TRANSPORTATION OF CLOSED DIFFERENTIAL FORMS WITH NON–HOMOGENEOUS CONVEX COSTS

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ABSTRACT. This manuscript extends a study initiated in [17] to incorporate non–homogeneous cost functions. The problems studied here are convex optimization problems, but the subdifferential of the actions we consider, are not easily characterized except when we deal with smooth cost functions with polynomial growth at infinity. We study minimization problems on the paths of k–forms, which involves dual maximization problems with constraints on the co–differential of the k–forms. When $k < n$, only some directional derivatives of a vector field are controlled. This is in contrast with prior studies of optimal transportation of volume forms ($k = n$), where the full gradient of a scalar function is controlled. An additional complication emerges due to the fact that our dual maximization problem cannot avoid the use of k–currents.

1. INTRODUCTION

This work continues our program on the theory of transportation of closed differential forms. The current manuscript studies actions defined on paths of closed differential forms, introduces various distances and improves on the study in [17] (for related work more centered on the symplectic case where $k = 2$, see [18]). We denote by Λ^k or $\Lambda^k(\mathbb{R}^n)$, the set of exterior k–forms over $\mathbb{R}^n$ (k–covectors of $\mathbb{R}^n$).

Consider a convex (in fact contractible will be sufficient) open bounded set $\Omega \subset \mathbb{R}^n$ and denote by ν the unit outward vector to the boundary $\partial\Omega$. Let d denote the exterior derivative operator on the set of differential forms on Ω and let δ denote the adjoint (or co-differential) of d. Let $\bar{f}_0, \bar{f}_1$ be two–closed k–forms on Ω (i.e. their distributional differential $d\bar{f}_0$ and $d\bar{f}_1$ are null) and the compatibility condition

$$(1.1) \qquad (\bar{f}_1 - \bar{f}_0) \wedge \nu = 0 \quad \text{on} \quad \partial\Omega$$

is satisfied when $1 \le k \le n-1$ while we impose that

$$(1.2) \qquad \int_\Omega (\bar{f}_1 - \bar{f}_0) dx = 0$$

when $k = n$. Accordingly, we denote by $\mathcal{H}$, the set of k–forms $h \in L^1(\Omega; \Lambda^k)$, which are closed in the weak sense, and such that when $1 \le k \le n-1$ then

$$(h - \bar{f}_0) \wedge \nu = 0 \quad \text{on} \quad \partial\Omega$$

while when $k = n$ it is rather required that

$$\int_\Omega (h - \bar{f}_0) dx = 0.$$

This is a subspace of the separable Banach $L^1(\Omega; \Lambda^k)$. If $s \to f_s$ is a path in $\mathcal{H}$, since on contractible domain every closed form is exact and $s \to -\partial_s f_s$ remains a path of closed forms,

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there exists a path $s \to A_s$ of $(k-1)$–forms such that $-\partial_s f = dA$. Let $p \in (1, \infty)$. In fact, we are interested in pairs (f, A) such that

$$(1.3) \qquad A \in L^p\big((0,1) \times \Omega; \Lambda^{k-1}\big), \quad f \in L^p\big((0,1) \times \Omega; \Lambda^k\big), \quad (f_0, f_1) = (\bar{f}_0, \bar{f}_1)$$

and

$$(1.4) \qquad \partial_s f + dA = 0 \ \text{ in } \ (0,1) \times \Omega \qquad \text{and} \qquad A \wedge \nu = 0 \ \text{ on } \ [0,1] \times \partial\Omega$$

in the weak sense (cf. Definition 2.2). The variable s has, a priori, no physical meaning and only serves as an interpolation variable between two prescribed closed forms. Let us denote by $P^p(\bar{f}_0, \bar{f}_1)$ the set of pairs (f, A) such that (1.3) and (1.4) holds.

