ERRATUM TO: TIGHT CONTACT STRUCTURES ON LENS SPACES

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The Proof of Lemma 3.6 in [E] is incorrect but all the main theorems in the paper are still correct. Lemma 3.6 can still be proven for the lens spaces $L(p,q)$ when $q=1$ or $p-1$. (We include the proof for these cases at the end of the paper since [Ko] used Lemma 3.6 when $q=p-1$.) However, the lemma may by bypassed in the essential arguments in [E]. Recall that the prime use of Lemma 3.6 was to show that we could assume that the generalized projective plane D in $L(p,q)$ (*i.e.* the two skeleton) had at most p elliptic points in its characteristic foliation. This is still true:

Lemma 0.1. *We may (topologically) isotop D in $L(p,q)$ so that $e_+(D) \leq p$.*

Note that with this lemma in hand Theorem 4.3, Corollary 4.5 and Theorem 4.10 from [E] are true and their proofs are essentially unchanged.

Proof. We assume that we have isotoped D so that $e_+(D)$ is minimal. Now if $e_+(D) > p$ then we derive a contradiction.

Recall that by choosing a point x on the one skeleton C of $L(p,q)$ we break ∂D into p intervals $B_1, \ldots, B_p$ using the p points $x_1, \ldots, x_p$ on ∂D that map to x when D is glued to C. We say that an interval I on ∂D is longer than k if for any choice of x the interior of the interval I contains at least k of the x_i's.

Note by Remark 3.9 in [E] we know that if we look at an outermost hyperbolic singularity on D (*i.e.* one whose unstable manifolds separate off a disk Δ containing one elliptic point) then $\Delta \cap \partial D$ is longer than 1. Now consider a hyperbolic point h whose unstable manifolds separate off a disk Δ from D that contains two elliptic points and one hyperbolic point h'. We claim that $I = \Delta \cap \partial D$ is longer than 2. To see this let $I = I_c \cup I_m \cup I_a$ where I_m and the unstable manifolds of h' bound a disk on D. We know that I_m is longer than 1 so if I is not longer than 2 then both I_c and I_a are not longer than 1. Moreover, in this situation it is not hard to see that the intervals I_c and I_a are disjoint when mapped to C. Thus just as in Figure 9 in [E] we may construct an overtwisted disk by extending the unstable manifolds of h and h' across C.

The above argument generalizes to prove: if h is a hyperbolic point whose unstable manifolds separate off a disk Δ containing a linear graph in the characteristic foliation containing k elliptic points, then $\Delta \cap \partial D$ is longer than k. There is one complication in this case that is not seen above. If h' is the hyperbolic point in Δ whose unstable manifolds separate off a disk Δ' containing $k-1$ elliptic points, then by induction on k we can assume that $\Delta' \cap \partial D$ is longer than $k-1$. So if out claim is not true then $I_c \cup I_a = S \cap \partial D$ is not longer than 1 and I_c and I_a are disjoint when mapped to C, where $S = \overline{\Delta \setminus \Delta'}$. As above we would like to say that we can construct an overtwisted disk as in Figure 9 in [E]. Unfortunately one of the intervals, say I_a, might not limit to a single elliptic point when pushed across C (strangely enough this does not happen in the situation above), but if this happens the other interval will limit to a single elliptic point when pushed across C. So if K is the union of all leaves in D_ξ that intersect I_c and end at a fixed elliptic point, then we can find a disk B close to $S \cup K$ such that $B \cap C$ is a neighborhood of I_a on C and B contains a single hyperbolic point h whose unstable manifolds intersect C in ∂I_a. Note we might have $B \cap D \neq \emptyset$ but then we may (topologically) isotope D keeping C fixed so that it has the same number and type of singularities and is disjoint from B (away from C). Now as in the proof of Lemma 3.8 in [E] we may use this disk B to decrease $e_+(D)$ by p (contradicting the minimality of $e_+(D)$). Combining this argument with the one in the proof of Theorem 4.6 in [E] one may easily remove the word "linear" from the above statement.

