

# LECTURES ON OPEN BOOK DECOMPOSITIONS AND CONTACT STRUCTURES

JOHN B. ETNYRE

## 1. INTRODUCTION

The main goal of this survey is to discuss the proof and examine some consequences of the following fundamental theorem of Giroux.

**Theorem 1.1** (Giroux 2000, [21]). *Let  $M$  be a closed oriented 3-manifold. Then there is a one to one correspondence between*

*{oriented contact structures on  $M$  up to isotopy}*

*and*

*{open book decompositions of  $M$  up to positive stabilization}.*

This theorem plays a pivotal role in studying cobordisms of contact structures and understanding filling properties of contact structures, see [2, 6, 13, 14, 16, 19]. This better understanding of fillings leads to various topological applications of contact geometry. Specifically, the much studied property P for knots was established by P. Kronheimer and T. Mrowka in [28]. A non-trivial knot has property P if non-trivial surgery on it never gives a homotopy sphere. In addition P. Ozsváth and Z. Szabó in [34] gave an alternate proof of a characterization of the unknot via surgery which was originally established in [29]. This characterization says that the unknot is the only knot on which  $p$ -surgery yields  $-L(p, 1)$ . Moreover, in [34] it is shown that the Thurston norm is determined by Heegaard Floer Homology.

Ideally the reader should be familiar with low-dimensional topology at the level of, say [36]. In particular, we will assume familiarity with Dehn surgery, mapping tori and basic algebraic topology. At various points we also discuss branch coverings, Heegaard splittings and other notions; however, the reader unfamiliar with these notions should be able to skim these parts of the paper without missing much, if any, of the main line of the arguments. Since diffeomorphisms of surfaces play a central role in much of the paper and specific conventions are important we have included an Appendix discussing basic facts about this. We also assume the reader has some familiarity with contact geometry. Having read [15] should be sufficient background for this paper. In order to accommodate the reader with little background in contact geometry we have included brief discussions, scattered throughout the paper, of all the necessary facts. Other good introductions to contact geometry are [1, 20], though a basic understanding of convex surfaces is also useful but is not covered in these sources.

---

1991 *Mathematics Subject Classification*. Primary 53D35; Secondary 57R17.

From lectures given at the 2004 Clay Mathematics Institute Summer School on Floer Homology, Gauge Theory, and Low Dimensional Topology at the Alfréd Rényi Institute; [www.claymath.org/programs/summer\\_school/2004/](http://www.claymath.org/programs/summer_school/2004/).

In the next three sections we give a thorough sketch of the proof of Theorem 1.1. In Section 2 we define open book decompositions of 3-manifolds, discuss their existence and various constructions. The following two sections discuss how to get a contact structure from an open book and an open book from a contact structure, respectively. Finally in Section 5 we will consider various applications of Theorem 1.1. While we prove various things about open books and contact structures our main goal is to prove the following theorem which is the basis for most of the above mentioned applications of contact geometry to topology.

**Theorem** (Eliashberg 2004 [6]; Etnyre 2004 [14]). *If  $(X, \omega)$  is a symplectic filling of  $(M, \xi)$  then there is a closed symplectic manifold  $(W, \omega')$  and a symplectic embedding  $(X, \omega) \rightarrow (W, \omega')$ .*

Acknowledgments: I am grateful to David Alexandre Ellwood, Peter Ozsváth, András Stipsicz, Zoltan Szabó, the Clay Mathematics Institute and the Alfréd Rényi Institute of Mathematics for organizing the excellent summer school on “Floer Homology, Gauge Theory, and Low Dimensional Topology” and for giving me an opportunity to give the lectures on which these notes are based. I also thank Emmanuel Giroux who gave a beautiful series of lectures at Stanford University in 2000 where I was first exposed to the strong relation between open books and contact structures. I am also grateful to Noah Goodman, Gordana Matić, András Némethi and Burak Ozbagci for many illuminating conversations. Finally I thank Paolo Lisca, Stephan Schoenberger and the referee for valuable comments on the first draft of this paper. This work was supported in part by NSF CAREER Grant (DMS-0239600) and FRG-0244663.

## 2. OPEN BOOK DECOMPOSITIONS OF 3-MANIFOLDS

Throughout this section (and these notes)

*$M$  is always a closed oriented 3-manifold.*

We also mention that when inducing an orientation on the boundary of a manifold we use the “outward normal first” convention. That is, given an oriented manifold  $N$  then  $v_1, \dots, v_{n-1}$  is an oriented basis for  $\partial N$  if  $\nu, v_1, \dots, v_{n-1}$  is an oriented basis for  $N$ .

**Definition 2.1.** *An open book decomposition of  $M$  is a pair  $(B, \pi)$  where*

- (1)  *$B$  is an oriented link in  $M$  called the binding of the open book and*
- (2)  *$\pi : M \setminus B \rightarrow S^1$  is a fibration of the complement of  $B$  such that  $\pi^{-1}(\theta)$  is the interior of a compact surface  $\Sigma_\theta \subset M$  and  $\partial \Sigma_\theta = B$  for all  $\theta \in S^1$ . The surface  $\Sigma = \Sigma_\theta$ , for any  $\theta$ , is called the page of the open book.*

One should note that it is important to include the projection in the data for an open book, since  $B$  does not determine the open book, as the following example shows.

**Example 2.2** Let  $M = S^1 \times S^2$  and  $B = S^1 \times \{N, S\}$ , where  $N, S \in S^2$ . There are many ways to fiber  $M \setminus B = S^1 \times S^1 \times [0, 1]$ . In particular if  $\gamma_n$  is an embedded curve on  $T^1$  in the homology class  $(1, n)$ , then  $M \setminus B$  can be fibered by annuli parallel to  $\gamma_n \times [0, 1]$ . There are diffeomorphisms of  $S^1 \times S^2$  that relate all of these fibrations but the fibrations coming from  $\gamma_0$  and  $\gamma_1$  are not isotopic. There are examples of fibrations that are not even diffeomorphic.

**Definition 2.3.** An abstract open book is a pair  $(\Sigma, \phi)$  where

- (1)  $\Sigma$  is an oriented compact surface with boundary and
- (2)  $\phi : \Sigma \rightarrow \Sigma$  is a diffeomorphism such that  $\phi$  is the identity in a neighborhood of  $\partial\Sigma$ . The map  $\phi$  is called the monodromy.

We begin by observing that given an abstract open book  $(\Sigma, \phi)$  we get a 3-manifold  $M_\phi$  as follows:

$$M_\phi = \Sigma_\phi \cup_\psi \left( \coprod_{|\partial\Sigma|} S^1 \times D^2 \right),$$

where  $|\partial\Sigma|$  denotes the number of boundary components of  $\Sigma$  and  $\Sigma_\phi$  is the mapping torus of  $\phi$ . By this we mean

$$\Sigma \times [0, 1] / \sim,$$

where  $\sim$  is the equivalence relation  $(\phi(x), 0) \sim (x, 1)$  for all  $x \in \Sigma$ . Finally,  $\cup_\psi$  means that the diffeomorphism  $\psi$  is used to identify the boundaries of the two manifolds. For each boundary component  $l$  of  $\Sigma$  the map  $\psi : \partial(S^1 \times D^2) \rightarrow l \times S^1 \subset \Sigma_\phi$  is defined to be the unique (up to isotopy) diffeomorphism that takes  $S^1 \times \{p\}$  to  $l$  where  $p \in \partial D^2$  and  $\{q\} \times \partial D^2$  to  $(\{q'\} \times [0, 1] / \sim) = S^1$ , where  $q \in S^1$  and  $q' \in \partial\Sigma$ . We denote the cores of the solid tori in the definition of  $M_\phi$  by  $B_\phi$ .

Two abstract open book decompositions  $(\Sigma_1, \phi_1)$  and  $(\Sigma_2, \phi_2)$  are called *equivalent* if there is a diffeomorphism  $h : \Sigma_1 \rightarrow \Sigma_2$  such that  $h \circ \phi_2 = \phi_1 \circ h$ .

**Lemma 2.4.** We have the following basic facts about open books and abstract open books:

- (1) An open book decomposition  $(B, \pi)$  of  $M$  gives an abstract open book  $(\Sigma_\pi, \phi_\pi)$  such that  $(M_{\phi_\pi}, B_{\phi_\pi})$  is diffeomorphic to  $(M, B)$ .
- (2) An abstract open book determines  $M_\phi$  and an open book  $(B_\phi, \pi_\phi)$  up to diffeomorphism.
- (3) Equivalent open books give diffeomorphic 3-manifolds.

**Exercise 2.5** Prove this lemma.

**Remark 2.6.** Clearly the two notions of open book decomposition are closely related. The basic difference is that when discussing open books (non-abstract) we can discuss the binding and pages up to *isotopy* in  $M$ , whereas when discussing abstract open books we can only discuss them up to *diffeomorphism*. Thus when discussing Giroux's Theorem 1.1 we need to use (non-abstract) open books; however, it is still quite useful to consider abstract open books and we will frequently not make much of a distinction between them.

**Example 2.7** Let  $S^3$  be the unit sphere in  $\mathbb{C}^2$ , and  $(z_1, z_2) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$  be coordinates on  $\mathbb{C}^2$ .

- (1) Let  $U = \{z_1 = 0\} = \{r_1 = 0\} \subset S^3$ . Thus  $U$  is a unit  $S^1$  sitting in  $S^3$ . It is easy to see that  $U$  is an unknotted  $S^1$  in  $S^3$ . The complement of  $U$  fibers:

$$\pi_U : S^3 \setminus U \rightarrow S^1 : (z_1, z_2) \mapsto \frac{z_1}{|z_1|}.$$

In polar coordinates this map is just  $\pi_U(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1$ . This fibration is related to the well known fact that  $S^3$  is the union of two solid tori. Pictorially we see this fibration in Figure 1.

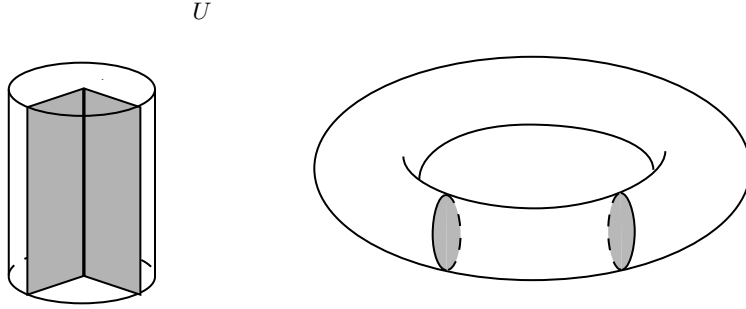


FIGURE 1.  $S^3$  broken into two solid tori (to get the one on the left identify top and bottom of the cylinder). The union of the shaded annuli and disks give two pages in the open book.

- (2) Let  $H^+ = \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}$  and  $H^- = \{(z_1, z_2) \in S^3 : z_1 \bar{z}_2 = 0\}$ .

**Exercise 2.8** Show  $H^+$  is the positive Hopf link and  $H^-$  is the negative Hopf link. See Figure 2. (Recall  $H^+$  gets an orientation as the boundary of a complex hypersurface in  $\mathbb{C}^2$ , and  $H^-$  may be similarly oriented.)

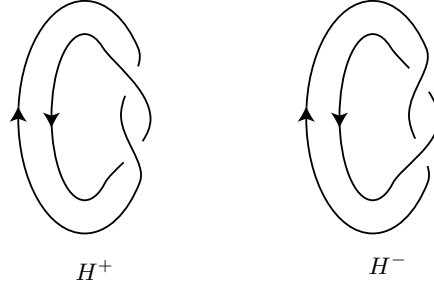


FIGURE 2. The two Hopf links.

We have the fibrations

$$\pi_+ : S^3 \setminus H^+ \rightarrow S^1 : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}, \text{ and}$$

$$\pi_- : S^3 \setminus H^- \rightarrow S^1 : (z_1, z_2) \mapsto \frac{z_1 \bar{z}_2}{|z_1 \bar{z}_2|}.$$

In polar coordinates these maps are just  $\pi_{\pm}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1 \pm \theta_2$ .

**Exercise 2.9** Picture these fibrations.

- (3) More generally, let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a polynomial that vanishes at  $(0, 0)$  and has no critical points inside  $S^3$  except possibly  $(0, 0)$ . Then  $B = f^{-1}(0) \cap S^3$  gives an open book of  $S^3$  with fibration

$$\pi_f : S^3 \setminus B \rightarrow S^1 : (z_1, z_2) \mapsto \frac{f(z_1, z_2)}{|f(z_1, z_2)|}.$$

This is called the Milnor fibration of the hypersurface singularity  $(0, 0)$  (note that  $(0, 0)$  does not have to be a singularity, but if it is not then  $B$  is always the unknot). See [32].

**Exercise 2.10** Suppose  $\Sigma$  is a surface of genus  $g$  with  $n$  boundary components and  $\phi$  is the identity map on  $\Sigma$ . Show  $M_\phi = \#_{2g+n-1} S^1 \times S^2$ .

HINT: If  $a$  is a properly embedded arc in  $\Sigma$  then  $a \times [0, 1]$  is an annulus in the mapping torus  $\Sigma_\phi$  that can be capped off into a sphere using two disks in the neighborhood of the binding.

**Theorem 2.11** (Alexander 1920, [4]). *Every closed oriented 3-manifold has an open book decomposition.*

We will sketch three proofs of this theorem.

*First Sketch of Proof.* We first need two facts

**Fact** (Alexander 1920, [4]). *Every closed oriented 3-manifold  $M$  is a branched cover of  $S^3$  with branched set some link  $L_M$ .*

**Fact** (Alexander 1923, [3]). *Every link  $L$  in  $S^3$  can be braided about the unknot.*

When we say  $L$  can be *braided about the unknot* we mean that if  $S^1 \times D^2 = S^3 \setminus U$  then we can isotop  $L$  so that  $L \subset S^1 \times D^2$  and  $L$  is transverse to  $\{p\} \times D^2$  for all  $p \in S^1$ .

Now given  $M$  and  $L_M \subset S^3$  as in the first fact we can braid  $L_M$  about the unknot  $U$ . Let  $P : M \rightarrow S^3$  be the branch covering map. Set  $B = P^{-1}(U) \subset M$ . We claim that  $B$  is the binding of an open book. The fibering of the complement of  $B$  is simply  $\pi = \pi_U \circ P$ , where  $\pi_U$  is the fibering of the complement of  $U$  in  $S^3$ .

**Exercise 2.12** Prove this last assertion and try to picture the fibration.

□

Before we continue with our two other proofs let's have some fun with branched covers.

**Exercise 2.13** Use the branched covering idea in the previous proof to find various open books of  $S^3$ .

HINT: Any cyclic branched cover of  $S^3$  over the unknot is  $S^3$ . Consider Figure 3. See also [23, 36].

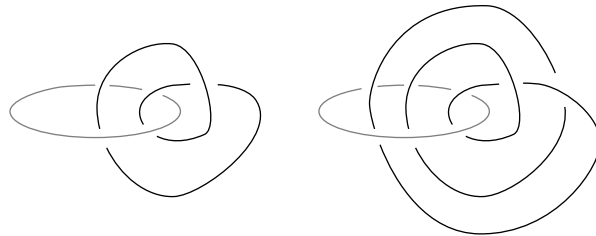


FIGURE 3. Here are two links each of which is a link of unknots. If we do a cyclic branched cover of  $S^3$  over the grey component and lift the black component to the cover it will become the binding of an open book decomposition of  $S^3$ .

*Second Sketch of Proof.* This proof comes from Rolfsen's book [36] and relies on the following fact.

**Fact** (Lickorish 1962, [30]; Wallace 1960, [40]). *Every closed oriented 3-manifold may be obtained by  $\pm 1$  surgery on a link  $L_M$  of unknots. Moreover, there is an unknot  $U$  such that  $L_M$  is braided about  $U$  and each component of  $L_M$  can be assumed to link  $U$  trivially one time. See Figure 4.*

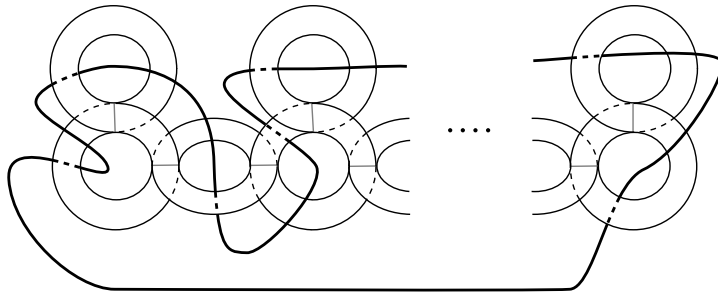


FIGURE 4. All the unknots in the link  $L_M$  can be isotoped to be on the annuli depicted here. The heavy black line is the unknot  $U$ .