Let $c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty]$ be a lower semicontinuous function such that when $\omega \in \Lambda^k$, $\xi \in \Lambda^{k-1}$ and $c(\omega, \xi) < \infty$ then

$$(1.5) \qquad\qquad c(\omega, \xi) = 0 \ \text{ if and only if } \ \xi = 0.$$

In order for c to induce a Riemannian or Finsler type metric, we further assume that

$$(1.6) \qquad\qquad c(\omega, \lambda\xi) = |\lambda|^p c(\omega, \xi).$$

For $f \in L^1(\Omega; \Lambda^k)$ and $A \in L^1(\Omega; \Lambda^{k-1})$ we set

$$(1.7) \qquad\qquad \|A\|_f = \left(\int_\Omega c(f, A) dx \right)^{\frac{1}{p}}$$

and define Finsler type metrics

$$(1.8) \qquad\qquad M_p(\bar{f}_0, \bar{f}_1) := \inf_{(f,A)} \left\{ \int_0^1 \|A_s\|_{f_s} ds \ \Big| \ (f, A) \in P^p(\bar{f}_0, \bar{f}_1) \right\}.$$

By Jensen's inequality

$$\left(\int_0^1 \|A_s\|_{f_s} ds \right)^p \le \int_0^1 \|A_s\|_{f_s}^p ds.$$

But using the standard "reparametrization of constant length" (cf. Lemma 5.2), one shows that in fact

$$(1.9) \qquad\qquad M_p^p(\bar{f}_0, \bar{f}_1) = \inf_{(f,A)} \left\{ \int_0^1 \|A_s\|_{f_s}^p ds \ \Big| \ (f, A) \in P^p(\bar{f}_0, \bar{f}_1) \right\}.$$

When $c(f, A) = |A|^p$, $p \in [1, \infty)$ and $rp = r + p$ then a sufficient condition for (f, A) to minimize (1.8) is (cf. [17])

$$(1.10) \qquad f_s = (1-s)\bar{f}_0 + s\bar{f}_1, \ \ \bar{f}_1 - \bar{f}_0 + dA = 0, \ \ A_s \equiv \delta g |\delta g|^{r-2}, \ g \in W^{1,r}(\Omega; \Lambda^k), \ dg \equiv 0$$

and so in this case, A is time independent. Further restricting p to $(1, \infty)$ turns (1.10) into a necessary condition, which uniquely characterizes the minimizers.

By Subsection B.3, any convex function $c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty)$ (hence assuming only finite values) satisfying (1.5) and (1.6) must be independent of ω. This is precisely the case already studied in [17]. This motivates our desire to study cost functions which take on infinite values. What matters the most in the choice of our cost function is the scaling condition (1.6), which is necessary to induce a metric.

An example of $c(\omega, \xi) = G(|\omega|, \xi)$ taking infinite value and studied in Subsection B.1 is

$$(1.11) \qquad c(\omega, \xi) = \begin{cases} \dfrac{|\xi|^p}{p\,(1-|\omega|^2)^{\frac{p-1}{2}}} & \text{if } \ |\omega| < 1 \\[2mm] 0 & \text{if } \ \xi = 0 \ \text{ and } \ |\omega| = 1 \\[1mm] \infty & \text{if } \ (\xi \ne 0 \ \text{ and } \ |\omega| = 1) \ \text{ or } \ (|\omega| > 1). \end{cases}$$

We can also consider cost functions of the form $G(|\omega|, \xi) + H(\xi)$,obtained by adding to the c in (1.11) a smooth function H. One could replace the denominator in the cost in (1.11) by $p\,(M - |\omega|^2)^{\frac{p-1}{2}}$, where M is a positive parameter. In this case, any minimizing path (f, A) in (1.9) must satisfy the requirement $|f| \le M$.

Let us for a moment keep our focus on the case $k = 2$. Given a non–degenerate closed smooth 2–form f, there exists a 1–form w such that

$$(1.12) \qquad\qquad A = w \lrcorner f \quad \text{and so} \quad dA = \mathcal{L}_w f,$$

where $\mathcal{L}_w$ is the Lie derivative acting on the set of 1–form (w has been identified with a vector field). A variant of (1.9) is

$$(1.13) \qquad\qquad \inf_{(f,w)} \left\{ \int_0^1 ||w \lrcorner f||^p_{f_s}\, ds \;\Big|\; \partial_s f + \mathcal{L}_w f = 0 \right\},$$

where the infimum is performed over the set of (f, w) such that $w : (0, 1) \times \Omega \to \Lambda^1$ is smooth and $s \to f_s$ are paths in $\mathcal{H}$ that start at $\bar{f}_0$ and end at $\bar{f}_1$. Unlike (1.9), (1.13) is not a convex minimization problem and so, it is not known to have minimizers. However, if a minimizer (f, A) of problem (1.9) is such that f_s is non–degenerate for almost every $s \in (0, 1)$, then $(f, v) := (f, A \lrcorner f^{-1})$ is a minimizer in (1.13).

