \times I$ and rounding the corners, we obtain the torus boundary of $M \setminus (V_2 \cup V_3 \cup (S^1 \times I))$; if we identify this torus with $\mathbf{R}^2/\mathbf{Z}^2$ in the same way as $\partial(M\backslash V_1)$, then the boundary slope is $-\frac{m_2+m_3+1}{3m_2+1} = -\frac{\frac{8}{5}m_2+1}{3m_2+1}$. When $m_2 = -5$, then the slope is $-\frac{1}{2}$, and any Legendrian divide gives rise to an overtwisted disk — this is because $A_1 : (1, 0)^T \mapsto (2, 1)^T$, which corresponds to a slope of $-\frac{1}{2}$ on $\partial(M\backslash V_1)$. When $m_2 < -5$, then the slope is $< -\frac{1}{2}$, which implies that on the $S^1 \times D^2 = M \setminus (V_2 \cup V_3 \cup (S^1 \times I))$, the twisting in the radial direction is almost π. In particular, there exists a convex torus with slope ∞ inside $S^1 \times D^2$, and hence a vertical Legendrian curve with zero twisting, obtained as a Legendrian divide of the convex torus. Any vertical annulus taken from this vertical Legendrian to V_2 (or V_3) will give bypasses.

We can now assume $m_2 = m_3 = -1$. The boundary slopes of $-\partial(M \setminus V_2)$ and $-\partial(M \setminus V_3)$ are $-\frac{1}{2}$ and $-\frac{1}{4}$, respectively. Again look at the vertical annulus $S^1 \times I$ spanning from V_2 to V_3, with Legendrian boundary. There are three possibilities: (1) There are no bypasses along $S^1 \times \{0\}$ (on the V_2 side) — in this case, we can cut along the annulus as before, and get boundary component with boundary slope $-\frac{1}{2}$, contradicting tightness. (2) There is one bypass along $S^1 \times \{0\}$ — cutting along the annulus again, we find that the boundary slope is -1. This means that the twisting of $T^2 \times I = M \setminus (V_1 \cup V_2 \cup V_3 \cup (S^1 \times I))$ is large, and that we have a convex torus in standard form with Legendrian divides of infinite slope as before. (3) There are two bypasses.

In any case, there exist vertical Legendrian curves with twisting number 0 with respect to the framing from the fibers. Since $A_1^{-1} : (0, 1)^T \mapsto$

$(1,2)^T$, we have bypasses for V_1 as well, and we can increase to $m_1 = 0$ using the Twist Number Lemma.

Next, taking a vertical annulus from V_2 to one such vertical Legendrian curve, we obtain two bypasses for V_2 and, similarly, we obtain four bypasses for V_3. By attaching these two bypasses, we obtain a thickening V_i' of V_i, $i = 1, 2, 3$, so that $\partial(M \setminus V_i)$ has two vertical dividing curves. $\qquad\qquad\square$

4. THE FIBRATION

In the section we use the structure of $M \setminus V_3$ as a punctured torus bundle over S^1 to complete the proof of our main theorem. The strategy is as follows: First we use the bundle structure to increase m_3 to 0 by finding a bypass along the boundary of the punctured torus fiber. We then show how to increase m_3 to 1 by making the boundary of the punctured torus fiber Legendrian with twist number 0. When $m_3 = 1$ the corresponding neighborhood of F_3 almost contains an overtwisted disk. By increasing it a little further it does contain one. This will complete the proof of our main theorem.

4.1. **Bundle structure.** We now describe the fibration as a punctured torus bundle over S^1. In the previous section, V_i was the neighborhood of a Legendrian curve F_i with twisting number m_i, and V_i' its thickening. Write $M \setminus (\cup_i V_i') = S^1 \times \Sigma$, where Σ is a 3-holed sphere. Let γ be an embedded arc on Σ connecting V_2' to V_1', $A = S^1 \times \gamma$ an annulus connecting V_2' to V_1', and V a neighborhood of A in $M \setminus (V_1' \cup V_2')$. Define $M' = V_1' \cup V_2' \cup V$, which is $M \setminus V_3$ with a $T^2 \times I$ layer removed from the boundary. Let D_1 and D_2 be meridional disks for V_1' and V_2', respectively. The slope of D_1 on $\partial(M \setminus V_1')$ is $-\frac{1}{2}$ and the slope of D_2 on $\partial(M \setminus V_2')$ is $\frac{1}{3}$. Thus we can take two copies of D_2 and three copies of D_1 and glue them together with six copies of $V \cap \Sigma$ to obtain a punctured torus T in M' with boundary on $\partial M'$. (See Figure 7.) Parallel copies of T will fiber M' (and $M \setminus V_3$) as a punctured torus bundle over S^1. The slope of ∂T on $-\partial(M \setminus V_3)$ is $-\frac{1}{6}$. Thus the slope on ∂V_3 is 1, so T provides a Seifert surface for F_3.