Now  $(U, \pi_U)$  is an open book for  $S^3$ . Let  $N$  be a small tubular neighborhood of  $L_M$ . Each component  $N_C$  of  $N$  corresponds to a component  $C$  of  $L_M$  and we can assume that  $N_C$  intersects the fibers of the fibration  $\pi_U$  in meridional disks. So the complement of  $U \cup N$  fibers so that each  $\partial N_C$  is fibered by meridional circles. To perform  $\pm 1$  surgery on  $L_M$  we remove each of the  $N_C$ 's and glue it back sending the boundary of the meridional disk to a  $(1, \pm 1)$  curve on the appropriate boundary component of  $\overline{S^3 \setminus N}$ . After the surgery we have  $M$  and inside  $M$  we have the union of surgery tori  $N'$ , the components of which we denote  $N'_C$  and the cores of which we denote  $C'$ . We denote the union of the cores by  $L'$ . Inside  $M$  we also have the “unknot”  $U$  (of course  $U$  may not be an unknot any more, for example it could represent non-trivial homology in  $M$ ). Since  $M \setminus (U \cup N') = S^3 \setminus (U \cup N)$  we have a fibration of  $M \setminus (U \cup N')$  and it is easy to see that the fibration induces on  $\partial N'_C$  a fibration by  $(1, \pm 1)$  curves. We can fiber  $N'_C \setminus C'$  by annuli so that the induced fibration on  $\partial N'_C$  is by  $(1, \pm 1)$  curves. Thus we may extend the fibration of  $M \setminus (U \cup N')$  to a fibration of  $M \setminus (U \cup L')$ , hence inducing an open book of  $M$ .

**Exercise 2.14** Convince yourself of these last statements.

**Remark 2.15.** We have produced an open book for  $M$  with planar pages!

□

*Third Sketch of Proof.* This proof is due to Harer. We need

**Fact** (Harer 1979, [25]). *An oriented compact 4-manifold has an achiral Lefschetz fibration with non-closed leaves over a disk if and only if it admits a handle decomposition with only 0-, 1-, and 2-handles.*

An achiral Lefschetz fibration of a 4-manifold  $X$  over a surface  $S$  is simply a map  $\pi : X \rightarrow S$  such that the differential  $d\pi$  is onto for all but a finite number of points  $p_1, \dots, p_k \in \text{int}(X)$ , where there are complex coordinate charts  $U_i$  of  $p_i$  and  $V_i$  of  $\pi(p_i)$  such that  $\pi_{U_i}(z_1, z_2) = z_1^2 + z_2^2$ . Note the definition implies that  $\pi$  restricted to  $X \setminus \pi^{-1}(\pi(\{p_1, \dots, p_k\}))$  is a locally trivial fibration. We denote a generic fiber by  $\Sigma_\pi$ .

**Fact** (Lickorish 1962, [30]; Wallace 1960, [40]). *Every closed oriented 3-manifold is the boundary of a 4-manifold built with only 0- and 2-handles.*

Given a 3-manifold  $M$  we use this fact to find a 4-manifold  $X$  with  $\partial X = M$  and  $X$  built with only 0- and 2-handles. Then the previous fact gives us an achiral Lefschetz fibration  $\pi : X \rightarrow D^2$ . Set  $B = \partial\pi^{-1}(x)$  for a non-critical value  $x \in \text{int}(D^2)$ . We claim that  $B$  is the binding of an open book decomposition for  $M$  and the fibration of the complement is the restriction of  $\pi$  to  $M \setminus B$ .  $\square$

**Definition 2.16.** *Given two abstract open books  $(\Sigma_i, \phi_i), i = 0, 1$ , let  $c_i$  be an arc properly embedded in  $\Sigma_i$  and  $R_i$  a rectangular neighborhood of  $c_i$ ,  $R_i = c_i \times [-1, 1]$ . The Murasugi sum of  $(\Sigma_0, \phi_0)$  and  $(\Sigma_1, \phi_1)$  is the open book  $(\Sigma_0 * \Sigma_1, \phi_0 \circ \phi_1)$  with page*

$$\Sigma_0 * \Sigma_1 = \Sigma_0 \cup_{R_1=R_2} \Sigma_1,$$

where  $R_0$  and  $R_1$  are identified so that  $c_i \times \{-1, 1\} = (\partial c_{i+1}) \times [-1, 1]$ , and the monodromy is  $\phi_0 \circ \phi_1$ .

**Theorem 2.17** (Gabai 1983, [18]).

$$M_{(\Sigma_0, \phi_0)} \# M_{(\Sigma_1, \phi_1)} \text{ is diffeomorphic to } M_{(\Sigma_0 * \Sigma_1, \phi_0 \circ \phi_1)}.$$

*Sketch of Proof.* The proof is essentially contained in Figure 5. The idea is that  $B_0 = R_0 \times [\frac{1}{2}, 1]$  is a 3-ball in  $M_{(\Sigma_1, \phi_1)}$  and similarly for  $B_1 = R_1 \times [0, \frac{1}{2}]$  in  $M_{(\Sigma_0, \phi_0)}$ . Now  $(\Sigma_0 * \Sigma_1) \times [0, 1]$  can be formed as shown in Figure 5.

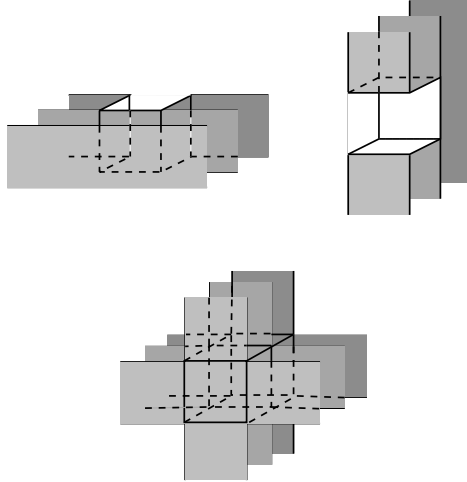


FIGURE 5. At the top left is a piece of  $\Sigma_0 \times [0, 1]$  near  $c_0$  with  $B_0$  cut out. The lightest shaded part is  $\Sigma_0 \times \{0\}$  the medium shaded part is  $\Sigma_0 \times \{\frac{1}{2}\}$  and the darkest shaded part is  $\Sigma_0 \times \{1\}$ . The top right is a similar picture for  $\Sigma_1$ . The bottom picture is  $(\Sigma_0 * \Sigma_1) \times [0, 1]$ .

Think about forming the mapping cylinder of  $\phi_0$  by gluing  $\Sigma_0 \times \{0\}$  to  $\Sigma_0 \times \{1\}$  using the identity and then cutting the resulting  $\Sigma_0 \times S^1$  along  $\Sigma_0 \times \{\frac{1}{4}\}$  and regluing using  $\phi_0$ . Similarly think about the mapping cylinder for  $\phi_1$  as  $\Sigma_1 \times S^1$  reglued along  $\Sigma_1 \times \{\frac{3}{4}\}$  and the mapping cylinder for  $\phi_0 \circ \phi_1$  as  $(\Sigma_0 * \Sigma_1) \times S^1$

reglued by  $\phi_0$  along  $(\Sigma_0 * \Sigma_1) \times \{\frac{1}{4}\}$  and by  $\phi_1$  along  $(\Sigma_0 * \Sigma_1) \times \{\frac{3}{4}\}$ . Thus we see how to fit all the mapping cylinders together nicely.

**Exercise 2.18** Think about how the binding fits in and complete the proof. □

**Definition 2.19.** A positive (negative) stabilization of an abstract open book  $(\Sigma, \phi)$  is the open book

- (1) with page  $\Sigma' = \Sigma \cup 1\text{-handle}$  and
- (2) monodromy  $\phi' = \phi \circ \tau_c$  where  $\tau_c$  is a right- (left-)handed Dehn twist along a curve  $c$  in  $\Sigma'$  that intersects the co-core of the 1-handle exactly one time.

We denote this stabilization by  $S_{(a,\pm)}(\Sigma, \phi)$  where  $a = c \cap \Sigma$  and  $\pm$  refers to the positivity or negativity of the stabilization. (We omit the  $a$  if it is unimportant in a given context.)

**Exercise 2.20** Show

$$S_{\pm}(\Sigma, \phi) = (\Sigma, \phi) * (H^{\pm}, \pi_{\pm})$$

where  $H^{\pm}$  is the positive/negative Hopf link and  $\pi_{\pm}$  is the corresponding fibration of its complement.

From this exercise and Theorem 2.17 we immediately have:

**Corollary 2.21.**

$$M_{(S_{\pm}(\Sigma, \phi))} = M_{(\Sigma, \phi)}.$$

**Exercise 2.22** Show how to do a Murasugi sum ambiently. That is show how to perform a Murasugi sum for open book decompositions (not abstract open books!). Of course one of the open books must be an open book for  $S^3$ .

HINT: See Figure 6.

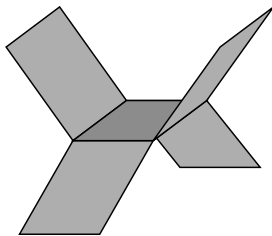


FIGURE 6. Two ambient surfaces being summed together.

**Exercise 2.23** Use Murasugi sums to show the right- and left-handed trefoil knots and the figure eight knot all give open book decompositions for  $S^3$ .

**Exercise 2.24** Use Murasugi sums to show all torus links give open book decompositions of  $S^3$ .

**Exercise 2.25** Show that every 3-manifold has an open book decomposition with connected binding.

**Exercise 2.26** Use the previous exercise to prove a theorem of Bing: A closed oriented 3-manifold is  $S^3$  if and only if every simple closed curve in  $M$  is contained in a 3-ball.

HINT: The only surface bundle over  $S^1$  yielding an orientable manifold that is not irreducible is  $S^1 \times S^2$ .



## 3. FROM OPEN BOOKS TO CONTACT STRUCTURES

**Definition 3.1.** An (oriented) contact structure  $\xi$  on  $M$  is an oriented plane field  $\xi \subset TM$  for which there is a 1-form  $\alpha$  such that  $\xi = \ker \alpha$  and  $\alpha \wedge d\alpha > 0$ . (Recall:  $M$  is oriented.)

**Remark 3.2.** What we have really defined is a *positive* contact structure, but since this is all we will talk about we will stick to this definition.

**Example 3.3** (1) On  $\mathbb{R}^3$  we have the standard contact structure  $\xi_{std} = \ker(dz + r^2 d\theta)$ . See Figure 7.

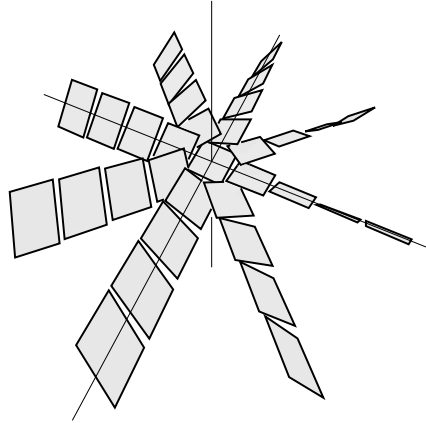


FIGURE 7. Standard contact structure on  $\mathbb{R}^3$ . (Picture by Stephan Schönenberger.)

(2) On  $S^3$ , thought of as the unit sphere in  $\mathbb{C}^2$ , we have  $\xi_{std}$  the set of complex tangents. That is  $\xi_{std} = TS^3 \cap i(TS^3)$ . We can also describe this plane field as  $\xi_{std} = \ker(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ , where we are using coordinates  $(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$  on  $\mathbb{C}^2$ .

We will need the following facts about contact structures. Most of these facts are proven or discussed in [1, 15, 20].

- All 3-manifolds admit a contact structure. See Theorem 3.13 below.
- Locally all contact structures look the same. This is called Darboux's Theorem and means that if  $p_i$  is a point in the contact manifold  $(M_i, \xi_i)$ ,  $i = 0, 1$ , then there is a neighborhood  $U_i$  of  $p_i$  and a diffeomorphism  $f : U_0 \rightarrow U_1$  such that  $f_*(\xi_0) = \xi_1$ . Such a diffeomorphism is called a *contactomorphism*.
- Given two contact manifolds  $(M_i, \xi_i)$ ,  $i = 0, 1$ , we can form their *contact connected sum*  $(M_0 \# M_1, \xi_0 \# \xi_1)$  as follows: there are balls  $B_i \subset M_i$  and an orientation reversing diffeomorphism  $f : \partial(\overline{M_0} \setminus B_0) \rightarrow \partial(\overline{M_1} \setminus B_1)$  such that  $M_0 \# M_1$  is formed by gluing  $M_0 \setminus B_0$  and  $M_1 \setminus B_1$  together using  $f$  and

$$\xi_0|_{M_0 \setminus B_0} \cup \xi_1|_{M_1 \setminus B_1}$$

extends to a well defined contact structure on  $M_0 \# M_1$ . See [5].

- Given a 1-parameter family of contact structures  $\xi_t, t \in [0, 1]$ , there is a 1-parameter family of diffeomorphisms  $\phi_t : M \rightarrow M$  such that  $(\phi_t)_*(\xi_0) = \xi_t$ . This is called Gray's Theorem.

Two contact structures  $\xi_0$  and  $\xi_1$  are called isotopic if there is a 1-parameter family of contact structures connecting them.

**Definition 3.4.** *A contact structure  $\xi$  on  $M$  is supported by an open book decomposition  $(B, \pi)$  of  $M$  if  $\xi$  can be isotoped through contact structures so that there is a contact 1-form  $\alpha$  for  $\xi$  such that*

- (1)  $d\alpha$  is a positive area form on each page  $\Sigma_\theta$  of the open book and
- (2)  $\alpha > 0$  on  $B$  (Recall:  $B$  and the pages are oriented.)

**Lemma 3.5.** *The following statements are equivalent*

- (1) *The contact manifold  $(M, \xi)$  is supported by the open book  $(B, \pi)$ .*
- (2)  *$(B, \pi)$  is an open book for  $M$  and  $\xi$  can be isotoped to be arbitrarily close (as oriented plane fields), on compact subsets of the pages, to the tangent planes to the pages of the open book in such a way that after some point in the isotopy the contact planes are transverse to  $B$  and transverse to the pages of the open book in a fixed neighborhood of  $B$ .*
- (3)  *$(B, \pi)$  is an open book for  $M$  and there is a Reeb vector field  $X$  for a contact structure isotopic to  $\xi$  such that  $X$  is (positively) tangent to  $B$  and (positively) transverse to the pages of  $\pi$ .*

The condition in part (2) of the lemma involving transversality to the pages is to prevent excess twisting near the binding and may be dispensed with for tight contact structures.

Recall that a vector field  $X$  is a Reeb vector field for  $\xi$  if it is transverse to  $\xi$  and its flow preserves  $\xi$ . This is equivalent to saying there is a contact form  $\alpha$  for  $\xi$  such that  $\alpha(X) = 1$  and  $\iota_X d\alpha = 0$ . The equivalence of (1) and (2) is supposed to give some intuition about what it means for a contact structure to be supported by an open book. We do not actually use (2) anywhere in this paper, but it is interesting to know that “supported” can be defined this way. Similarly condition (3) should be illuminating if you have studied Reeb vector fields in the past.

*Proof.* We begin with the equivalence of (1) and (2). Suppose  $(M, \xi)$  is supported by  $(B, \pi)$ . So  $M \setminus B$  fibers over  $S^1$ . Let  $d\theta$  be the coordinate on  $S^1$ . We also use  $d\theta$  to denote the pullback of  $d\theta$  to  $M \setminus B$ , that is, for  $\pi^*d\theta$ . Near each component for the binding we can choose coordinates  $(\psi, (r, \theta))$  on  $N = S^1 \times D^2$  in such a way that  $d\theta$  in these coordinates and  $\pi^*d\theta$  agree. Choosing the neighborhood  $N$  small enough we can assume that  $\alpha(\frac{\partial}{\partial \psi}) > 0$  (since  $\alpha$  is positive on  $B$ ). Choose an increasing non-negative function  $f : [0, \epsilon] \rightarrow \mathbb{R}$  that equals  $r^2$  near 0 and 1 near  $\epsilon$ , where  $\epsilon$  is chosen so that  $\{(\psi, (r, \theta)) | r < \epsilon\} \subset N$ . Now consider the 1-form  $\alpha_R = \alpha + Rf(r)d\theta$ , where  $R$  is any large constant. (Here we of course mean that outside the region  $\{(\psi, (r, \theta)) | r < \epsilon\}$  we just take  $f$  to be 1.) Note that  $\alpha_R$  is a contact 1-form for all  $R > 0$ . Indeed

$$\alpha_R \wedge d\alpha_R = \alpha \wedge d\alpha + Rf d\theta \wedge d\alpha + Rf' \alpha \wedge dr \wedge d\theta.$$

The first term on the right is clearly positive since  $\alpha$  is a contact form. The second term is also positive since  $d\alpha$  is a volume form for the pages,  $d\theta$  vanishes on the pages and is positive on the oriented normals to the pages. Finally the last term

is non-negative since  $dr \wedge d\theta$  vanishes on  $\frac{\partial}{\partial\psi}$  while  $\alpha(\frac{\partial}{\partial\psi}) > 0$ . As  $R \rightarrow \infty$  we have a 1-parameter family of contact structures  $\xi_R = \ker \alpha_R$  that starts at  $\xi = \xi_0$  and converges to the pages of the open book away from the binding while staying transverse to the binding (and the pages near the binding).

Now for the converse we assume (2). Let  $\xi_s$  be a family of plane fields isotopic to  $\xi$  that converge to a singular plane field tangent to the pages of the open book (and singular along the binding) as  $s \rightarrow \infty$ . Let  $\alpha_s$  be contact forms for the  $\xi_s$ . We clearly have that  $\alpha_s > 0$  on  $B$ .