We now know that $e_+(D) \leq p$ since the total "length" of ∂D is p. □

Theorem 4.6 (and hence Corollary 4.7 and Theorems 4.8 and 4.9) in [E] follow from the above proof as follows: Note that under the hypothesis of Theorem 4.6 there must be p elliptic and $p-1$ hyperbolic singularities in D_ξ. If we take an outermost hyperbolic point h then its unstable manifolds separate D into two disks Δ_0 and Δ_1 and by the remark at then end of the above proof $I_0 = \Delta_0 \cap \partial D$ is longer than 1 while $I_1 = \Delta_1 \cap \partial D$ is longer than $p-1$. Now take a point x on C that is in the intersection of C with the unstable manifolds of h. The interior of I_0 contains at least 1 of the points x_i while the interior of I_1 contains at least $p-1$ of the points x_i. That means that at least p of the x_i's are accounted for on the interiors of the respective intervals but one of the x_i's in on the boundary of both intervals. This contradicts the fact that there are only p, x_i's. Thus there can be no tight contact structure satisfying the hypothesis of Theorem 4.6.

We have now demonstrated that the main theorems in [E] are correct; but, in order to repair a gap in [Ko] caused the incorrect proof of Lemma 3.6 in [E] we show that the lemma is indeed correct in the cases relevant to [Ko].

Proof of Lemma 3.6 in [E] *for* $q = 1$ *or* $p-1$. We will show how to isotope D to a disk D' with transverse boundary in ∂V_1, whose graph of singularities relates to D's as shown in Figure 2 of [E]. Since the graph of singularities in D_ξ must be a tree, a sequence of such moves will clearly yield the conclusion of the lemma.

Assume that part of the graph of singularities in D_ξ is as shown on the left hand side of Figure 2 in [E] and let h be the hyperbolic singularity whose stable separatrix we wish to move. The unstable separatrices of h cut D into two pieces: one, Δ, containing only one elliptic singularity e and one ,$D \setminus \Delta$, containing all the other singularities. Let U be a (closed) neighborhood, in V_1, of $\partial V_1 \cup \Delta$ for which $U \cap D$ contains only the singularities h and e. We may assume that U is diffeomorphic to $(\partial V_1) \times [0,1]$ and that the characteristic foliations on both boundary components of ∂U are non singular. Moreover we need $D \cap U$ to have transverse boundary. Here is where we must assume that $q = 1$ or $p-1$ since $(\partial U) \setminus (\partial V_1)$ will naturally have four singularities that need to be canceled. We would like to do this cancellation in the complement of $D \cap U$ so as to keep it transverse. By taking V_0 to be a sufficiently small neighborhood of its core C when $q = 1$ or $p-1$ we may achieve this (as the reader my easily verify by looking at the monodromy on ∂D induced by the characteristic foliation of ∂V_1). Now let

$A = D \cap U$, $D' = \overline{D \setminus A}$, $c = \partial D'$ and x be the intersection of c with the stable seperatrix of h that we wish to move. Note there is a region $R \subset c$ such that if D'_ξ is glued to A_ξ via a diffeomorphism $\psi : c \to c$ that takes x into R the resulting singular foliation is as seen on the right hand side of Figure 2 in [E]. We now show how to "realize" such a diffeomorphism by isotoping A.

Let T' be the boundary component of U that lies in the interior of V_1. Since the characteristic foliation of T' is non singular and contains no leaves parallel to c (since ξ is tight) we may use T'_ξ, thought of as a flow, to define a Poincaré return map $\phi : c \to c$. We can assume that ϕ has irrational rotation number since by isotoping T' in the neighborhood of a meridional curve (away from D) we will change ϕ. During this isotopy the rotation numbers for the corresponding ϕ's will change and thus at some point be irrational.

If we cut T' along c we get annulus A'. Gluing one boundary component of A' to $c \subset A$ and the other to $c \subset D'$ and rounding corners we have a new meridional disk D_1 whose characteristic foliation is D'_ξ glued to A_ξ via ϕ. (Note that using Makar-Limanov's corner rounding method [ML] one can round both corners above without altering the topological type of the characteristic foliation.) It is useful to think of D_1 as obtained from D by pushing part of the interior of D once around V_1.

If ϕ does not take x into R then we do not have the desired characteristic foliation. But since ϕ has irrational rotation number the orbit of x under ϕ is dense in c. Thus there is some power, say n, of ϕ that will take x into R. If we take n disjoint copies of T' then we can do the above procedure using all n copies of T' to obtain a disk D_n. Of course the characteristic foliation on D_n will not be exactly D'_ξ glued to A_ξ via ϕ^n since the Poincaré return maps on the copies of T' are not exactly ϕ. But if we take the copies of T' to be sufficiently close to T' then the gluing map will be close enough to ϕ^n to still take x into R. Thus D_n will be the desired new meridional disk whose characteristic foliation is related to D's as seen if Figure 2 in [E]. □

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References

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