We can show (for example by using Kirby calculus) that M, represented in Figure 3(a), is diffeomorphic to Figure 3(b), and that $M \setminus V_3$ is diffeomorphic to the complement of the right-handed trefoil knot in S^3. Thus we may identify the monodromy of the punctured torus bundle as

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$
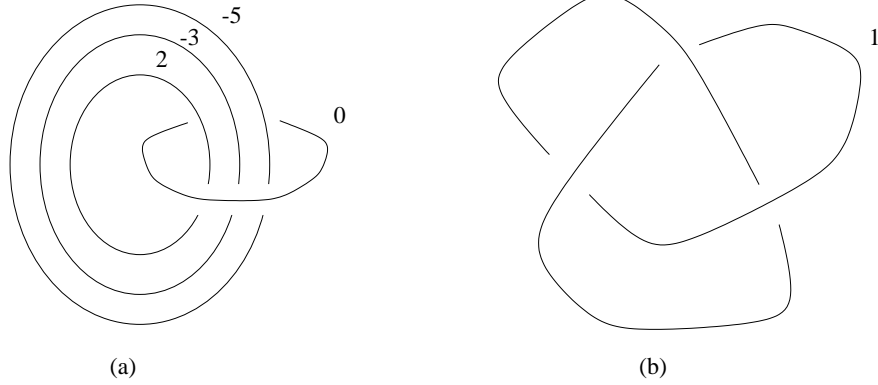
FIGURE 3. Two surgery pictures for M.

4.2. **Dividing curves on the punctured torus fiber.** Recall we have arranged that $m_3 = -1$, so the dividing curves on ∂V_3 have slope -1. Thus, if we make ∂T a Legendrian curve on ∂V_3, then $tb(\partial T) = -2$. Here is a preliminary lemma:

Lemma 8. *We can always find a bypass along ∂T (after possibly isotoping T).*

Proof. We begin by showing that either we can find a bypass for ∂T or arrange that the dividing curves on T consist of exactly two parallel arcs. If there are no bypasses, then the dividing set must consist of two parallel arcs and an even number, say $2m$, of closed parallel curves. Consider $M'\backslash T = T \times [0,1]$ with T identified with $T \times \{0\}$. Here we are viewing M' as $M\backslash V_3$. Let α be a closed Legendrian curve on T parallel to the dividing curves. Then consider $\mathcal{A} = \alpha \times [0,1]$ which we may assume is convex with Legendrian boundary. The dividing curves do not intersect $\alpha \times \{0\}$, but they intersect $\alpha \times \{1\}$ at least $2m$ times. Thus we may find a bypass D for $T \times \{1\}$. The inner boundary T' of a neighborhood of $(T \times \{1\}) \cup D$ is a convex torus with either a bypass for $\partial T'$ or two fewer dividing curves than T. Repeating this argument m times will result in a bypass for ∂T or a convex torus whose dividing set consist of exactly two parallel arcs.

Now suppose Γ_T consists of two parallel arcs, since otherwise we are done. Then $\Gamma_{T \times \{0\}}$ consists of two arcs and $\Gamma_{T \times \{1\}}$ consists of the image of these two arcs under the monodromy map for the bundle. Let α be a closed curve on $T \times \{0\}$ parallel to the dividing curves, and $\mathcal{A} = \alpha \times [0,1]$. We may arrange for $\alpha \times \{0,1\}$ to be Legendrian and for $\mathcal{A}$ to be convex. Now the dividing curves on $\mathcal{A}$ do not intersect $\alpha \times \{0\}$ but intersect $\alpha \times \{1\}$ at least two times. Thus we have a bypass D for the dividing curves on $T \times \{1\}$.