Thinking of  $M \setminus B$  as the mapping torus  $\Sigma_\phi$  we can use coordinates  $(x, \theta) \in \Sigma \times [0, 1]$  (we use  $\theta$  for the coordinate on  $[0, 1]$  since on the mapping torus  $\Sigma_\phi$  this is the pullback of  $\theta$  on  $S^1$  under the fibration) and write

$$\alpha_s = \beta_s(\theta) + u_s(\theta) d\theta,$$

where  $\beta_s(\theta)$  is a 1-form on  $\Sigma$  and  $u_s(\theta)$  is a function on  $\Sigma$  for each  $s$  and  $\theta$ . Let  $N$  be a tubular neighborhood of  $B$  on which  $\xi_s$  is transverse to the pages of the open book and let  $N'$  be a tubular neighborhood of  $B$  contained in  $N$ . We can choose  $N'$  so that  $\overline{M \setminus N'}$  is a mapping torus  $\Sigma'_{\phi'}$  where  $\Sigma' \subset \Sigma$  is  $\Sigma$  minus a collar neighborhood of  $\partial\Sigma$  and  $\phi' = \phi|_{\Sigma'}$  is the identity near  $\partial\Sigma'$ . For  $s$  large enough  $u_s(\theta) > 0$  on  $\overline{M \setminus N'}$  for all  $\theta$ , since  $\alpha_s$  converges uniformly to some positive multiple of  $d\theta$  on  $\overline{M \setminus N'}$  as  $s \rightarrow \infty$ . Thus, for large  $s$ , on  $\overline{M \setminus N'} = \Sigma'_{\phi'}$  we can divide  $\alpha_s$  by  $u_s(\theta)$  and get a new family of contact forms

$$\alpha'_s = \beta'_s(\theta) + d\theta.$$

We now claim that  $d\alpha'_s|_{\text{page}} = d\beta'_s$  is a positive volume form on  $\Sigma'$ . To see this note that

$$\alpha'_s \wedge d\alpha'_s = d\theta \wedge (d\beta'_s(\theta) - \beta'_s(\theta) \wedge \frac{\partial\beta'_s}{\partial\theta}(\theta)).$$

So clearly

$$(1) \quad d\beta'_s(\theta) - \beta'_s(\theta) \wedge \frac{\partial\beta'_s}{\partial\theta}(\theta) > 0.$$

To see that  $d\beta'_s(\theta) > 0$  for  $s$  large enough, we note that the second term in this equation vanishes to higher order than the first as  $s$  goes to infinity. From this one can easily conclude that  $d\beta'_s(\theta) \geq 0$  for  $s$  large enough.

**Exercise 3.6** Verify this last statement.

HINT: Assume, with out loss of generality, the 1-forms  $\alpha_s$  are analytic and are analytic in  $s$ .

By adding a small multiple of a 1-form, similar to the one constructed on the mapping torus in the proof of Theorem 3.13 below, we easily see that for a fixed  $s$ , large enough, we can assume  $d\beta'_s(\theta) > 0$ .

**Exercise 3.7** Show how to add this 1-form to  $\alpha'_s$  preserving all the properties of  $\alpha'_s$  in  $N'$  but still having  $d\alpha'_s > 0$  on  $\Sigma'_\theta$ .

HINT: Make sure the 1-form and its derivative are very small and use a cutoff function that is  $C^1$ -small too.

We may now assume that  $d\alpha'_s$  is a volume form on the pages of  $\Sigma'_{\phi'}$ . Denote  $\alpha'_s$  by  $\alpha$ . We are left to verify  $\alpha$  can be modified to have the desired properties in  $N$ .

**Exercise 3.8** Show that we may assume each component of  $N$  is diffeomorphic to  $S^1 \times D^2$  with coordinates  $(\psi, (r, \theta))$  such that the pages of the open book go to constant  $\theta$  annuli in  $S^1 \times D^2$  and the contact structure  $\ker \alpha$  on  $M$  maps to  $\ker(d\psi + f(r) d\theta)$ , for some function  $f(r)$ .

HINT: This is more than a standard neighborhood of a transverse curve. Think about the foliation on the pages of the open book near the binding and on the constant  $\theta$  annuli.

Under this identification  $\alpha$  maps to some contact form  $\alpha' = h(d\psi + f(r) d\theta)$  near the boundary of  $S^1 \times D^2$ , where  $h$  is function on this neighborhood. By scaling  $\alpha$  if necessary we may assume that  $h > 1$  where it is defined.

**Exercise 3.9** Show that  $d\alpha'$  is a volume form on the (parts of the) constant  $\theta$  annuli (where it is defined) if and only if  $h_r > 0$ .

Since that we know  $d\alpha$  is a volume form on the pages of the mapping torus,  $h_r > 0$  where it is defined. Moreover we can extend it to all of  $S^1 \times D^2$  so that it is equal to 1 on  $r = 0$  and so that  $h_r > 0$  everywhere. Thus the contact form equal to  $\alpha$  off of  $N'$  and equal to  $h(d\psi + f(r) d\theta)$  on each component of  $N$  is a globally defined contact form for  $\xi_s$  and satisfies conditions (1) of the lemma.

**Exercise 3.10** Try to show that the condition in part (2) of the lemma involving transversality to the pages of the open book near  $B$  is unnecessary if the contact structure is tight.

HINT: There is a unique universally tight contact structure on a solid torus with a fixed non-singular characteristic foliation on the boundary that is transverse to the meridional circles.

We now establish the equivalence of (1) and (3). Assume (3) and let  $X$  be the vector field discussed in (3). Since  $X$  is positively tangent to the binding we have  $\alpha > 0$  on oriented tangent vectors to  $B$ . Moreover, since  $X$  is positively transverse to the pages of the open book we have  $d\alpha = \iota_X \alpha \wedge d\alpha > 0$  on the pages. Thus  $(M, \xi)$  is supported by  $(B, \pi)$ . Conversely assume (1) is true and let  $\alpha$  be the contact form implicated in the definition of supporting open book. Let  $X$  be the Reeb vector field associated to  $\alpha$ . It is clear that  $X$  is positively transverse to the pages of the open book since  $d\alpha$  is a volume form on the pages. Thus we are left to check that  $X$  is positively tangent to  $B$ . To this end consider coordinates  $(\psi, (r, \theta))$  on a neighborhood of a component of  $B$  such that constant  $\theta$ 's give the pages of the open book in the neighborhood. Switching  $(r, \theta)$  coordinates to Cartesian coordinates  $(x, y)$  we can write  $X = f \frac{\partial}{\partial \psi} + g \frac{\partial}{\partial x} + h \frac{\partial}{\partial y}$ , where  $f, g, h$  are functions. We need to see that  $g$  and  $h$  are zero when  $(x, y) = (0, 0)$ . This is clear, for if say  $g > 0$  at some point  $(c, (0, 0))$  then it will be positive in some neighborhood of this point in particular at  $(c, (0, \pm\epsilon))$  for sufficiently small  $\epsilon$ . But at  $(c, (0, \epsilon))$  the  $\frac{\partial}{\partial x}$  component of  $X$  must be negative, not positive, in order to be positively transverse to the pages. Thus  $g$  and  $h$  are indeed zero along the binding.  $\square$

**Example 3.11** Let  $(U, \pi_U)$  be the open book for  $S^3$ , where  $U$  is the unknot and

$$\pi_U : S^3 \setminus U \rightarrow S^1 : (r_1, \theta_1, r_2, \theta_2) \mapsto \theta_1.$$

(Recall that we are thinking of  $S^3$  as the unit sphere in  $\mathbb{C}^2$ .) This open book supports the standard contact structure  $\xi_{std} = \ker(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ . To see this notice that for fixed  $\theta_1$  the page  $\pi_U^{-1}(\theta_1)$  is parameterized by

$$f(r, \theta) = (\sqrt{1 - r^2}, \theta_1, r, \theta).$$

Thus  $df^*(r_1^2 d\theta_1 + r_2^2 d\theta_2) = 2r dr \wedge d\theta$  which is the volume form on the disk. Moreover the positively oriented tangent to  $U$  is  $\frac{\partial}{\partial\theta_2}$  and  $\alpha(\frac{\partial}{\partial\theta_2}) > 0$

**Exercise 3.12** Show that  $(H^+, \pi_+)$  also supports  $\xi_{std}$  but that  $(H^-, \pi_-)$  does not.

**Theorem 3.13** (Thurston-Winkelnkemper 1975, [38]). *Every open book decomposition  $(\Sigma, \phi)$  supports a contact structure  $\xi_\phi$  on  $M_\phi$ .*

*Proof.* Recall

$$M_\phi = \Sigma_\phi \cup_\psi \left( \coprod_{|\partial\Sigma|} S^1 \times D^2 \right),$$

where  $\Sigma_\phi$  is the mapping torus of  $\phi$ . We first construct a contact structure on  $\Sigma_\phi$ . To this end we consider the set

$$S = \{1\text{-forms } \lambda : \begin{array}{l} (1) \lambda = (1+s)d\theta \text{ near } \partial\Sigma \text{ and} \\ (2) d\lambda \text{ is a volume form on } \Sigma \end{array}\}$$

where near each boundary component of  $\Sigma$  we use coordinates  $(s, \theta) \in [0, 1] \times S^1$ .

**Exercise 3.14** Show this set is convex.

To show this set is non-empty let  $\lambda_1$  be any 1-form on  $\Sigma$  that has the right form near the boundary. Note that

$$\int_\Sigma d\lambda_1 = \int_{\partial\Sigma} \lambda_1 = 2\pi|\partial\Sigma|.$$

Let  $\omega$  be any volume form on  $\Sigma$  whose integral over  $\Sigma$  is  $2\pi|\partial\Sigma|$  and near the boundary of  $\Sigma$  equals  $ds \wedge d\theta$ . We clearly have

$$\int_\Sigma (\omega - d\lambda_1) = 0$$

and  $\omega - d\lambda_1 = 0$  near the boundary. Thus the de Rham theorem says we can find a 1-form  $\beta$  vanishing near the boundary such that  $d\beta = \omega - d\lambda_1$ . One may check  $\lambda = \lambda_1 + \beta$  is a form in  $S$ .

Now given  $\lambda \in S$  note that  $\phi^*\lambda$  is also in  $S$ . Consider the 1-form

$$\lambda_{(t,x)} = t\lambda_x + (1-t)(\phi^*\lambda)_x$$

on  $\Sigma \times [0, 1]$  where  $(x, t) \in \Sigma \times [0, 1]$  and set

$$\alpha_K = \lambda_{(t,x)} + Kdt.$$

**Exercise 3.15** Show that for sufficiently large  $K$  this form is a contact form.

It is clear that this form descends to a contact form on the mapping torus  $\Sigma_\phi$ . We now want to extend this form over the solid tori neighborhood of the binding. To this end consider the map  $\psi$  that glues the solid tori to the mapping torus. In coordinates  $(\varphi, (r, \vartheta))$  on  $S^1 \times D^2$  where  $D^2$  is the unit disk in the  $\mathbb{R}^2$  with polar coordinates we have

$$\psi(\varphi, r, \vartheta) = (r-1+\epsilon, -\varphi, \vartheta).$$

This is a map defined near the boundary of  $S^1 \times D^2$ . Pulling back the contact form  $\alpha_K$  using this maps gives

$$\alpha_\psi = K d\vartheta - (r+\epsilon) d\varphi.$$

We need to extend this over all of  $S^1 \times D^2$ . We will extend using a form of the form

$$f(r) d\varphi + g(r) d\vartheta.$$

**Exercise 3.16** Show this form is a contact form if and only if  $f(r)g'(r) - f'(r)g(r) > 0$ . Said another way, that

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix}, \begin{pmatrix} f'(r) \\ g'(r) \end{pmatrix}$$

is an oriented basis for  $\mathbb{R}^2$  for all  $r$ .

Near the boundary  $\alpha_\psi$  is defined with  $f(r) = -(r + \epsilon)$  and  $g(r) = K$ . Near the core of  $S^1 \times D^2$  we would like  $f(r) = 1$  and  $g(r) = r^2$ .

**Exercise 3.17** Show that  $f(r)$  and  $g(r)$  can be chosen to extend  $\alpha_\psi$  across the solid torus.

HINT: Consider the parameterized curve  $(f(r), g(r))$ . This curve is defined for  $r$  near 0 and 1; can we extend it over all of  $[0, 1]$  so that the position and tangent vector are never collinear?

□

**Proposition 3.18** (Giroux 2000, [21]). *Two contact structures supported by the same open book are isotopic.*

*Proof.* Let  $\alpha_0$  and  $\alpha_1$  be the contact forms for  $\xi_0$  and  $\xi_1$ , two contact structures that are supported by  $(B, \pi)$ . In the proof of Lemma 3.5 we constructed a contact form  $\alpha_R = \alpha + Rf(r)d\theta$  from  $\alpha$ . (See the proof of the lemma for the definitions of the various terms.) In a similar fashion we can construct  $\alpha_{0R}$  and  $\alpha_{1R}$  from  $\alpha_0$  and  $\alpha_1$ . These are all contact forms for all  $R \geq 0$ . Now consider

$$\alpha_s = s\alpha_{1R} + (1 - s)\alpha_{0R}.$$

**Exercise 3.19** For large  $R$  verify that  $\alpha_s$  is a contact form for all  $0 \leq s \leq 1$ .

HINT: There are three regions to consider when verifying that  $\alpha_s$  is a contact form. The region near the binding where  $f(t) = r^2$ , the region where  $f$  is not 1 and the region where  $f$  is 1. Referring back to the proof of Lemma 3.5 should help if you are having difficulty when considering any of these regions.

Thus we have an isotopy from  $\alpha_0$  to  $\alpha_1$ .

□

We now know that for each open book  $(B, \pi)$  there is a unique contact structure supported by  $(B, \pi)$ . We denote this contact structure by  $\xi_{(B, \pi)}$ . If we are concerned with the abstract open books we denote the contact structure by  $\xi_{(\Sigma, \phi)}$ .

**Theorem 3.20.** *We have*

$$\xi_{(\Sigma_0, \phi_0)} \# \xi_{(\Sigma_1, \phi_1)} = \xi_{(\Sigma_0, \phi_0) * (\Sigma_1, \phi_1)}$$

□

This theorem follows immediately from the proof of Theorem 2.17 concerning the effect of Murasugi sums on the 3-manifold. The theorem seems to have been known in some form or another for some time now but the first reference in the literature is in Torisu's paper [39].

**Exercise 3.21** Go back through the proof of Theorem 2.17 and verify that the contact structures are also connect summed.

HINT: If you have trouble see the proof of Theorem 4.6.

**Corollary 3.22** (Giroux 2000, [21]). *Let  $a$  be any arc in  $\Sigma$ , then*

$$M_{S_{(\pm, a)}(\Sigma, \phi)} \text{ is diffeomorphic to } M_{(\Sigma, \phi)}$$

and

$$\xi_{S_{(+, a)}(\Sigma, \phi)} \text{ is isotopic to } \xi_{(\Sigma, \phi)}$$

(where the corresponding manifolds are identified using the first diffeomorphism).

**Remark 3.23.** The contact structure  $\xi_{S_{(-, a)}(\Sigma, \phi)}$  is not isotopic to  $\xi_{(\Sigma, \phi)}$ ! One can show that these contact structures are not even homotopic as plane fields.

*Proof.* The first statement was proven in the previous section. For the second statement recall if  $(H^+, \pi_+)$  is the open book for the positive Hopf link then  $\xi_{(H^+, \pi_+)}$  is the standard contact structure on  $S^3$ . Thus

$$\xi_{S_{(+, a)}(\Sigma, \phi) * (H^+, \pi_+)} = \xi_{(\Sigma, \phi)} \# \xi_{(H^+, \pi_+)} = \xi_{(\Sigma, \phi)}$$

where all the equal signs mean isotopic. The last equality follows from the following exercise.

**Exercise 3.24** Show the contact manifold  $(S^3, \xi_{std})$  is the union of two standard (Darboux) balls.

**Exercise 3.25** Show  $\xi|_{M \setminus B}$  is tight. (If you do not know the definition of *tight* then see the beginning of the next section.)

HINT: Try to show the contact structure pulled back to the cover  $\Sigma \times \mathbb{R}$  of  $M \setminus B$  is tight. This will be much easier after you know something about convex surfaces and, in particular, Giroux's tightness criterion which is discussed at the beginning of the next section.

□

Note that we now have a well defined map

$$(2) \quad \Psi : \mathcal{O} \rightarrow \mathcal{C}$$

where

$$\mathcal{C} = \{\text{oriented contact structures on } M \text{ up to isotopy}\}$$

and

$$\mathcal{O} = \{\text{open book decompositions of } M \text{ up to positive stabilization}\}.$$

In the next section we will show that  $\Psi$  is onto and one-to-one.

#### 4. FROM CONTACT STRUCTURES TO OPEN BOOKS

We begin this section by recalling a few more basic facts and definitions from contact geometry. Again for more details see [1, 15, 20]. This is not meant to be an introduction to contact geometry, but simply to remind the reader of some important facts or to allow the reader with little background in contact geometry to follow some of the arguments below. First some facts about Legendrian curves.

- A curve  $\gamma$  in  $(M, \xi)$  is Legendrian if it is always tangent to  $\xi$ .
- Any curve may be  $C^0$  approximated by a Legendrian curve.

- If  $\gamma \subset \Sigma$  is a simple closed Legendrian curve in  $\Sigma$  then  $tw(\gamma, \Sigma)$  is the twisting of  $\xi$  along  $\gamma$  relative to  $\Sigma$ . Said another way, both  $\xi$  and  $\Sigma$  give  $\gamma$  a framing (that is, a trivialization of its normal bundle) by taking a vector field normal to  $\gamma$  and tangent to  $\xi$  or  $\Sigma$ , respectively; then  $tw(\gamma, \Sigma)$  measures how many times the vector field corresponding to  $\xi$  rotates as  $\gamma$  is traversed measured with respect to the vector field corresponding to  $\Sigma$ .

We now turn to surfaces in contact 3-manifolds and the fundamental dichotomy in 3-dimensional contact geometry: tight vs. overtwisted.