There is a sharp contrast between the search of optimal paths in the set of closed k–forms, when $1 \le k \le n - 1$, and that of the case $k = n$. This, can well be illustrated by comparing the case $k = 2$, expressed in terms of electro–magnetism, to the case $k = n$, expressed as a mass transport problem. Consider a bounded open convex (or contractible) set $O \subset \mathbb{R}^3$ and set

$$\Omega := (0, T) \times O.$$

Define $\mathcal{S}$ to be the set of pairs of electro/magnetic time dependent vector fields

$$(B, E) : (0, T) \times O \to \mathbb{R}^6$$

which are integrable, satisfy a certain boundary conditions (omitted now but formulated in Subsection E to match (1.1)) and satisfy Gauss's law for magnetism and the Maxwell–Faraday induction equations

$$(1.14) \qquad\qquad \nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0.$$

When $k = 2$, (1.8) is equivalent to the search of paths of minimal actions on $\mathcal{S}$ (cf. Subsection E). Any starting (resp. ending) point $(\bar{B}_0, \bar{E}_0)$ (resp. $(\bar{B}_1, \bar{E}_1)$) in $\mathcal{S}$ is identified with a starting (resp. ending) point $\bar{f}_0$ (resp. $\bar{f}_1$) in the set of closed 2–forms. Similarly, a path $s \in [0, 1] \to (B(s), E(s))$ which interpolates between (B_0, E_0) and (B_1, E_1), corresponds to a path $s \in [0, 1] \to f(s)$, lying in the set of closed 2–forms $\mathcal{H}$, which interpolates between $\bar{f}_0$ and $\bar{f}_1$. If $f(s)$ is not degenerate then there exists $w : (0, 1) \times \Omega \to \Lambda^1$ such that $\partial_s f + \mathcal{L}_w f = 0$. Here, it is worth stressing that in contrast with the study of n–forms (i.e. volume forms), intensively studied in the past few years in the theory of optimal transportation, s does not represent a time variable. In the theory of optimal transportation, given two volume forms $\bar{\mu}_0$ and $\bar{\mu}_1$ of same mass, we want to minimize an action over the set of paths $t \to \mu(t)$ which interpolate between $\bar{\mu}_0$ and $\bar{\mu}_1$. For each path $t \to \mu(t)$, there exists a velocity vector field v such that the continuity equation $\partial_t \mu + \mathcal{L}_v \mu = 0$ is satisfied. The action to minimize is an integral over the set of time of an expression either written in terms of $(\mu(t), v(t))$ or equivalently in terms of $(\mu(t), A(t)) = (\mu(t), \mu(t)v(t))$. In the case of 2–forms, the time t appears in (1.14) to ensure that $f(s)$ is a closed form for each s, but w is not the physical velocity. Now, the action to be minimized is an integral over the set of parameters s, of an expression which depends on either $(f(s), w(s))$ (cf. 1.13) or equivalently $(f(s), A(s)) = (f(s), w(s) \lrcorner f(s))$ (cf. 1.9).

This manuscript contributes to the identification of a non–trivial class of metrics on set of closed k–differential forms, with potential impacts on the study of evolutive equations on the set of closed k–differential forms. The non–homogeneous costs allow for a much richer class of metrics, but come at the expense of yielding transportation problems for which the subdifferentials of the actions are not easily characterized. We then face the study of dual problems which involve k–differential forms, whose differential are not a–priory locally summable. This means that unlike the case when $k = n$, a difficulty we have to deal with when $k < n$, is to face a dual problem involving functions for which not all partial derivatives are summable. This means we cannot rely on any classical Sobolev type inequality and need to prove a result such as Lemma 4.7. In this Lemma, we show that up to a translation in one–dimensional interpolation variables, any path on the set of measures of k–differential forms, is controlled by its derivative with respect to the interpolation variable and the L^r–norm of its co-differential. The point is that we obtain an inequality which does not need to involve the L^r–norms of both the differential and the co-differential of our k–forms. The proof of the Lemma relies on the use of a subtle Gaffney type inequality and the result is central to obtain needed coercitivity properties of a functional we study in a dual problem. An extremely challenging problem we leave open and which we hope to be the purpose of future investigations, is the regularity properties of geodesics of minimal length. Problem A.2 comments on a systems of PDEs induced by these geodesics.