Assume first that the intersection number is exactly two. Then the annulus $\mathcal{A}$ may be split into two annuli, one of which, call it $\mathcal{A}'$, intersects $T \times \{1\}$ and contains D, which is of degenerate type. The boundary of a small neighborhood of $\mathcal{A}'$ in $T \times (0, 1)$ is an annulus $\mathcal{A}''$ and has dividing curves as shown in Figure 4. The boundary of $\mathcal{A}''$ sits on $T \times \{1\}$. If we cut out the annulus in $T \times \{1\}$ that cobounds a solid torus with $\mathcal{A}''$, glue in $\mathcal{A}''$, and smooth corners, then the dividing curves on the new $T \times \{1\}$ are shown in Figure 4. In particular we have
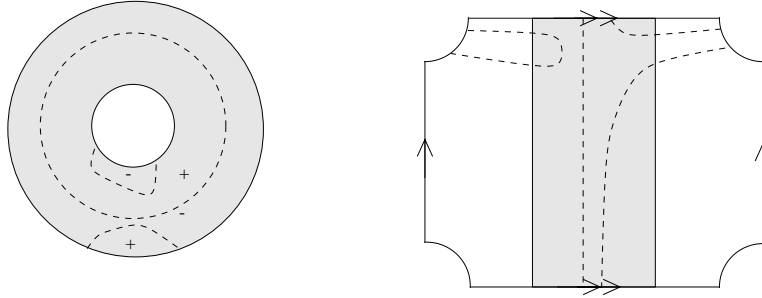

FIGURE 4. The annulus $\mathcal{A}''$ and the new $T \times \{1\}$.

a bypass for ∂T.

If the intersection number is greater than two, then simply attach the bypass D onto $T \times \{1\}$ — either the new $T \times \{1\}$ has a bypass, in which case we are done, or we can take a new $\mathcal{A}$ which has fewer (but ≥ 2 intersections). We continue until the intersection number becomes two. $\square$

4.3. **Twist number increase.** We are now ready to finish the first part of the program outlined in the beginning of this section.

Lemma 9. *We may increase m_3 to 0 while keeping $m_1 = 0$ and $m_2 = -1$. Moreover, V_2 may be further thickened to V_2'' so that $-\partial(M \setminus V_2'')$ has dividing curves of slope -1.*

Proof. Since we have a bypass on ∂T we may apply the Twist Number Lemma to increase the twist number of F_3 to $m_3 = 0$. So we have two vertical dividing curves on ∂V_3 and on $\partial(M \setminus V_3)$ they have slope 0. Now repeating the argument in Lemma 7 we see that we can arrange $m_1 = 0$ and $m_2 = -1$. Note the dividing curves on $-\partial(M \setminus V_2)$ and $\partial(M \setminus V_1)$ have slope $-\frac{1}{2}$ and 0, respectively. Thus taking a vertical annulus between ∂V_1 and ∂V_2, we find a vertical bypass along V_2. With

this bypass we can thicken V_2 to V_2'' whose dividing curves have slope -1. $\qquad\square$

Note that now V_1, V_2, V_3 are neighborhoods of Legendrians with $m_1 = 0$, $m_2 = -1$, $m_3 = 0$, and V_2'' has slope -1 on $-\partial(M \setminus V_2'')$.

4.4. Thinning before thickening.
Note if we had another bypass along ∂T we could increase m_3 to 1 and from here we could then find an overtwisted disk (see below). However, we know of no direct way to prove this bypass exists. Therefore we use the following strategy, which can be called as 'thinning before thickening'. Notice we have thickened V_3 so that $m_3 = 1$; we will now backtrack by having V_3 relinquish some of its thickness (we peel off a toric annulus from V_3 and reattach to V_1 and V_2), and then thicken again to obtain a contradiction.

Let D_i be the meridional disk to V_i for $i = 1, 2$. If we arrange that the Legendrian rulings on ∂V_i are horizontal and the D_i are convex, then $\#\Gamma_{D_1} = 1$ and $\#\Gamma_{D_2} = 2$. The dividing sets divide D_1 into two regions, one positive and one negative, and D_2 into three regions — without loss of generality we can assume that, two regions are positive and one is negative. Note that the signs of these regions depend on an orientation on the disks. Pick an orientation for the fibers of the Seifert fibration, and the disks will be oriented so that their intersection with a fiber is positive.