- If  $\Sigma$  is a surface in  $(M, \xi)$  then if at each point  $x \in \Sigma$  we consider  $l_x = \xi_x \cap T_x \Sigma$  we get a singular line field on  $\Sigma$ . This (actually any) line field can be integrated to give a singular foliation on  $\Sigma$ . This singular foliation is called the *characteristic foliation* and is denoted  $\Sigma_\xi$ .
- The contact structure  $\xi$  is called *overtwisted* if there is an embedded disk  $D$  such that  $D_\xi$  contains a closed leaf. Such a disk is called an overtwisted disk. If there are no overtwisted disks in  $\xi$  then the contact structure is called *tight*.
- The standard contact structures on  $S^3$  and  $\mathbb{R}^3$  are tight.
- If  $\xi$  is a tight contact structure and  $\Sigma$  is a surface with Legendrian boundary then we have the weak-Bennequin inequality

$$tw(\partial\Sigma, \Sigma) \leq \chi(\Sigma).$$

We now begin a brief discussion of convex surfaces. These have proved to be an invaluable tool in studying 3-dimensional contact manifolds.

- A surface  $\Sigma$  in  $(M, \xi)$  is called *convex* if there is a vector field  $v$  transverse to  $\Sigma$  whose flow preserves  $\xi$ . A vector field whose flow preserves  $\xi$  is called a *contact vector field*.
- Any closed surface is  $C^\infty$  close to a convex surface. If  $\Sigma$  has Legendrian boundary such that  $tw(\gamma, \Sigma) \leq 0$  for all components  $\gamma$  of  $\partial\Sigma$  then after a  $C^0$  perturbation of  $\Sigma$  near the boundary (but fixing the boundary)  $\Sigma$  will be  $C^\infty$  close to a convex surface.
- Let  $\Sigma$  be convex, with  $v$  a transverse contact vector field. The set

$$\Gamma_\Sigma = \{x \in \Sigma | v(x) \in \xi_x\}$$

is a multi-curve on  $\Sigma$  and is called the *dividing set*.

- Let  $\mathcal{F}$  be a singular foliation on  $\Sigma$  and let  $\Gamma$  be a multi-curve on  $\Sigma$ . The multi-curve  $\Gamma$  is said to *divide*  $\mathcal{F}$  if
  - (1)  $\Sigma \setminus \Gamma = \Sigma_+ \amalg \Sigma_-$
  - (2)  $\Gamma$  is transverse to  $\mathcal{F}$  and
  - (3) there is a vector field  $X$  and a volume form  $\omega$  on  $\Sigma$  so that
    - (a)  $X$  is tangent to  $\mathcal{F}$  at non-singular points and  $X = 0$  at the singular points of  $\mathcal{F}$  (we summarize this by saying  $X$  *directs*  $\mathcal{F}$ )
    - (b) the flow of  $X$  expands (contracts)  $\omega$  on  $\Sigma_+$  ( $\Sigma_-$ ) and
    - (c)  $X$  points out of  $\Sigma_+$ .
- If  $\Sigma$  is convex then  $\Gamma_\Sigma$  divides  $\Sigma_\xi$ .
- On any compact subset of  $\Sigma_+$  we can isotop  $\xi$  to be arbitrarily close to  $T\Sigma_+$  while keeping it transverse to  $\Gamma_\Sigma$ .
- If  $\Sigma$  is convex and  $\mathcal{F}$  is any other foliation divided by  $\Gamma_\Sigma$  then there is a  $C^0$  small isotopy, through convex surfaces, of  $\Sigma$  to  $\Sigma'$  so that  $\Sigma'_\xi = \mathcal{F}$ .



- If  $\gamma$  is a properly embedded arc or a closed curve on  $\Sigma$ , a convex surface, and all components of  $\Sigma \setminus \gamma$  contain some component of  $\Gamma_\Sigma \setminus \gamma$  then  $\Sigma$  may be isotoped through convex surfaces so that  $\gamma$  is Legendrian. This is called *Legendrian realization*.
- If  $\Sigma_1$  and  $\Sigma_2$  are convex,  $\partial\Sigma_1 = \partial\Sigma_2$  is Legendrian and the surfaces meet transversely, then the dividing curves interlace as shown in Figure 8 and we can round the corner to get a single smooth convex surface with dividing curves shown in Figure 8.

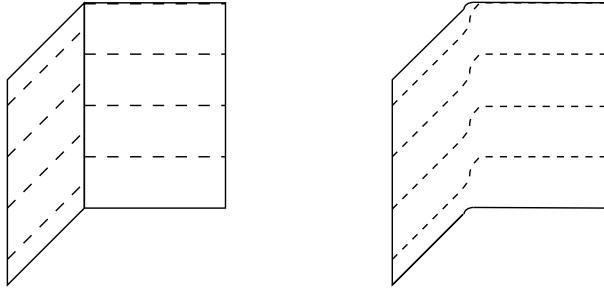


FIGURE 8. Two convex surfaces intersecting along their boundary, left, and the result of rounding their corners, right.

- If  $\gamma$  is a Legendrian simple closed curve on a convex surface  $\Sigma$  then  $tw(\gamma, \Sigma) = -\frac{1}{2}(\gamma \cap \Gamma_\Sigma)$ .
- If  $\Sigma$  is a convex surface then a small neighborhood of  $\Sigma$  is tight if and only if  $\Sigma \neq S^2$  and no component of  $\Gamma_\Sigma$  is a contractible circle or  $\Sigma = S^2$  and  $\Gamma_\Sigma$  is connected. This is called Giroux's tightness criterion.

Now that we know about convex surfaces we can discuss a fourth way to say that a contact structure is supported by an open book decomposition.

**Lemma 4.1.** *The contact structure  $\xi$  on  $M$  is supported by the open book decomposition  $(B, \pi)$  if and only if for every two pages of the open book that form a smooth surface  $\Sigma'$  the contact structure can be isotoped so that  $\Sigma'$  is convex with dividing set  $B \subset \Sigma'$  and  $\xi$  is tight when restricted to the components of  $M \setminus \Sigma'$ .*

This criterion for an open book to be supported by a contact structure is one of the easiest to check and is quite useful in practice. This lemma is essentially due to Torisu [39].

*Proof.* Assume  $\xi$  is supported by  $(B, \pi)$ . Let  $V_0$  and  $V_1$  be the closures of the complement of  $\Sigma'$  in  $M$ .

**Exercise 4.2** Show  $V_i$  is a handlebody.

HINT: Each  $V_i$  is diffeomorphic to  $\Sigma \times [0, 1]$  where  $\Sigma$  is a surface with boundary (*i.e.* the page).

It is not hard to show that  $\Sigma'$  is convex. Indeed  $\Sigma'$  is the union of two pages  $\Sigma_1$  and  $\Sigma_2$ . Each  $\Sigma_i$  has a transverse contact vector field  $v_i$ . Along  $\partial\Sigma_i$  the  $v_i$  point in opposite directions.

**Exercise 4.3** Show  $v_1$  and  $-v_2$  can be altered in a neighborhood of  $\partial\Sigma_1 = \partial\Sigma_2 = B$  so that they give a contact vector field  $v$  on  $\Sigma'$  so that  $B$  is the dividing set.

HINT: If you have trouble see [22].

In Exercise 3.25 you checked that  $\xi$  restricted to  $M \setminus B$  is tight. It is easy to contact isotop  $B$  to be disjoint from  $V_i$  so  $\xi$  restricted to  $V_i$  is tight.

The other implication immediately follows from the next lemma.  $\square$

**Lemma 4.4** (Torisu 2000, [39]). *Given an open book decomposition  $(B, \pi)$  of  $M$  there is a unique contact structure  $\xi$  that makes  $\Sigma'$  (the smooth union of two pages) convex with dividing set  $B$  and that is tight when restricted to each component of  $M \setminus \Sigma'$ .*

*Sketch of Proof.* Let  $\Sigma \subset \Sigma'$  be a page of the open book. Let  $\alpha_1, \dots, \alpha_n$  be a collection of disjoint properly embedded arcs in  $\Sigma$  that cut  $\Sigma$  into a 2-disk. Since each component of  $M \setminus \Sigma'$  is a handlebody  $V_i = \Sigma \times [0, 1]$  we can consider the disks  $D_j = \alpha_j \times [0, 1]$ . These disks cut  $V_i$  into a 3-ball. We can Legendrian realize  $\partial D_j$  on  $\Sigma'$  and make all the disks  $D_j$  convex. Now cutting  $V_i$  along these disks and rounding corners we get a tight contact structure on the 3-ball. Eliashberg [8] has shown that there is a unique tight contact structure on the 3-ball with fixed characteristic foliation. From this it follows that there is a unique tight contact structure on  $V_i$  with any fixed characteristic foliation divided by  $B$ . Finally this implies there is at most one contact structure on  $M$  satisfying the conditions in the lemma. The existence of one contact structure satisfying these conditions is given by Theorem 3.13.

**Exercise 4.5** Fill in the details of this proof.

HINT: If you have trouble read the section on convex surfaces in [15] or see [39].

$\square$

We are now ready to show that the map  $\Psi$  from open books to contact structures is onto.

**Theorem 4.6** (Giroux 2000, [21]). *Every oriented contact structure on a closed oriented 3-manifold is supported by an open book decomposition.*

*Proof.* We begin the proof with a definition.

**Definition 4.7.** *A contact cell decomposition of a contact 3-manifold  $(M, \xi)$  is a finite CW-decomposition of  $M$  such that*

- (1) *the 1-skeleton is a Legendrian graph,*
- (2) *each 2-cell  $D$  satisfies  $tw(\partial D, D) = -1$ , and*
- (3)  *$\xi$  is tight when restricted to each 3-cell.*

**Lemma 4.8.** *Every closed contact 3-manifold  $(M, \xi)$  has a contact cell decomposition.*

*Proof.* Cover  $M$  by a finite number of Darboux balls (this is clearly possible since  $M$  is compact). Note that since Darboux balls are by definition contactomorphic to a ball in the standard contact structure on  $\mathbb{R}^3$  we know  $\xi$  restricted to the Darboux balls is tight. Now take any finite CW-decomposition of  $M$  such that each 3-cell sits in some Darboux ball. Isotop the 1-skeleton to be Legendrian (this can be done preserving the fact that 3-cells sit in Darboux balls). Note that we have a CW-decomposition satisfying all but condition (2) of contact cell decomposition. To achieve this condition consider a 2-cell  $D$ . By the weak-Bennequin inequality we have  $tw(\partial D, D) \leq -1$ . Thus we can perturb each 2-cell to be convex (care must be taken at the boundary of the 2-cells). Since  $\Gamma_D$  contains no simple closed curves

and  $tw(\partial D, D) = -\frac{1}{2}(\Gamma_D \cap \partial D)$  we know that there are  $\frac{1}{2}(\Gamma_D \cap \partial D)$  components to  $\Gamma_D$ . If  $tw(\partial D, D) \neq -1$  there is more than one component to  $\Gamma_D$  and we can thus use Legendrian realization to realize arcs separating all the components of  $\Gamma_D$  by Legendrian arcs. If we add these arcs to the 2-skeleton then condition (2) of the definition is also satisfied.  $\square$

Suppose we have a contact cell decomposition of  $(M, \xi)$ . Denote its 1-skeleton by  $G$ . Given the (or any) Legendrian graph  $G$  the *ribbon of  $G$*  is a compact surface  $R = R_G$  satisfying

- (1)  $R$  retracts onto  $G$ ,
- (2)  $T_p R = \xi_p$  for all  $p \in G$ ,
- (3)  $T_p R \neq \xi_p$  for all  $p \in R \setminus G$ .

Clearly any Legendrian graph has a ribbon. Let  $B = \partial R$  and note that  $B$  is a transverse link.

**Claim.**  $B$  is the binding of an open book decomposition of  $M$  that supports  $\xi$ .

Clearly the proof follows from this claim.  $\square$

*Proof of Claim.* Since  $B$  is a transverse link there is a contactomorphism from each component of a neighborhood  $N(B)$  of  $B$  to an  $\epsilon$ -neighborhood of the  $z$ -axis in  $(\mathbb{R}^3, \ker(dz + r^2 d\theta)) / \sim$  where  $(r, \theta, z) \sim (r, \theta, z + 1)$ . Let  $X(B) = \overline{M} \setminus N(B)$  be the complement of  $N(B)$  and  $R_X = R \cap X(B)$ . We can choose a neighborhood  $N(R) = R_X \times [-\delta, \delta]$  of  $R_X$  in  $X(B)$  such that  $\partial R_X \times \{pt\}$  thought of as sitting in  $N(B)$  is a line with constant  $\theta$  value. Clearly  $N(R)$  is an  $R_X$  bundle over  $[-\delta, \delta]$ . Set  $X(R) = X(B) \setminus N(R)$ . See Figure 9.

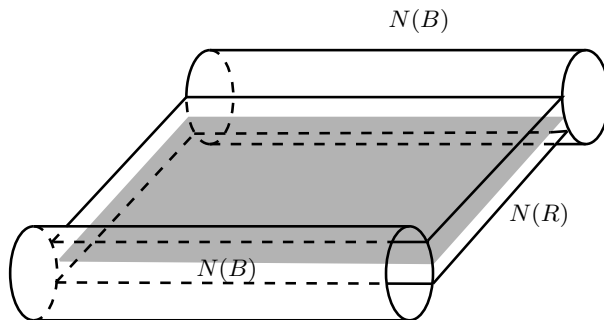


FIGURE 9. The neighborhoods  $N(B)$  and  $N(R)$ . The grey part is  $R_X$ .

We first show that  $X(R)$  is diffeomorphic to  $R_X \times [0, 1]$  and that  $X(B)$  is formed by identifying  $R_X \times \{0\}$  ( $R_X \times \{1\}$ ) in  $X(R)$  and  $R_X \times \{\delta\}$  ( $R_X \times \{-\delta\}$ ) in  $N(R)$ . Clearly this implies that  $X(B)$ , the complement of a neighborhood of  $B$ , is fibered and the fibration can be extended over  $N(B) \setminus B$  so that the boundary of the fibers is  $B$ . Note that  $\partial X(R) = A \cup F$  where  $A = (\partial X(R)) \cap N(B)$  is a disjoint union of annuli (one for each component of  $N(B)$ ) that are naturally fibered by circles of constant  $\theta$  value in  $N(B)$ . The subsurface  $F$  is defined to be the closure of the complement of  $A$  in  $\partial X(R)$ . Note that we can write  $F = F^- \cup F^+$ , where  $F^\pm$  is identified with  $R_X \times \{\pm\delta\}$  in  $N(R)$ .

**Exercise 4.9** Show that  $\partial X(R)$  is a convex surface with dividing set  $\Gamma_{\partial X(R)}$  equal to the union of the cores of  $A$  and such that  $F^\pm \subset (\partial X(R))_\pm$ . (Note,  $\partial X(R)$  is only piecewise smooth, but if we rounded the edges it would be convex.)

**Remark 4.10.** Throughout this part of the proof we will be discussing manifolds whose boundaries have corners. We do not want to smooth the corners. However, sometimes to understand the annuli  $A$  better we will think about rounding the corners, but once we have understood  $A$  sufficiently we actually will not round the corners.

Let  $D_1, \dots, D_k$  be the two cells in the contact cell decomposition of  $(M, \xi)$ . Recall that  $\partial D_i$  is Legendrian and has twisting number  $-1$ . Thus since  $R$  twists with the contact structure along the 1-skeleton  $G$  we can assume that  $B$  intersects  $D_i$  exactly twice for all  $i$ . Let  $D'_i = D_i \cap X(R)$ .

**Exercise 4.11** Show that it can be arranged that the  $D'_i$ 's intersect the region  $A \subset \partial X(R)$  in exactly two properly embedded arcs, and each arc runs from one boundary component of  $A$  to another.

**Exercise 4.12** Show that the interior of  $X(R)$  cut along all the  $D'_i$ 's is homeomorphic to  $M$  minus its 2-skeleton. That is,  $X(R)$  cut along the  $D'_i$ 's is a union of balls.

Using the Legendrian realization principle we can assume  $\partial D'_i$  is Legendrian. (Again as in the remark above we only Legendrian realize  $\partial D'_i$  to see what happens to the dividing curves when we cut  $X(R)$  along these disks. Once we have seen this we don't actually do the Legendrian realization.) Let's consider what happens to  $X(R)$  when we cut along  $D'_1$ ; denote the resulting manifold by  $X_1$ . Note: in  $\partial X_1$  there are two copies of  $D'_1$ . Let  $A_1$  be  $A$  sitting in  $\partial X_1$  union the two copies of  $D_1$ . See Figure 10.

**Exercise 4.13** Show that the components of  $A_1$  are annuli and that they have a natural fibration by  $S^1$  that is naturally related to the fibration on  $A$ .

Note that  $\partial X_1 \setminus A_1$  naturally breaks into two surfaces  $F_1^+$  and  $F_1^-$ , where  $F_1^\pm$  is obtained from  $F^\pm$  by cutting along a properly embedded arc.

**Exercise 4.14** Show  $\partial X_1$  is convex (once the corners are rounded) and its dividing set is the union of the cores of  $A_1$  and  $F_1^\pm \subset (\partial X_1)_\pm$ .

If we continue to cut along the  $D'_i$ 's we eventually get to  $X_k$  once we have cut along all the disks. From above we know  $X_k$  is a disjoint union of balls (all contained in 3-cells of our contact cell decomposition). Moreover, on  $\partial X_k$  we have that  $A_k$  a union of annuli whose cores give the dividing curves for  $\partial X_k$ . By the definition of contact cell decomposition we know that the contact structure when restricted to each component of  $X_k$  is tight. Thus we know  $A_k$  has exactly one component on the boundary of each component of  $X_k$ . Thus each component of  $X_k$  is a ball  $B^3$  with an annulus  $S$  that has a natural fibration by circles. Clearly  $B^3$  has a natural fibration by  $D^2$ 's that extends the fibration of  $S$  by circles. That is,  $B^3 = D^2 \times [0, 1]$  with  $(\partial D^2) \times [0, 1] = S$ .

**Exercise 4.15** Show that as we glue  $X_k$  together along the two components of  $D'_k$  in  $\partial X_k$  to get  $X_{k-1}$  we can glue the fibration of  $X_k$  by  $D^2$ 's together to get a fibration of  $X_{k-1}$  by surfaces that extend the fibration of  $A_{k-1}$  by circles.

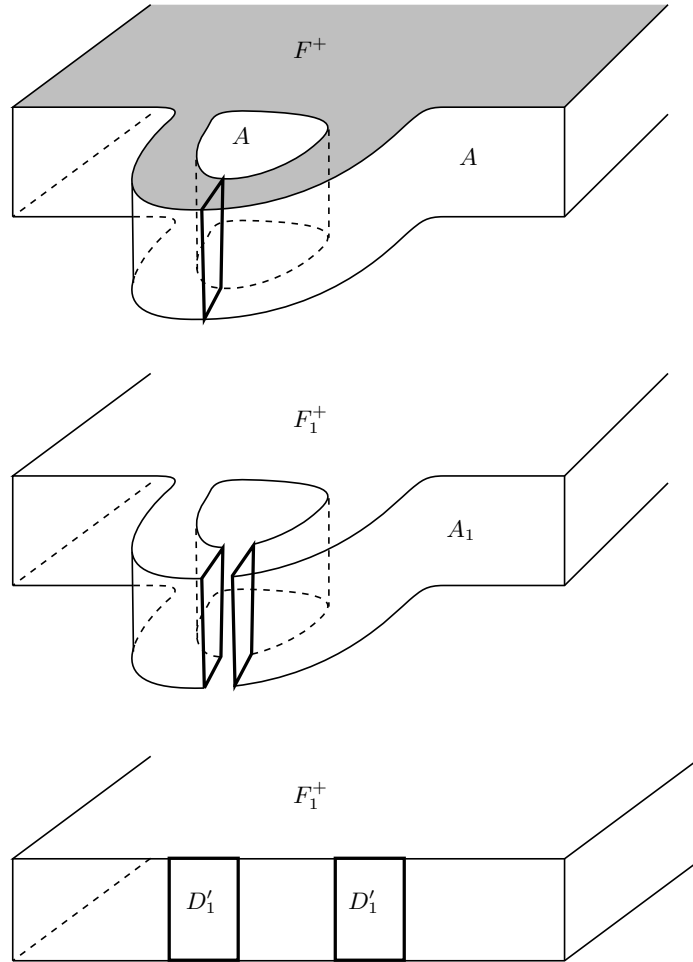


FIGURE 10. Top picture is  $X(R)$  with the boundary of  $D'_1$  drawn darkly. The middle picture is  $X_1$  right after the cut and the bottom picture is  $X_1$  after isotoping a little.

Thus, continuing in this fashion we get back to  $X(R)$  and see that it is fibered by surfaces that extend the fibration of  $A$  by circles. This clearly implies that  $X(R) = F^- \times [0, 1] = R_X \times [0, 1]$  and the surfaces  $R_X \times \{0\}$  and  $R_X \times \{1\}$  are glued to the boundary of  $N(R)$  as required above. Hence we have shown that  $X(B)$  is fibered over the circle by surfaces diffeomorphic to  $R$  and that the fibers all have boundary  $B$ . That is, we have demonstrated that  $B$  is the binding of an open book.

We now must show that this open book supports the contact structure  $\xi$ . Looking back through the proof it is not hard to believe that one may isotop the contact planes to be arbitrarily close to the pages of the open book, but it seems a little difficult to prove this directly. We will show the open book is compatible with the contact structure by showing that there is a Reeb vector field that is tangent to the binding and transverse to the pages. Recall that the neighborhood  $N(B)$  of the binding is contactomorphic to an  $\epsilon$ -neighborhood of the  $z$ -axis in  $(\mathbb{R}^3, \ker(dz +$

$r^2 d\theta)/\sim$  where  $(r, \theta, z) \sim (r, \theta, z+1)$ . Moreover, we can assume the pages intersect this neighborhood as the constant  $\theta$  annuli.

**Exercise 4.16** In the explicit model for  $N(B)$  find a Reeb vector field that is tangent to the binding and positively transverse to the pages of the open book in the neighborhood. Also make sure the boundary of  $N(B)$  is preserved by the flow of the Reeb field.

We can think of the Reeb fields just constructed as giving a contact field in the neighborhood of the boundary of  $R_X$  (recall this is the ribbon of the Legendrian 1-skeleton  $G$  intersected with the complement of  $N(B)$ ).

**Exercise 4.17** Show that this contact vector field defined in a neighborhood of  $\partial R_X$  can be extended to a contact vector field  $v$  over the rest of  $R_X$  so that it is transverse to  $R_X$  and there are no dividing curves. (This is OK since  $R_X$  is not a closed surface.) Note that since there are no dividing curves  $v$  is also transverse to  $\xi$ .

Use  $v$  to create the neighborhood  $N(R)$  of  $R_X$ . Since  $v$  is never tangent to the contact planes along  $R_X$  we can assume that this is the case in all of  $N(R)$ .

**Exercise 4.18** Show that a contact vector field which is never tangent to the contact planes is a Reeb vector field.

Thus we have a Reeb vector field defined on  $N(B) \cup N(R)$  that has the desired properties.

We now need to extend the Reeb vector field  $v$  over  $X(R)$ . From the construction of  $v$  we can assume we have  $v$  defined near the boundary of  $X(R)$  and as a vector field defined there it satisfies the following:

- (1)  $v$  is tangent to  $A \subset \partial X(R)$ .
- (2) There is a neighborhood  $N(A)$  of  $A$  in  $X(R)$  such that  $\partial N(A) = A \cup A^+ \cup A^- \cup A'$  where  $A^\pm = N(A) \cap F^\pm$  and  $A'$  is a parallel copy of  $A$  on the interior of  $X(R)$ .  $v$  is defined in  $N(A)$ , tangent to  $A \cup A'$ ,  $\pm v$  points transversely out of  $N(A)$  at  $A^\pm$  and  $v$  is transverse to the pages of the open book intersected with  $N(A)$ . Moreover the flow of  $v$  will take  $A^-$  to  $A^+$ .
- (3)  $\pm v$  points transversely out of  $X(R)$  along  $F^\pm$ .

We now want to construct a model situation into which we can embed  $X(R)$ . To this end let  $\Sigma = R_X \cup A' \cup -R_X$ , where  $A' = (\partial R_X) \times [0, 1]$  and the pieces are glued together so that  $\Sigma$  is diffeomorphic to the double of  $R_X$ . On  $\Sigma$  let  $\mathcal{F}$  be the singular foliation  $(R_X)_\xi$  on each of  $R_X$  and  $-R_X$  and extend this foliation across  $A'$  so that it is non singular there and the leaves of the foliation run from one boundary component to another. Let  $\Gamma$  be the union of the cores of the annuli that make up  $A'$ . It is easy to see that  $\mathcal{F}$  is divided by  $\Gamma$ . Given this one can create a vertically invariant contact structure  $\xi'$  on  $\Sigma \times \mathbb{R}$  such that  $(\Sigma \times \{t\})_{\xi'} = \mathcal{F}$  and the dividing set on  $\Sigma \times \{t\}$  is  $\Gamma$ , for all  $t \in \mathbb{R}$ . (See [22].) Note that  $\frac{\partial}{\partial t}$  restricted to  $R_X \times \mathbb{R}$  is a Reeb vector field since it is a contact vector field and positively transverse to  $\xi'$  in this region. Pick a diffeomorphism  $f : F^- \rightarrow (R_X \times \{0\}) \subset \Sigma \times \{0\}$  that sends  $(R_X)_\xi$  to  $\mathcal{F}$ . (Recall  $F^+ \cup F^- = (\partial X(R)) \setminus A$ .)

**Exercise 4.19** Show that the flow of  $v$  on  $R_X$  and  $\frac{\partial}{\partial t}$  on  $\Sigma \times \mathbb{R}$  allow you to extend  $f$  to a contact embedding of  $N(A)$  into  $\Sigma \times \mathbb{R}$ .

Thus we can use the flow of  $v$  and  $\frac{\partial}{\partial t}$  to extend  $f$  to a contact embedding of a neighborhood of  $\partial X(R)$  in  $X(R)$  into  $\Sigma \times \mathbb{R}$ .

**Exercise 4.20** Make sure you understand how to get the embedding near  $F^+$ .  
 HINT: From the previous exercise we have a neighborhood of the boundary of  $F^+$  embedded into  $R_X \times \{t_0\}$ , for some  $t_0$ . Show that there is an obvious way to extend this to an embedding of all of  $F^+$  to  $R_X \times \{t_0\}$ .

Of course this extension of  $f$ , which we also call  $f$ , takes the Reeb field  $v$  to the Reeb field  $\frac{\partial}{\partial t}$ . We can clearly extend  $f$  to an embedding, but not necessarily a contact embedding, of all of  $X(R)$  into  $\Sigma \times \mathbb{R}$ . The following exercises allow us to isotop  $f$ , relative to a neighborhood of the boundary, to a contact embedding and thus we may extend  $v$  to all of  $X(R)$  by  $\frac{\partial}{\partial t}$ . This gives us a Reeb vector field on  $M$  which demonstrates that the open book supports  $\xi$ .

**Exercise 4.21** Let  $H$  be a handlebody and  $D_1, \dots, D_g$  be properly embedded disks that cut  $H$  into a 3-ball. Given any singular foliation  $\mathcal{F}$  on the boundary of  $H$  that is divided by  $\Gamma$  for which  $\partial D_i \cap \Gamma = 2$ , for all  $i$ , then there is at most one tight contact structure on  $H$ , up to isotopy, that induces  $\mathcal{F}$  on  $\partial H$ .

HINT: This is a simple exercise in convex surface theory. See [15].

**Exercise 4.22** Show the contact structure  $\xi'$  on  $\Sigma \times \mathbb{R}$  is tight. (This is easy using Giroux's tightness criterion.) Also show the contact structure  $\xi$  restricted to  $X(R)$  is tight.

HINT: The second part is not so easy. The idea is that if you can cut up a handlebody by disks, as in the previous exercise, and the 3-ball you end up with has a tight contact structure on it, then the original contact structure on the handlebody is tight. See [26].

□

We have the following immediate useful corollaries.

**Corollary 4.23.** *If  $L$  is a Legendrian link in  $(M, \xi)$  then there is an open book decomposition supporting  $\xi$  such that  $L$  sits on a page of the open book and the framing given by the page and by  $\xi$  agree.*

*Proof.* Simply include the Legendrian link  $L$  in the 1-skeleton of the contact cell decomposition. □

**Example 4.24** Figure 11 illustrates Corollary 4.23 for two knots in  $S^3$  with its standard contact structure.

Using positive stabilizations we can see the following.

**Corollary 4.25.** *Any contact manifold is supported by an open book with connected binding.*

**Corollary 4.26** (Contact Bing). *A contact manifold  $(M, \xi)$  is the standard tight contact structure on  $S^3$  if and only if every simple closed curve is contained in a Darboux ball.*

**Exercise 4.27** Prove this last corollary.

HINT: There is a unique tight contact structure on  $B^3$  inducing a fixed characteristic foliation on the boundary [8].

**Theorem 4.28** (Giroux 2002, [21]). *Two open books supporting the same contact manifold  $(M, \xi)$  are related by positive stabilizations.*

To prove this theorem we need the following lemma.

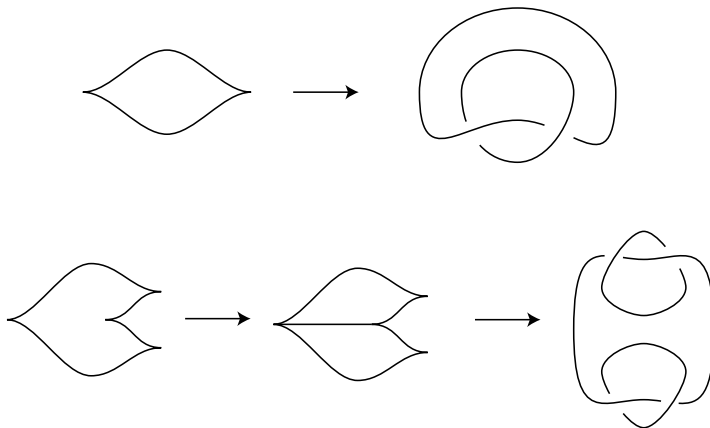


FIGURE 11. On the top left we have a Legendrian unknot that is the 1-skeleton of a contact cell decomposition of  $S^3$ . The resulting open book is shown on the upper right. On the bottom left we start with a Legendrian unknot, moving to the right we add a Legendrian arc to get the 1-skeleton of a contact cell decomposition. The bottom right shows the resulting open book.

**Lemma 4.29.** *Any open book supporting  $(M, \xi)$ , after possibly positively stabilizing, comes from a contact cell decomposition.*

*Proof.* Let  $\Sigma$  be a page of the open book and  $G$  be the core of  $\Sigma$ . That is,  $G$  is a graph embedded in  $\Sigma$  onto which  $\Sigma$  retracts. We can Legendrian realize  $G$ .

**Remark 4.30.** The Legendrian realization principle is for curves, or graphs, on a closed convex surface or a convex surface with Legendrian boundary. The pages of an open book are convex but their boundary is transverse to the contact structure so we cannot apply the Legendrian realization principle as it is usually stated. Nonetheless since we can keep the characteristic foliation near the boundary fixed while trying to realize a simple closed curve or graph, we can still realize it. But recall the curve or graph must be non-isolating. In this context this means that all components of the complement of the curve in the surface should contain a boundary component. To see this review Giroux's proof of realization.

Note that  $\Sigma$  is the ribbon of  $G$ . Let  $N$  be a neighborhood of  $\Sigma$  such that  $\partial\Sigma \subset \partial N$ . Let  $\alpha_i$  be a collection of properly embedded arcs on  $\Sigma$  that cut  $\Sigma$  into a disk. Let  $\tilde{A}_i$  be  $\alpha_i \times [0, 1]$  in  $M \setminus \Sigma = \Sigma \times [0, 1]$  and  $A'_i = \tilde{A}_i \cap \overline{(M \setminus N)}$ . Note that  $A'_i$  intersects  $\partial\Sigma$  on  $\partial N$  exactly twice. Thus if we extend the  $A_i$ 's into  $N$  so their boundaries lie on  $G$  then the twisting of  $\Sigma$ , and hence  $\xi$ , along  $\partial A_i$  with respect to  $A_i$  is  $-1, 0$  or  $1$ . If all the twisting is  $-1$  then we have a contact cell decomposition (recall that the contact structure restricted to the complement of  $\Sigma$  is tight). Thus we just need to see how to reduce the twisting of  $\xi$  along  $\partial A_i$ .

Suppose  $\partial A_i$  has twisting  $0$ . Positively stabilize  $\Sigma$  as shown in Figure 12. Note the curve  $C$  shown in the picture can be assumed to be Legendrian and bounds a disk  $D$  in  $M$ . Now isotop  $G$  across  $D$  to get a new Legendrian graph with all the  $A_j$ 's unchanged except that  $A_i$  is replaced with the disk  $A'_i$  obtained from  $A_i$  by isotoping across  $D$ . We also add  $C$  to  $G$  and add  $D$  to the 2-skeleton.



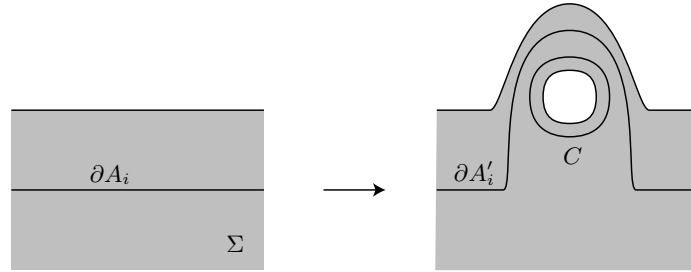


FIGURE 12. On the left is part of  $\Sigma$  and  $\partial A_i$  near  $\partial\Sigma$ . On the right is the stabilized  $\Sigma, C$  and  $\partial A'_i$ . (This picture is abstract. If we were drawing it ambiently the added 1-handle would have a full left-handed twist in it.)

**Exercise 4.31** Show that the twisting of  $\xi$  along  $\partial A'_i$  is one less than the twisting along  $\partial A_i$ .

Clearly the twisting of  $\xi$  along  $D$  is  $-1$ . Thus we can reduce the twisting of  $\xi$  along  $\partial A_i$  as needed and after sufficiently many positive stabilizations we have an open book that comes from a contact cell decomposition.  $\square$

*Proof of Theorem 4.28.* Given two open books  $(B, \pi)$  and  $(B', \pi')$  supporting  $(M, \xi)$  we can assume they both come from contact cell decompositions by using Lemma 4.29. Now, given two contact cell decompositions one can show that they are related by a sequence of the following:

- (1) A subdivision of a 2-cell by a Legendrian arc intersecting the dividing set one time.
- (2) Add a 1-cell  $c'$  and a 2-cell  $D$  so that  $\partial D = c' \cup c$  where  $c$  is part of the original 1-skeleton and  $tw(\partial D, D) = -1$ .
- (3) Add a 2-cell  $D$  whose boundary is already in the 1-skeleton and  $tw(\partial D, D) = -1$ .