Note we can write $V_3 = C \cup U$, where C is a solid torus with convex boundary and dividing curves of slope -1 (on ∂C), $U = T^2 \times [0, 1]$, $T^2 \times \{0\} = \partial C$, and $T^2 \times \{1\}$ is convex with vertical dividing curves. We do this by stabilizing (reducing the twist number) F_3 with $m_3 = 0$ in V_3. In U we can find a T^2, say $T^2 \times \{\frac{1}{2}\}$, with dividing curves of slope -5 (which correspond to vertical dividing curves from the point of view of $-\partial(M\setminus V_3)$). By performing the correct stabilization we can arrange that on a convex annulus in $T^2 \times [\frac{1}{2}, 1]$ of slope -5, with Legendrian boundary on $T^2 \times \{\frac{1}{2}\}$ and $T^2 \times \{1\}$, there is a negative bypass along the boundary component on $T^2 \times \{1\}$. We can use this bypass to thicken V_i, $i = 1, 2$, to V_i' so that $\partial(M \setminus V_i')$ have vertical slopes. Using the relative Euler class we can easily see that each meridional disk D_i' in V_i' has an extra positive region. Specifically, if e is the relative Euler class on $V = V_1' \setminus V_1$ (as defined in Section 2) and μ and λ are the horizontal and vertical curves on ∂V then one may easily compute that $e(\lambda \times [0, 1]) = 1$ and $e((-\mu - \lambda) \times [0, 1]) = 0$. Thus $e(c \times [0, 1]) = 1$ where c is the boundary of the meridional disk in V_1. From this and Equation 1, we can conclude that the dividing curves are as in Figure 5 (a), and hence a convex meridional disk in V_1' will have one negative and two positive
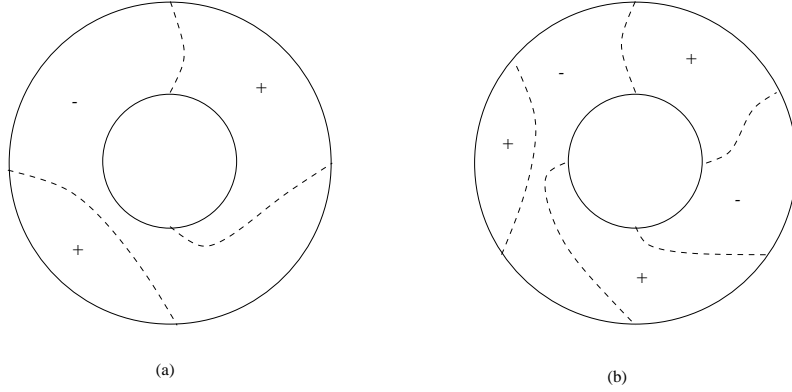
FIGURE 5. (a) Dividing curves on $c \times [0,1]$ in $V_1' \setminus V_1$, (b) Dividing curves on a meridional disk for V_2' intersected with $V_2' \setminus V_2$

regions in the complement of the dividing curves. Similarly, a convex meridional disk in V_2' will have one negative and three positive regions since a relative Euler class argument will show its intersection with $V_2' \setminus V_2$ is shown in Figure 5 (b)

4.5. **Tight structures on** $S^1 \times \Sigma$. We would now like to piece these meridional disks together to form a punctured torus fiber for $M \setminus V_3'$, but to do this we first need to understand the complement of the singular fibers. Recall all the boundary slopes of $S^1 \times \Sigma = M \setminus (V_1' \cup V_2' \cup V_3')$ are infinite. Take $\Sigma = \{0\} \times \Sigma$, which we assume is convex with Legendrian boundary. All the boundary components of Σ have exactly two half-elliptic points.

Lemma 10. *Each dividing curve of Σ must connect one boundary component to another boundary component.*

Proof. There are several possible configurations of the dividing curves. See Figure 6.

We argue that if there is a ∂-compressible dividing arc (as in (B), (C), (D)) on Σ, then M is overtwisted. The ∂-compressible dividing arc implies the existence of a bypass (of degenerate type) along some $\partial(M \setminus V_i')$. Hence, there exists a layer $L_1 = T^2 \times I \subset S^1 \times \Sigma$ with one boundary component $\partial(M \setminus V_i')$ and boundary slopes 0 and ∞. However, since there are vertical Legendrian curves with twisting number 0 outside of $V_i' \cup L_1$, we obtain another layer $L_2 = T^2 \times I$, this time with boundary slopes ∞ and 0. Therefore, $V_i' \cup L_1 \cup L_2$ has too much radial twisting, and is overtwisted. The only possible configuration without a ∂-compressible dividing arc is (A). □
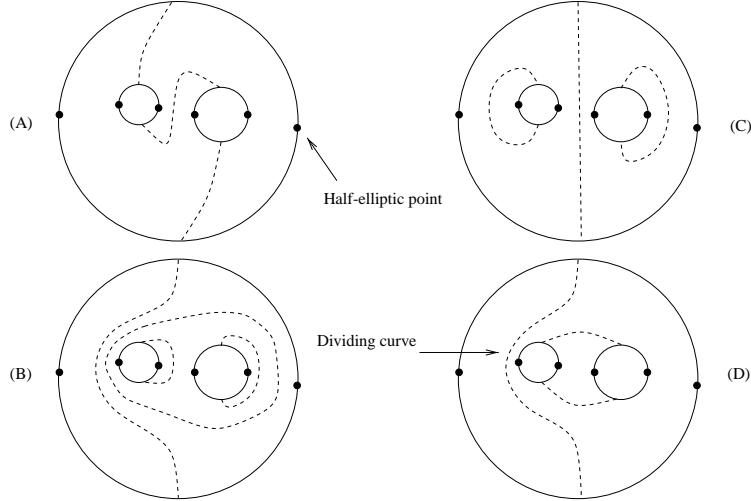
FIGURE 6. Dividing curves of Σ

If we take signs into consideration, there are two possible tight structures on $S^1 \times \Sigma = M \setminus (V_1' \cup V_2' \cup V_3')$, depending on whether in Figure 6(A), the dividing curve from the top hits V_2' or V_3'.