Thus the theorem follows from the following exercises.

**Exercise 4.32** Show that (3) does not change the open book associated to the cell decomposition.

**Exercise 4.33** Show that (1) and (2) positively stabilize the open book associated to the cell decomposition.

HINT: Show that an arc  $a$  is added to the 1-skeleton and a disk  $D$  to the 2-skeleton so that  $\partial D$  is part of the old 1-skeleton union  $a$ ,  $\Gamma_D$  is a single arc and  $a \cap \Gamma_D$  is one point. Show that adding such an arc to the one skeleton is equivalent to positively stabilizing the open book.

$\square$

## 5. SYMPLECTIC COBORDISMS AND CAPS.

A contact manifold  $(M, \xi)$  is called *(weakly) symplectically fillable* if there is a compact symplectic 4-manifold  $(X, \omega)$  such that  $\partial X = M$  and  $\omega|_\xi \neq 0$ . Many applications of contact geometry to topology (see [28, 34] and the discussion in the introduction) rely on the following theorem.

**Theorem 5.1** (Eliashberg 2004 [6]; Etnyre 2004 [14]). *If  $(X, \omega)$  is a symplectic filling of  $(M, \xi)$  then there is a closed symplectic manifold  $(W, \omega')$  and a symplectic embedding  $(X, \omega) \rightarrow (W, \omega')$ .*

Partial results aimed towards this theorem were obtained by many people. Specifically, Lisca and Matić established this result for Stein fillable manifolds in [31] and later work of Akbulut and Ozbagci [2] coupled with work of Plamenevskaya [35] provided an alternate proof in this case (for unfamiliar terminology see the next paragraph). For strongly fillable manifolds this was proven by Gay in [19] and follows trivially from Theorem 1.3 in [16]. The full version of this theorem also follows fairly easily from [37].

In the process of proving Theorem 5.1 we will need to take a few detours. The first concerns various types of symplectic fillings and the second concerns Legendrian/contact surgery. These two detours occupy the next two subsections. We return to the proof of Theorem 5.1 in Subsection 5.3. In the final section we discuss the relation between open book decompositions and overtwistedness.

**5.1. Symplectic fillings.** A contact manifold  $(M, \xi)$  is said to be *strongly symplectically filled* by the symplectic manifold  $(X, \omega)$  if  $X$  is compact,  $\partial X = M$  and there is a vector field  $v$  transversely pointing out of  $X$  along  $M$  such that the flow of  $v$  dilates  $\omega$  (that is to say the Lie derivative of  $\omega$  along  $v$  is a positive multiple of  $\omega$ ). The symplectic manifold  $(X, \omega)$  is said to have convex boundary if there is a contact structure  $\xi$  on  $\partial X$  that is strongly filled by  $(X, \omega)$ . We say that  $(X, \omega)$  is a *strong concave filling* if  $(X, \omega)$  and  $v$  are as above except that  $v$  points into  $X$ . Note that given a symplectic manifold  $(X, \omega)$  with a dilating vector field  $v$  transverse to its boundary then  $\iota_v \omega$  is a contact form on  $\partial X$ . If  $v$  points out of  $X$  then the contact form gives an oriented contact structure on  $\partial X$  and if  $v$  points into  $X$  then it gives an oriented contact structure on  $-\partial X$ . (Recall Remark 3.2)

Given a contact manifold  $(M, \xi)$  let  $\alpha$  be a contact form for  $\xi$ , then consider  $W = M \times \mathbb{R}$  and set  $\omega_W = d(e^t \alpha)$ , where  $t$  is the coordinate on  $\mathbb{R}$ . It is easy to see that  $\omega_W$  is a symplectic form on  $W$  and the vector field  $v = \frac{\partial}{\partial t}$  is a dilating vector field for  $\omega_W$ . The symplectic manifold  $(W, \omega_W)$  is called the *symplectization* of  $(M, \xi)$ .

**Exercise 5.2** Given any other contact form  $\alpha'$  for  $\xi$  (note that this implies that  $\alpha' = g\alpha$  for some function  $g : M \rightarrow (0, \infty)$ ) show there is some function  $f$  such that  $\alpha' = F^*(\iota_v \omega_W)$  where  $F : M \rightarrow W : x \mapsto (x, f(x))$ .

It can be shown that if  $(X, \omega)$  is a strong symplectic filling (strong concave filling) of  $(M, \xi)$  then there is a neighborhood  $N$  of  $M$  in  $X$ , a function  $f$ , a one-sided neighborhood  $N_f$  of the graph of  $f$  in  $W$  with  $N_W$  lying below (above) the graph and a symplectomorphism  $\psi : N_W \rightarrow N$ . See Figure 13. Thus we have a model for a neighborhood of a contact manifold in a strong symplectic filling.

**Exercise 5.3** If  $(X_1, \omega_1)$  is a strong symplectic filling of  $(M, \xi)$  and  $(X_2, \omega_2)$  is a strong concave filling of  $(M, \xi)$  then show  $X = X_1 \cup X_2$  has a symplectic structure  $\omega$  such that  $\omega|_{X_1} = \omega_1$  and  $\omega|_{X_2 \setminus N} = c\omega_2$  where  $N$  is a neighborhood of  $\partial X_2$  in  $X_2$  and  $c > 0$  is a constant.

HINT: Look at Figure 13.

Thus we can use strong symplectic fillings to glue symplectic manifolds together. This is not, in general, possible with a weak symplectic filling.

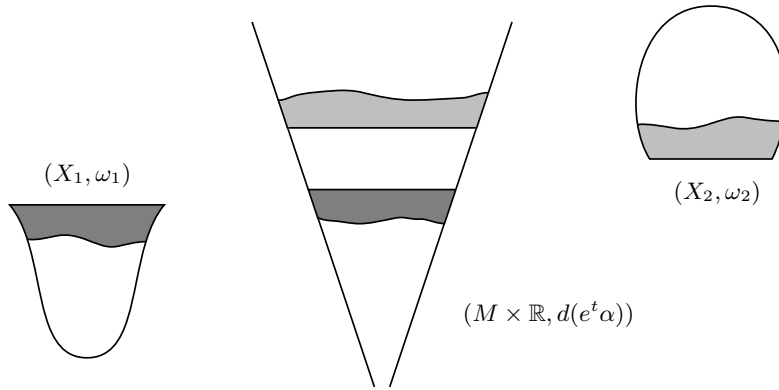


FIGURE 13. The symplectization of  $(M, \xi)$ , middle, and a symplectic manifold with convex, left, and concave, right, boundary.

Recall that a Stein manifold is a triple  $(X, J, \psi)$  where  $J$  is a complex structure on  $X$  and  $\omega_\psi(v, w) = -d(d\psi \circ J)(v, w)$  is non-degenerate. A contact manifold  $(M, \xi)$  is called *Stein fillable* (or *holomorphically fillable*) if there is a Stein manifold  $(X, J, \psi)$  such that  $\psi$  is bounded from below,  $M$  is a non-critical level of  $\psi$  and  $-(d\psi \circ J)$  is a contact form for  $\xi$ . It is customary to think of  $X$  as  $\psi^{-1}((-\infty, c])$  where  $M = \psi^{-1}(c)$ . Thus we can think of  $X$  as a compact manifold (Stein manifolds themselves are never compact).

In [11, 41] it was shown how to attach a 1-handle to the boundary of a symplectic manifold with convex boundary and extend the symplectic structure over the 1-handle so as to get a new symplectic manifold with convex boundary. They also showed the same could be done when a 2-handle is attached along a Legendrian knot with framing one less than the contact framing. In fact we have the following characterization of Stein manifolds.

**Theorem 5.4** (Eliashberg 1990, [11]). *A 4-manifold  $X$  is Stein if and only if  $X$  has a handle decomposition with only 0-handles, 1-handles and 2-handles attached along Legendrian knots with framing one less than the contact framing.*  $\square$

Summarizing the relations between various notions of filling and tightness we have

$$\text{Tight} \supset \text{Weakly Fillable} \supset \text{Strongly Fillable} \supset \text{Stein Fillable}.$$

The first two inclusions are strict, see [16] and [7] respectively. It is unknown whether or not the last inclusion is strict. We have the following useful fact.

**Theorem 5.5** (Eliashberg 1991, [9]; Ohta and Ono 1999, [33]). *If  $M$  is a rational homology sphere then any weak filling of  $(M, \xi)$  can be deformed into a strong filling.*  $\square$

**5.2. Contact surgery.** Let  $L$  be a Legendrian knot in a contact 3-manifold  $(M, \xi)$ . It is well known (see [15, 20]) that  $L$  has a neighborhood  $N_L$  that is contactomorphic to a neighborhood of the  $x$ -axis in

$$(\mathbb{R}^3, \ker(dz - ydx)) / \sim,$$

where  $\sim$  identifies  $(x, y, z)$  with  $(x + 1, y, z)$ . With respect to these coordinates on  $N_L$  we can remove  $N_L$  from  $M$  and topologically glue it back with a  $\pm 1$ -twist (that is, we are doing Dehn surgery along  $L$  with framing the contact framing  $\pm 1$ ). Call the resulting manifold  $M_{(L, \pm 1)}$ . There is a unique way, up to isotopy, to extend  $\xi|_{M \setminus N_L}$  to a contact structure  $\xi_{(L, \pm 1)}$  over all of  $M_{(L, \pm 1)}$  so that  $\xi_{(L, \pm 1)}|_{N_L}$  is tight (see [27]). The contact manifold  $(M, \xi)_{(L, \pm 1)} = (M_{(L, \pm 1)}, \xi_{(L, \pm 1)})$  is said to be obtained from  $(M, \xi)$  by  $\pm 1$ -contact surgery along  $L$ . It is customary to refer to  $-1$ -contact surgery along  $L$  as *Legendrian surgery* along  $L$ .

**Question 1.** *Is  $\xi_{(L, -1)}$  tight if  $\xi$  is tight?*

If the original contact manifold  $(M, \xi)$  is not closed then it is known that the answer is sometimes NO, see [26]. But there is no known such example on a closed manifold. It is known, by a combination of Theorems 5.4 and 5.5, that Legendrian surgery (but not  $+1$ -contact surgery!) preserves any type of symplectic fillability. (Similarly,  $+1$ -contact surgery preserves non-fillability.) We have the following result along those lines.

**Theorem 5.6** (Eliashberg 1990, [11]; Weinstein 1991, [41]). *Given a contact 3-manifold  $(M, \xi)$  let  $(W = M \times [0, 1], \omega = d(e^t \alpha))$  be a piece of the symplectization of  $(M, \xi)$  discussed in the last section. Let  $L$  be a Legendrian knot sitting in  $(M, \xi)$  thought of as  $M \times \{1\}$ . Let  $W'$  be obtained from  $W$  by attaching a 2-handle to  $W$  along  $L \subset M \times \{1\}$  with framing one less than the contact framing. Then  $\omega$  may be extended over  $W'$  so that the upper boundary is still convex and the induced contact manifold is  $(M_{(L, -1)}, \xi_{(L, -1)})$ . Moreover, if the 2-handle was added to a Stein filling (respectively weak filling, strong filling) of  $(M, \xi)$  then the resulting manifold would be a Stein filling (respectively weak filling, strong filling) of  $(M_{(L, -1)}, \xi_{(L, -1)})$ .  $\square$*

We now want to see how contact surgery relates to open book decompositions. The main result along these lines is the following.

**Theorem 5.7.** *Let  $(\Sigma, \phi)$  be an open book supporting the contact manifold  $(M, \xi)$ . If  $L$  is a Legendrian knot on the page of the open book then*

$$(M, \xi)_{(L, \pm 1)} = (M_{(\Sigma, \phi \circ D_L^\mp)}, \xi_{(\Sigma, \phi \circ D_L^\mp)}).$$

*Proof.* We begin by ignoring the contact structures and just concentrating on the manifold. We have a simple closed curve  $L$  on the page  $\Sigma$  of the open book. Recall  $M \setminus \text{nbhd } B$  is the mapping cylinder  $\Sigma_\phi$ . We will think of  $L$  as sitting on  $\Sigma \times \{\frac{1}{2}\}$  in  $\Sigma \times [0, 1]$ , then by moding out by the identification  $(\phi(x), 0) \sim (x, 1)$  we will have  $L$  on a page in  $\Sigma_\phi \subset M$ .

**Exercise 5.8** Show that cutting  $\Sigma_\phi$  open along  $\Sigma \times \{\frac{1}{2}\}$  and regluing using  $D_L^\pm$  will give you a manifold diffeomorphic to  $\Sigma_{\phi \circ D_L^\pm}$ .

Let  $N_\Sigma$  be a closed tubular neighborhood of  $L$  in  $\Sigma \times \{\frac{1}{2}\}$ . Then a neighborhood  $N$  of  $L$  in  $M$  is given by  $N_\Sigma \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ . We can assume the support of  $D_L^\mp$  is in  $N_\Sigma$ . Thus if  $N'$  is neighborhood of  $L$  in  $\Sigma_{\phi \circ D_L^\mp}$  corresponding to  $N$  in  $\Sigma_\phi$  then

$$\Sigma_\phi \setminus N = \Sigma_{\phi \circ D_L^\mp} \setminus N'.$$

So clearly  $M_{(\Sigma, \phi)} \setminus N$  is diffeomorphic to  $M_{(\Sigma, \phi \circ D_L^\mp)} \setminus N'$ . Said another way  $M_{(\Sigma, \phi \circ D_L^\mp)}$  is obtained from  $M_{(\Sigma, \phi)}$  by removing a solid torus and gluing it back in, *i.e.* by a Dehn surgery along  $L$ . We are left to see that the Dehn surgery is a  $\pm 1$  Dehn

surgery with respect to the framing on  $L$  coming from the page on which it sits. See Figure 14 while reading the rest of this paragraph. Note that we get  $N'$  from  $N$  by

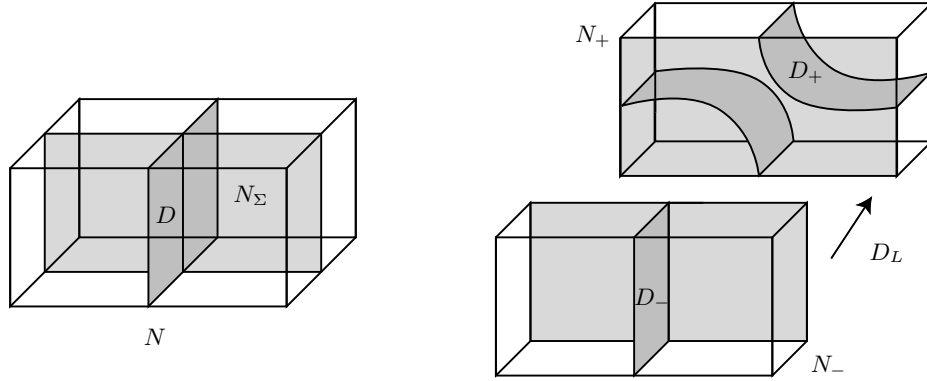


FIGURE 14. On the right is the neighborhood  $N$  with its meridional disk  $D$ . On the right is the neighborhood  $N'$ . (The right- and left-hand sides of each cube are identified to get a solid torus.)

cutting  $N$  along  $N_\Sigma$  and regluing using  $D_L^\mp$ . We get a meridional disk  $D$  for  $N$  by taking an arc  $a$  on  $N_\Sigma \times \{\frac{1}{2} - \epsilon\}$  running from one boundary component to the other and setting  $D = a \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ . Let  $N_- = N_\Sigma \times [\frac{1}{2} - \epsilon, \frac{1}{2}]$  and  $N_+ = N_\Sigma \times [\frac{1}{2}, \frac{1}{2} + \epsilon]$ . Thus we get  $N'$  by gluing  $\Sigma \times \{\frac{1}{2}\}$  in  $N_+$  to  $\Sigma \times \{\frac{1}{2}\}$  in  $N_-$ . Set  $D_- = D \cap N_-$ . Then  $a_- = D_- \cap \Sigma \times \{\frac{1}{2}\}$  in  $N_-$  is taken to  $a_+ \subset \Sigma \times \{\frac{1}{2}\} \subset N_+$ . So  $D'$ , the meridional disk in  $N'$ , is  $D_- \cup a_+ \times [\frac{1}{2} - \epsilon, \frac{1}{2}]$ . Thus on  $\partial M_{(\Sigma, \phi)} \setminus N = \partial M_{(\Sigma, \phi \circ D_L^\mp)} \setminus N'$  the curve  $\partial D'$  is homologous to  $\partial D \pm L'$ , where  $L'$  is a parallel copy of  $L$  lying on  $\partial N$ . Thus  $M_{(\Sigma, \phi \circ D_L^\mp)}$  is obtained from  $M_{(\Sigma, \phi)}$  by a  $\pm 1$  Dehn surgery on  $L$ .

We now must see that the contact structure one gets from  $\pm 1$ -contact surgery on  $L$  is the contact structure supported by the open book  $(\Sigma, \phi \circ D_L^\mp)$ . To do this we consider the definition of compatible open book involving the Reeb vector field.

**Exercise 5.9** Think about trying to show that  $\xi$  can be isotoped arbitrarily close to the pages of the open book. Intuitively this is not too hard to see (but as usual, making a rigorous proof out of this intuition is not so easy).

Since  $(\Sigma, \phi)$  is compatible with  $\xi$  there is a Reeb vector field  $X$  for  $\xi$  such that  $X$  is positively transverse to the pages and tangent to the binding. Notice our neighborhood  $N = N_\Sigma \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$  is such that  $X$  is transverse to all the  $N_\Sigma \times \{t\}$  and the flow of  $X$  preserves these pages. As usual we will now consider a model situation. Consider  $(\mathbb{R}^3, \ker(dz - y dx)) / \sim$ , where  $(x, y, z) \sim (x + 1, y, z)$ . It is easy to arrange that the foliation on  $N_\Sigma$  is the same as the foliation on  $\{(x, y, z) \mid -\delta \leq y \leq \delta\}$ . Thus we can assume the contact structure on  $N$  is contactomorphic to the contact structure on  $N_m = \{(x, y, z) \mid -\epsilon \leq z \leq \epsilon, -\delta \leq y \leq \delta\}$ . Moreover, we can assume this contactomorphism takes a Reeb vector field for  $\xi$  to  $\frac{\partial}{\partial z}$ , the Reeb vector field for  $dz - y dx$ . (We do this by picking an identification of  $N_\Sigma$  and the annulus in the  $xy$ -plane that preserves the characteristic foliation and then using the Reeb vector fields to extend this identification.) Let  $\psi : \partial N_m \rightarrow \partial N$  be the diffeomorphism that agrees with the above contactomorphism everywhere except

on  $S_u = \{(x, y, z) | z = \epsilon, -\delta \leq y \leq \delta\} \subset \partial N_m$ , where it differs by  $D_L^\pm$ . Note that gluing  $N_m$  to  $M \setminus N$  using  $\psi$  will yield the manifold  $M_{(L, \pm 1)}$ . Use  $\psi$  to pullback the characteristic foliation on  $\partial(M \setminus N)$  to  $\partial N_m$ . Note that the characteristic foliation agrees with the characteristic foliation on  $\partial N_m \setminus S_u$  induced by  $\ker(dz - y dx)$ .

**Exercise 5.10** Show there is a function  $f : \{(x, y) | -\delta \leq y \leq \delta\} \rightarrow \mathbb{R}$  that equals  $\epsilon$  near  $|y| = \delta$  such that the characteristic foliation on the graph of  $f$  agrees with the foliation on  $S_u$  that is pulled back from  $N$  by  $\psi$ . (By “agrees with” I mean that  $S_u$  and the graph of  $f$  are isotopic rel boundary so that the isotopy takes the pulled back foliation to the characteristic foliation on the graph of  $f$ .)

HINT: Figure out what the pullback foliation is first. Then experiment with perturbing the graph of the constant function.

Now let  $N'_m$  be the region bounded by  $\partial N_m \setminus S_u$  union the graph of  $f$ . There is a natural way to think of  $\psi$  as a map from  $\partial N'_m$  to  $\partial(M \setminus N)$  that preserves the characteristic foliation. Furthermore we can extend  $\psi$  to a neighborhood of the boundary so that it preserves Reeb vector fields. The contact structures  $\xi_{M \setminus N}$  and  $\ker(dz - y dx)|_{N'_m}$  glue to give a contact structure on  $M_{(L, \pm 1)}$ . We can also glue up the Reeb vector fields to get a Reeb vector field on  $M_{(L, \pm 1)}$  that is transverse to the pages of the obvious open book and tangent to the binding.  $\square$

An easy corollary of this theorem is the following.

**Theorem 5.11** (Giroux 2002, [21]). *A contact manifold  $(M, \xi)$  is Stein fillable if and only if there is an open book decomposition for  $(M, \xi)$  whose monodromy can be written as a composition of right-handed Dehn twists.*

*Proof.* We start by assuming that there is an open book  $(\Sigma, \phi)$  supporting  $(M, \xi)$  for which  $\phi$  is a composition of right-handed Dehn twists. Let us begin by assuming that  $\phi$  is the identity map on  $\Sigma$ . In Exercise 2.10 you verified that  $M = \#_{2g+n-1} S^1 \times S^2$ , where  $g$  is the genus of  $\Sigma$  and  $n$  is the number of boundary components. Eliashberg has shown that  $\#_{2g+n-1} S^1 \times S^2$  has a unique strong symplectic filling [10]. This filling  $(W, \omega)$  is also a Stein filling. Thus we are done if  $\phi$  is the identity map.

Now assume  $\phi = D_\gamma^+$  where  $\gamma$  is a simple non-separating closed curve on  $\Sigma$ . We can use the Legendrian realization principle to make  $\gamma$  a Legendrian arc on a page of the open book. (Recall that even though our convex surface does not have Legendrian boundary we can still use the Legendrian realization principle. See Remark 4.30.) (Note that we required  $\gamma$  to be non-separating so that we could use the Legendrian realization principle.) We know  $(M_{(\Sigma, id)}, \xi_{(\Sigma, id)})$  is Stein filled by  $(W, \omega)$  so by Theorems 5.6 and 5.7 we can attach a 2-handle to  $W$  to get a Stein filling of  $(M_{(\Sigma, D_\gamma^+)}, \xi_{(\Sigma, D_\gamma^+)})$ . If  $\phi$  is a composition of more than one right-handed Dehn twist along non-separating curves in  $\Sigma$  we may clearly continue this process to obtain a Stein filling of  $(M_{(\Sigma, \phi)}, \xi_{(\Sigma, \phi)})$ . The only thing left to consider is when one or more of the curves on which we Dehn twist is separating. Suppose  $\gamma$  is separating. If both components of the complement of  $\gamma$  contain parts of the boundary of the page then we can still realize  $\gamma$ . Thus we only have a problem when there is a subsurface  $\Sigma'$  of  $\Sigma$  such that  $\partial \Sigma' = \gamma$ . In this case we can use the “chain relation” (see Theorem 6.5 in the Appendix) to write  $D_\gamma^+$  as a composition of positive Dehn twists along non-separating curves in  $\Sigma'$ .

For the other implication we assume that  $(M, \xi)$  is Stein fillable by say  $(W, J, \psi)$ . According to Eliashberg’s Theorem 5.4,  $W$  has a handle decomposition with only

1-handles and 2-handles attached along Legendrian knots with framing one less than the contact framing. Let  $W'$  be the union of the 0- and 1-handles. Clearly  $M' = \partial W' = \#_k S^1 \times S^2$  and the induced contact structure  $\xi'$  is tight. So we have a Legendrian link  $L$  in  $M'$  on which we can perform Legendrian surgery to obtain  $(M, \xi)$ . Now according to Corollary 4.23 there is an open book decomposition  $(\Sigma, \phi)$  for  $(M', \xi')$  such that  $L$  sits on a page of the open book. At the moment  $\phi$  might not be the composition of right-handed Dehn twists. According to a theorem of Eliashberg [8] and Colin [5]  $M'$  has a unique tight contact structure. We know there is a surface  $\Sigma'$  such that  $M' = M_{(\Sigma', id)}$  and the supported contact structure is tight. According to Giroux's Stabilization Theorem 4.28 we can positively stabilize  $(\Sigma', id)$  and  $(\Sigma, \phi)$  so that they become isotopic. Let  $(\Sigma'', \phi'')$  be their common stabilization. Since the stabilizations were all positive  $\phi''$  is a composition of positive Dehn twists. Moreover  $L$  sits on a page of this open book. As in the previous paragraph of the proof, performing Legendrian surgery on  $L$  will change the open book  $(\Sigma'', \phi'')$  by composing  $\phi''$  by right-handed Dehn twists. Thus we eventually get an open book for  $(M, \xi)$  whose monodromy consists of a composition of right-handed Dehn twists.  $\square$

**5.3. Proof of Theorem 5.1.** We are ready to begin the proof of Theorem 5.1.

**Lemma 5.12** (Etnyre 2004, [14]; cf. Stipsicz 2003, [37]). *Suppose  $(X, \omega)$  is a weak filling of  $(M, \xi)$ . Then there is a compact symplectic manifold  $(X', \omega')$  such that  $(X, \omega)$  embeds in  $(X', \omega')$  and the boundary of  $(X', \omega')$  is strongly convex.*

*Proof.* Let  $(B, \pi)$  be an open book for  $(M, \xi)$ . Using positive stabilizations of the open book we can assume the binding is connected. Let  $\phi$  be the monodromy of the open book. It is well known, see Lemma 6.7 in the Appendix, that  $\phi$  can be written

$$\phi = D_c^m \circ D_{\gamma_1}^{-1} \circ \dots \circ D_{\gamma_n}^{-1},$$

where  $\gamma_i$  are non-separating curves on the interior of the page  $\Sigma$  and  $c$  is a curve on  $\Sigma$  parallel to the boundary of  $\Sigma$ .

We know  $M \setminus B$  is the mapping cylinder  $\Sigma_\phi$  and an identical argument to the one in the first paragraph of the proof of Theorem 5.7 says we may think of  $\Sigma_\phi$  as

$$\prod_{i=1}^n \Sigma_i / \sim,$$

where  $\Sigma_i = \Sigma \times [\frac{i-1}{n}, \frac{i}{n}]$  and  $\sim$  is the equivalence relation that glues  $\Sigma \times \{\frac{i}{n}\}$  in  $\Sigma_i$  to  $\Sigma \times \{\frac{i}{n}\}$  in  $\Sigma_{i+1}$  by  $D_{\gamma_i}^{-1}$  and  $\Sigma \times 1$  to  $\Sigma \times 0$  by  $D_c^m$ . See Figure 15.

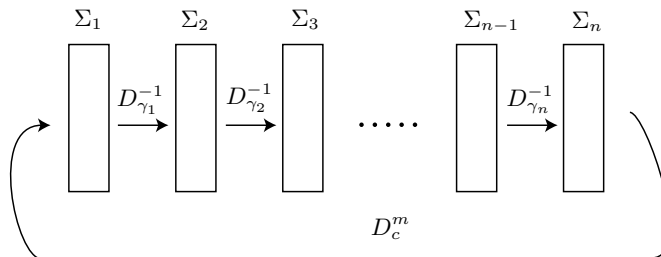


FIGURE 15. Breaking up the monodromy.

Choose a point  $p \in (0, \frac{1}{n})$ . We can Legendrian realize  $\gamma_1$  on the surface  $\Sigma \times \{p\}$  in  $\Sigma_1 \subset M$ . If we cut  $\Sigma_\phi$  along  $\Sigma \times \{p\}$  and reglue using the diffeomorphism  $D_{\gamma_1}$  then the new open book decomposition will have page  $\Sigma$  and monodromy  $D_c^m \circ D_{\gamma_2}^{-1} \circ \dots \circ D_{\gamma_n}^{-1}$ . Continuing in this way we can get an open book with page  $\Sigma$  and monodromy  $D_c^m$ . Denote the contact manifold supported by this open book by  $(M', \xi')$ . By Theorem 5.7 we know we can get from  $(M, \xi)$  to  $(M', \xi')$  by a sequence of Legendrian surgeries. Thus by Theorem 5.6 we can add 2-handles to  $(X, \omega)$  in a symplectic way to get a symplectic manifold  $(X'', \omega'')$  with weakly convex boundary equal to  $(M', \xi')$ .

Let  $a_1, \dots, a_{2g}$  be the curves on  $\Sigma$  pictured in Figure 16. We can Legendrian

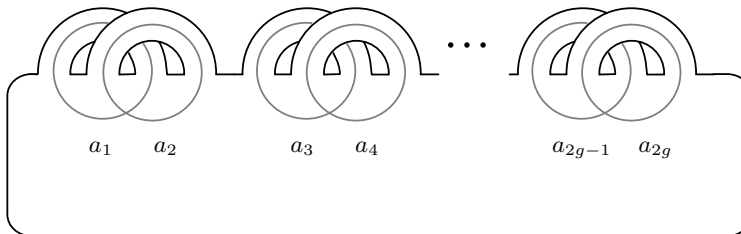


FIGURE 16. The curves  $a_1, \dots, a_{2g}$  on  $\Sigma$ .

realize these curves on separate pages of the open book for  $(M', \xi')$  and do Legendrian surgery on them to get the contact manifold  $(M'', \xi'')$ . Moreover, we can add 2-handles to  $(X'', \omega'')$  to obtain the symplectic manifold  $(X', \omega')$  with weakly convex boundary  $(M'', \xi'')$ . The open book supporting  $(M'', \xi'')$  has page  $\Sigma$  and monodromy  $D_{a_1} \circ \dots \circ D_{a_{2g}} \circ D_c^m$ .

**Exercise 5.13** Show  $M''$  is topologically obtained from  $S^3$  by  $\frac{1}{m}$  Dehn surgery on the knot in Figure 17. Thus  $M''$  is a homology sphere.

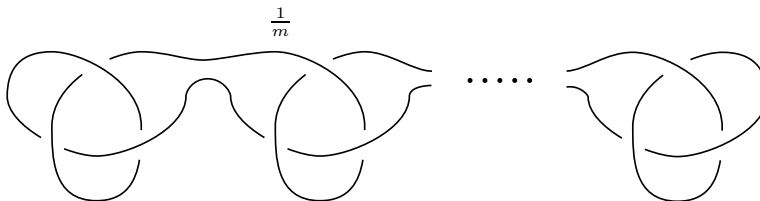


FIGURE 17. Topological description of  $M''$ .

Now Theorem 5.5 says for a homology sphere a weak filling can be deformed into a strong filling. Thus we may deform  $(X', \omega')$  into a strong filling of  $(M'', \xi'')$  and clearly  $(X, \omega)$  symplectically embeds into this manifold.  $\square$

**Lemma 5.14** (Etnyre and Honda 2002, [16]; Gay 2002, [19]). *Given any contact manifold  $(M, \xi)$  there is a strong concave filling of  $(M, \xi)$ .*

This Lemma was also proven by Lisca and Matić [31] for Stein fillable contact structures. An alternate proof in the Stein case was provided by the work of Akbulut and Ozbagci [2] coupled with that of Plamenevskaya [35]. The proof below is in the spirit of Gay's work.



*Proof.* We start with the symplectic manifold  $(W, \omega_W) = (M \times [0, 1], d(e^t \alpha))$ , where  $\alpha$  is a contact form of  $\xi$  and  $t$  is the coordinate on  $[0, 1]$ . It is easy to see that  $M \times \{0\}$  is a strongly concave boundary component of  $W$  and  $M \times \{1\}$  is a strongly convex boundary. Our strategy will be to cap off the convex boundary component so we are left with only the concave boundary component. Throughout this proof we will call the concave boundary of a symplectic manifold the *lower boundary component* and the convex boundary component the *upper boundary component*.

Let  $(\Sigma, \phi)$  be an open book for  $(M, \xi)$  with a connected boundary. As in the proof of Lemma 5.12 we can add 2-handles to  $W$  to get a symplectic manifold  $(W', \omega')$  with lower boundary  $M$  and upper boundary a contact manifold with open book having page  $\Sigma$  and monodromy  $D_c^m$  where  $c$  is a boundary parallel curve in  $\Sigma$ .

We want to argue that we can assume that  $m = 1$ . If  $m < 1$  then by adding symplectic 2-handles to  $(W', \omega')$  we can get to the situation where  $m = 1$ . (Note  $c$  is separating, but we can handle this as we did in the proof of Theorem 5.11.) Throughout the rest of the proof as we add handles to  $(W', \omega')$  we still denote the resulting manifold by  $(W', \omega')$ . We are left to consider the situation where  $m > 1$ . For this we observe that we can increase the genus of  $\Sigma$  as follows.

**Exercise 5.15** Show that if we add a symplectic 1-handle to  $(W', \omega')$  this has the effect on the upper boundary of  $W'$  of connect summing with the standard (unique tight) contact structure on  $S^1 \times S^2$ .

But we know connect summing the contact manifold can be achieved by Murasugi summing their open books. The tight contact structure on  $S^1 \times S^2$  has open book with page an annulus and monodromy the identity map. Thus adding symplectic 1-handles to  $W'$  has the effect on the open book of the upper boundary component of adding a 1-handle to  $\Sigma$  and extending the old monodromy over this handle by the identity. So by adding 1-handles to  $W$  we can arrange that the open book for the upper boundary component has page  $\Sigma'$  shown in Figure 18 and monodromy

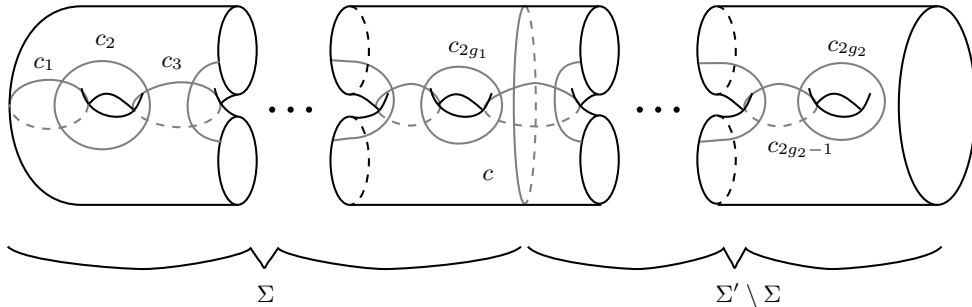


FIGURE 18. The surfaces  $\Sigma$  and  $\Sigma'$  and the curves  $c$  and  $c_1, \dots, a_{2g_2}$ .