Lemma 11. *For each of the two configurations of dividing curves on* Σ, *the tight contact structure on* $S^1 \times \Sigma = M \setminus (V_1' \cup V_2' \cup V_3')$ *is unique.*

Proof. We cut $S^1 \times \Sigma$ along Σ, round the edges, and examine the dividing curves on the boundary of the resulting genus 2 handlebody. We then cut along the meridional disks of the handlebody, which we may assume have Legendrian boundary, and eventually obtain a 3-ball. Since each meridional disk of the handlebody intersects the dividing set only twice, the configuration of dividing curves on the meridional disks is unique. Therefore the initial configuration of dividing curves on Σ uniquely determines the tight structure on $S^1 \times \Sigma$. □

4.6. **The final stretch.** The lemma implies that the tight structure on $S^1 \times \Sigma$ is a translation-invariant tight structure on $T^2 \times I$ with infinite boundary slopes, with the standard neighborhood of a vertical Legendrian curve with zero twisting removed. We view this $T^2 \times I$ (minus $S^1 \times D^2$) as the region between $\partial V_1'$ and $\partial V_2'$ (minus V_3'). We may think of the I-factor as being quite small, and then isotop one of the Legendrian divides on $\partial V_1'$ to one of the Legendrian divides on $\partial V_2'$ and finally identify small neighborhoods (in the tori) of these divides. We may then isotop $\partial V_i'$, $i = 1, 2$, away from these neighborhoods so that the meridional disk D_i' in V_i' has Legendrian boundary. Forming the fiber T in the fibration of $M \setminus V_3'$ from three copies of D_1 and two

copies of D_2, we can arrange that ∂T is Legendrian and the dividing curves on T are as in Figure 7. Thus there are six bypasses for ∂T. We
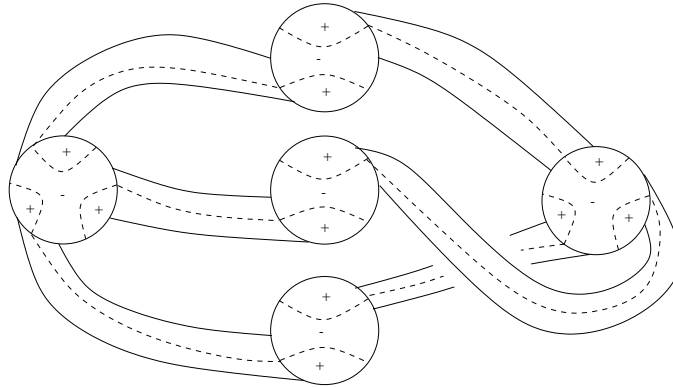

FIGURE 7. The fiber T with six bypasses (note the boundaries of the disks are actually touching, the strips seen here just indicate how the boundaries of the disks connect)

can use these to increase the twisting number of ∂T to 0 (*i.e.*, ∂T will be a Legendrian divide on some thickening of ∂V_3) which corresponds to an increase of m_3 to 1. Thus the slope of the dividing curves on $\partial(M \setminus V_3)$ is $-\frac{1}{6}$. Now repeat the argument in Lemma 7 — take a vertical annulus from $\partial(M \setminus V_2)$ to $\partial(M \setminus V_3)$, and start with m_2 small. Since the denominator of $-\frac{1}{6}$ is never equal to $3m_2 + 1$, we eventually arrive at $m_2 = -1$. This implies the existence of a vertical bypass for $\partial(M \setminus V_3)$, with which we can increase the slope of the dividing curves on $\partial(M \setminus V_3)$ to $-\frac{1}{5}$. Thus the Legendrian divide on our thickened solid torus bounds a meridional disk in V_3, so we have found an overtwisted disk. This completes the proof of the main theorem. $\square$

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STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail address: etnyre@math.stanford.edu
URL: http://math.stanford.edu/~etnyre

UNIVERSITY OF GEORGIA, ATHENS, GA 30602
E-mail address: honda@math.uga.edu
URL: http://www.math.uga.edu/~honda