$D_c^m$ . Let  $c_1, \dots, c_{2g_1}$  be the curves in  $\Sigma$  shown in Figure 18;  $g_1$  is the genus of  $\Sigma$ . The Chain Relation (see Theorem 6.5) says  $D_c^m = (D_{c_1} \circ \dots \circ D_{c_{2g_1}})^{m(4g_1+2)}$ . Now let  $c_{g_1+1}, \dots, c_{2g_2}$  be the curves shown in Figure 18;  $g_2$  is the genus of  $\Sigma'$ .

**Exercise 5.16** Show that we can assume  $m$  is such that we can choose the genus  $g_2$  of  $\Sigma'$  so that  $m(4g_1 + 2) = 4g_2 + 2$ .

HINT: Attach symplectic 2-handles.

Thus we can attach symplectic 2-handles to  $(W', \omega')$  so that the upper boundary  $(M', \xi')$  has an open book decomposition with page  $\Sigma'$  and monodromy  $D_{c'} = (D_{c_1} \circ \dots \circ D_{c_{2g_2}})^{4g_2+2}$ , where  $c'$  is a curve on  $\Sigma'$  parallel to the boundary.

**Exercise 5.17** Show  $M'$  is an  $S^1$  bundle over  $\Sigma''$  with Euler number  $-1$ , where  $\Sigma''$  is  $\Sigma'$  with a disk capping of its boundary.

**Exercise 5.18** Let  $C$  be the  $D^2$  bundle over  $\Sigma''$  with Euler number 1. Construct a natural symplectic structure  $\omega_C$  on  $C$ .

HINT: On the circle bundle  $\partial C$  there is a connection 1-form  $\alpha$  that is also a contact form on  $\partial C$ . Use this to construct the symplectic form on  $C$ . Note that if you think about the symplectization of a contact structure you can easily get a symplectic structure on  $C$  minus the zero section. Some care is needed to extend over the zero section.

**Exercise 5.19** Show  $(C, \omega_C)$  has a strongly concave boundary,  $\partial C = -M'$ , and  $\xi'$  is the induced contact structure.

HINT: The contact structure induced on  $\partial C$  is transverse to the circle fibers. If you remove a neighborhood of one of the fibers the resulting manifold is  $\Sigma' \times S^1$ . This is the mapping cylinder of the identity on  $\Sigma'$ . Show that the contact planes can be isotoped arbitrarily close to the pages. Now consider how the neighborhood of the fiber is glued back in.

We now simply glue  $(C, \omega_C)$  to the top of  $(W', \omega')$  to get our concave filling of  $(M, \xi)$ .  $\square$

We are now ready to prove the main result of this section.

*Proof of Theorem 5.1.* We start with a weak symplectic filling  $(X, \omega)$  of  $(M, \xi)$ . Now apply Lemma 5.12 to embed  $(X, \omega)$  symplectically into  $(X', \omega')$  where  $(X', \omega')$  has a strongly convex boundary  $(M', \xi')$ . Now use Lemma 5.14 to find a symplectic manifold  $(X'', \omega'')$  that is a strong concave filling of  $(M', \xi')$ . Using an exercise from Subsection 5.1 we can glue  $(X', \omega')$  and  $(X'', \omega'')$  together to get a closed symplectic manifold  $(W, \omega_W)$  into which  $(X, \omega)$  embeds.  $\square$

**5.4. Sobering arcs and overtwisted contact structures.** Theorem 5.11 gives a nice characterization of Stein fillable contact structures in terms of open book decompositions. It turns out there is a similar characterization of overtwisted contact structures due to Goodman [24]. Suppose we are given an oriented surface  $\Sigma$ . Given two properly embedded oriented arcs  $a$  and  $b$  on  $\Sigma$  with  $\partial a = \partial b$  we can isotop them relative to the boundary so that the number of intersection points between the arcs is minimized. At a boundary point  $x$  of  $a$  define  $\epsilon(x)$  to be  $+1$  if the oriented tangent to  $a$  at  $x$  followed by the oriented tangent to  $b$  at  $x$  is an oriented basis for  $\Sigma$ , otherwise we set  $\epsilon(x) = -1$ . Let  $i(a, b) = \frac{1}{2}(\epsilon(x) + \epsilon(y))$ , where  $x$  and  $y$  are the boundary points of  $a$ .

**Definition 5.20.** Let  $\Sigma$  be an oriented surface and  $\phi : \Sigma \rightarrow \Sigma$  a diffeomorphism that fixes the boundary. An arc  $b$  properly embedded in  $\Sigma$  is a sobering arc for the pair  $(\Sigma, \phi)$  if  $i(b, \phi(b)) \geq 0$  and there are no positive intersection points of  $b$  with  $\phi(b)$  (after isotoping to minimize the number of intersection points).

Note the definition of sobering arc does not depend on an orientation on  $b$ .

**Theorem 5.21** (Goodman 2004, [24]). If  $(\Sigma, \phi)$  admits a sobering arc then the corresponding contact structure  $\xi_{(\Sigma, \phi)}$  is overtwisted.  $\square$

We will not prove this theorem but indicate by an example how one shows a contact structure is overtwisted if a supporting open book admits a sobering arc. Indeed, consider  $(A, \phi)$  where  $A = S^1 \times [-1, 1]$  and  $\phi$  is a left-handed Dehn twist about  $S^1 \times \{0\}$ . Of course this is the open book describing the negative Hopf link in  $S^3$ . Earlier we claimed that the associated contact structure is overtwisted; we now find the overtwisted disk. (Actually we find a disk whose Legendrian boundary violates the Bennequin inequality, but from this one can easily locate an overtwisted disk.) The arc  $b = \{pt\} \times [-1, 1] \subset A$  is obviously a sobering arc. Let  $T$  be the union of two pages of the open book. Clearly  $T$  is a torus that separates  $S^3$  into two solid tori  $V_0$  and  $V_1$ . We can think of  $V_0$  as  $A \times [0, \frac{1}{2}]$  (union part of the neighborhood of the binding if you want to be precise) and  $V_1$  is  $A \times [\frac{1}{2}, 1]$ . In  $V_0$  we can take  $D_0 = b \times [0, \frac{1}{2}]$  to be the meridional disk and in  $V_1$  we take  $D_1 = b \times [\frac{1}{2}, 1]$  to be the meridional disk. If we think of  $T$  as the boundary of  $V_1$  then  $\partial D_1$  is simply a meridional curve, that is, a  $(1, 0)$  curve. Note that  $\partial D_0$  does not naturally sit on  $T$ . We must identify  $A \times \{\frac{1}{2}\} \subset V_0$  with  $A \times \{\frac{1}{2}\} \subset V_1$  using the identity map and  $A \times \{0\} \subset V_0$  with  $A \times \{1\} \subset V_1$  via  $\phi$ . Thus  $\partial D_1$  in  $T$  is a  $(-1, 1)$  curve. In particular these two meridional curves intersect once on  $T$ . We can Legendrian realize a  $(0, 1)$  curve on  $T$ . Call the Legendrian curve  $L$ . Note since  $(0, 1) = (1, 0) + (-1, 1)$ ,  $L$  bounds a disk  $D$  in  $S^3$ . With respect to the framing induced by  $T$  the twisting of  $L$  is 0.

**Exercise 5.22** Show that the framing induced on  $L$  by  $T$  is one larger than the framing induced by  $D$ . Thus  $tw(\xi, D) = 1$ .

So we see that  $\partial D$  violates the Bennequin inequality and thus  $\xi$  is overtwisted.

**Exercise 5.23** Starting with  $D$  find an overtwisted disk.

In general, in the proof of the above theorem you will not always be able to find an explicit overtwisted disk, but in a manner similar to what we did above you will always be able to construct a Legendrian knot bounding a surface that violates the Bennequin inequality.

Lastly we have the following theorem.

**Theorem 5.24** (Goodman 2004, [24]). *A contact structure is overtwisted if and only if it is there is an open book decomposition supporting the contact structure that admits a sobering arc.*

The if part of this theorem is the content of the previous theorem. The only if part follows from:

**Exercise 5.25** Show that any overtwisted contact structure is supported by an open book that has been negatively stabilized. Show that this implies there is a sobering arc. HINT: You will need to use Eliashberg's classification of overtwisted contact structures by their homotopy class of plain field [12]. If you are having trouble you might want to consult [13] or, of course, the original paper [24].

## 6. APPENDIX

We recall several important facts about diffeomorphisms of surfaces. First, given an embedded curve  $\gamma$  in an oriented surface  $\Sigma$  let  $N = \gamma \times [0, 1]$  be a (oriented) neighborhood of the curve. We then define the *right-handed Dehn twists along  $\gamma$* , denoted  $D_\gamma$ , to be the diffeomorphism of  $\Sigma$  that is the identity on  $\Sigma \setminus N$  and on  $N$

is given by  $(\theta, t) \mapsto (\theta + 2\pi t, t)$ , where  $\theta$  is the coordinate on  $\gamma = S^1$  and  $t$  is the coordinate on  $[0, 1]$  and we have chosen the product structure so that  $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}$  is an oriented basis for  $N \subset \Sigma$ . (Note that to make  $D_\gamma$  a diffeomorphism one needs to “smooth” it near  $\partial N$ .) A *left-handed Dehn twist* about  $\gamma$  is  $D_\gamma^{-1}$ .

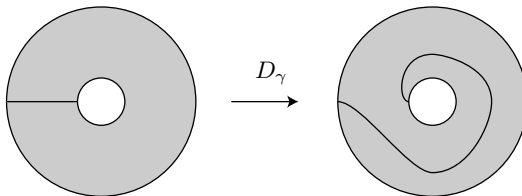


FIGURE 19. A right-handed Dehn twist.

**Exercise 6.1** Show the following

- (1)  $D_\gamma$  does not depend on an orientation on  $\gamma$ .
- (2) If  $\gamma$  and  $\gamma'$  are isotopic then  $D_\gamma$  and  $D_{\gamma'}$  are isotopic diffeomorphisms.

**Theorem 6.2** (Lickorish 1962, [30]). *Any diffeomorphism of a compact oriented surface can be written as a composition of Dehn twists about non-separating curves and curves parallel to the boundary of the surface.*  $\square$

There are several important relations among Dehn twists. For example:

- (1) For any  $\gamma$  and diffeomorphism  $f$  we have  $f \circ D_\gamma \circ f^{-1} = D_{f(\gamma)}$ .
- (2) If  $\gamma$  and  $\delta$  are disjoint then  $D_\gamma \circ D_\delta = D_\delta \circ D_\gamma$ .
- (3) If  $\gamma$  and  $\delta$  intersect in one point then  $D_\delta \circ D_\gamma(\delta)$  is isotopic to  $\gamma$ .
- (4) If  $\gamma$  and  $\delta$  intersect in one point then  $D_\delta \circ D_\gamma \circ D_\delta = D_\gamma \circ D_\delta \circ D_\gamma$ .

**Exercise 6.3** Prove the above relations.

**Exercise 6.4** Show that given two non-separating curves  $\gamma$  and  $\delta$  on  $\Sigma$  there is a diffeomorphism of  $\Sigma$  taking  $\gamma$  to  $\delta$ .

In many of the above applications of open books we needed the following fundamental relation called the Chain Relation.

**Theorem 6.5.** *Let  $\gamma_1, \dots, \gamma_k$  be a chain of simple closed curves in  $\Sigma$ , that is, the curves satisfy  $\gamma_i \cdot \gamma_j$  is 1 if  $|i - j| = 1$  and is 0 otherwise, where  $\cdot$  means geometric intersection. Let  $N$  be a neighborhood of the union of the  $\gamma_i$ 's. If  $k$  is odd then  $N$  has two boundary components  $d_1$  and  $d_2$ . If  $k$  is even then  $N$  has one boundary component  $d$ . We have the following relations*

$$(D_{\gamma_1} \circ \dots \circ D_{\gamma_k})^{2k+2} = D_d, \quad \text{if } k \text{ is even, and}$$

$$(D_{\gamma_1} \circ \dots \circ D_{\gamma_k})^{k+1} = D_{d_1} \circ D_{d_2} \quad \text{if } k \text{ is odd.}$$

$\square$

**Exercise 6.6** Try to prove this theorem. Note: for  $k = 1$  it is trivial, for  $k = 2, 3$  it is quite easy to explicitly check the relation.

An important consequence of this theorem is the following lemma.

**Lemma 6.7.** *Let  $\Sigma$  be a surface with one boundary component, then any diffeomorphism of  $\Sigma$  can be written as the composition of right-handed Dehn twists about non-separating curves on the interior of  $\Sigma$  and arbitrary Dehn twists about a curve parallel to the boundary of  $\Sigma$ .*

*Proof.* We find a chain of curves  $\gamma_1, \dots, \gamma_{2g}$  in  $\Sigma$  such that  $\Sigma$  is a neighborhood of their union. Thus the chain relation tells us that  $(D_{\gamma_1} \circ \dots \circ D_{\gamma_{2g}})^{4g+2} = D_d$ . So clearly we can replace  $D_{\gamma_i}^{-1}$  by a composition of right-handed Dehn twists and one left-handed Dehn twist about  $d$ . Now by the exercises above any left-handed Dehn twist about a separating curve can be written as right-handed Dehn twists about non-separating curves and a left-handed Dehn twist about  $d$ .  $\square$

## REFERENCES

- [1] B. Aebischer, et. al., *Symplectic Geometry*, Progress in Math. **124**, Birkhäuser, Basel, Boston and Berlin, 1994.
- [2] S. Akbulut, B. Ozbagci, *On the topology of compact Stein surfaces*, Int. Math. Res. Not. 2002, no. **15**, 769–782.
- [3] J. W. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. USA, **9** (1923), 93–95.
- [4] J. W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc., **26** (1920), 370–372.
- [5] V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 659–663.
- [6] Y. Eliashberg, *A few remarks about symplectic filling*, Geom. Topol. **8** (2004) 277–293.
- [7] Y. Eliashberg, *Unique holomorphically fillable contact structure on the 3-torus*, Internat. Math. Res. Notices **2** (1996), 77–82.
- [8] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier **42** (1992), 165–192.
- [9] Y. Eliashberg, *On symplectic manifolds with some contact properties*, J. Diff. Geom. **33** (1991), 233–238.
- [10] Y. Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, Vol. II (Ed. Donaldson and Thomas), Cambridge, 1990.
- [11] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension  $> 2$* , Int. J. of Math. **1** (1990), 29–46.
- [12] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623–637.
- [13] J. Etnyre, *Planar open book decompositions and contact structures*, IMRN **2004:79** (2004) 4255–4267.
- [14] J. Etnyre, *On Symplectic Fillings*, Algebr. Geom. Topol. **4** (2004), 73–80.
- [15] J. Etnyre, *Introductory Lectures on Contact Geometry*, Topology and geometry of manifolds (Athens, GA, 2001), 81–107, Proc. Sympos. Pure Math., **71**, Amer. Math. Soc., Providence, RI, 2003.
- [16] J. Etnyre, K. Honda, *On Symplectic Cobordisms*, Math. Annalen **323** (2002), no. 1, 31–39.
- [17] J. Etnyre, K. Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. **148** (2002), no. 3, 609–626.
- [18] D. Gabai, *The Murasugi sum is a natural geometric operation*, Low-dimensional topology (San Francisco, Calif., 1981), 131–143, Contemp. Math., **20**, Amer. Math. Soc., Providence, RI, 1983.
- [19] D. Gay, *Explicit concave fillings of contact three-manifolds*, Proc. Cam. Phil. Soc. **133** (2002), 431–441.
- [20] H. Geiges, *Contact Geometry*, to appear in the Handbook of Differential Geometry, vol. 2.
- [21] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

- [22] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66** (1991), no. 4, 637–677.
- [23] D. L. Goldsmith, *Symmetric fibered links*, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), pp. 3–23. Ann. of Math. Studies, No. **84**, Princeton Univ. Press, Princeton, N.J., 1975.
- [24] N. Goodman, *Overtwisted Open Books From Sobering Arcs*, arXiv:math.GT/0407420
- [25] J. Harer, *Pencils of curves on 4-manifolds*, UC Berkeley PhD thesis 1979.
- [26] K. Honda, *Gluing tight contact structures*, Duke Math. J., **115** (2002), no. 3, 435–478.
- [27] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368.
- [28] P. Kronheimer, T. Mrowka, *Witten’s conjecture and property P*, Geom. Topol. **8** (2004), 295–310.
- [29] P. Kronheimer, T. Mrowka, P. Ozsváth, Z. Szabó. *Monopoles and lens space surgeries*, preprint (arXiv: math.GT/0310164).
- [30] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** 1962 531–540.
- [31] P. Lisca, G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. **129** (1997), no. 3, 509–525.
- [32] J. Milnor, *Singular points of complex hypersurfaces*, Volume 61, Ann. of Math. Stud. Princeton Univ. Press. 1968.
- [33] H. Ohta, K. Ono, *Simple singularities and topology of symplectically filling 4-manifold*, Comment. Math. Helv. **74** (1999), 575–590.
- [34] P. Ozsváth, Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.
- [35] O. Plamenevskaya, *Contact structures with distinct Heegaard-Floer invariants*, Math. Res. Lett. **11** (2004), no. 4, 547–561.
- [36] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series **7** (1976), Publish or Perish Inc.
- [37] A. Stipsicz, *On the geography of Stein fillings of certain 3-manifolds*, Michigan Math. J. **51** (2003), no. 2, 327–337.
- [38] W. P. Thurston, H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.
- [39] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices 2000, no. 9, 441–454.
- [40] A. H. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** 1960 503–528
- [41] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkiado Math. Journal **20** (1991), 241–251.

UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104

*E-mail address:* etnyre@math.upenn.edu

*URL:* <http://www.math.upenn.edu/~etnyre>