This manuscript is divided into two parts, the first one containing our central results. The second part is an appendix consisting of examples and technical regularization Lemmas, needed to circumvent the lack of smoothness property of the functions we are dealing with. The appendix ends with a section alluding to the interpretation of our work in the context of electromagnetism.

In Section 3, we consider cost functions c on $\Lambda^k \times \Lambda^{k-1}$ which assume only finite values, are smooth, strictly convex, with a polynomial growth at infinity. We do not impose that $c(\omega, \cdot)$ is p–homogeneous and use standard methods to characterize the subdifferential of the actions along paths of minimal length. This Section will later be useful when studying cost functions which take infinite value. Section 4 is a preliminary section which deals with paths of bounded variations on metric spaces, the metric space in our case being the set of k–currents. We later use these to study Finsler type metrics on the set of k–forms. In Section 5 not only the set where c assumes the value $+\infty$) is not empty but also c^*, the dual of c, is assumed to have a lower bound which may be linear: $c^*(b, B) \geq \gamma_6\big(|b| + |B|^r\big)$. This creates a difficulty, usually not faced in the optimal transportation theory, which led to incorporating the two lengthy Sections C and D. We identify and exploit a dual maximization problem to characterize the paths minimizing our action. When $k = n$, in the dual problem, all the partial derivatives of a scalar function are controlled. When $k < n$ we face serious technical difficulties since the control of the co–differential of a $(k + 1)$–forms is equivalent to the control of some directional derivatives. We anticipate that the level of complications will substantially increase if we extend the class of cost functions c to include those which are polyconvex or even quasiconvex in a sense to be specified. These considerations, which constitute a new type of challenges, will be addressed in a forthcoming paper [15]. We close our description by drawing the attention of the reader to a recent paper by Y. Brenier and X. Duan [7], one of the very few related to our context, which considers gradient flows of entropy functionals on the set of differential forms.

Throughout the manuscript, it would have been sufficient to assume that Ω is a contractible domain of smooth boundary and not necessarily a convex set. In order to reduce the level of technicality, we chose not to state some of our results under the sharpest assumptions.

2. Preliminaries for the smooth case

For simplicity, throughout the manuscript, $\Omega \subset \mathbb{R}^n$ is assumed to be an open bounded convex set and ν denote the outward unit normal to $\partial\Omega$. Let $1 \leq k \leq n$ be an integer. We assume that $r, p \in (1, \infty)$ are conjugate of each other in the sense that $r + p = rp$.

Definition 2.1. *Let $f \in L^1\left(\Omega; \Lambda^k\right)$, let $A \in L^1\left(\Omega; \Lambda^{k-1}\right)$ and $B \in L^1\left(\Omega; \Lambda^{k+1}\right)$.*

(i) *We write $-df = A$ (resp. $-\delta f = B$) in Ω in the weak sense if for any $h \in C_c^\infty\left(\Omega; \Lambda^k\right)$*

$$\int_\Omega \langle f; h \rangle = \int_\Omega \langle A; \delta h \rangle \qquad \left(resp. \int_\Omega \langle f; h \rangle = \int_\Omega \langle B; dh \rangle \right).$$

(ii) *Similarly if we want to express in the weak sense*

$$(2.1) \qquad (i) \begin{cases} -dA = f & in\ \Omega \\ \nu \wedge A = 0 & on\ \partial\Omega \end{cases} \left(resp. \quad (ii) \begin{cases} -\delta B = g & in\ \Omega \\ \nu \lrcorner B = 0 & on\ \partial\Omega \end{cases} \right),$$

we impose that for any $h \in C^\infty\left(\bar{\Omega}; \Lambda^k\right)$

$$\int_\Omega \langle f; h \rangle = \int_\Omega \langle A; \delta h \rangle \qquad \left(resp. \int_\Omega \langle f; h \rangle = \int_\Omega \langle B; dh \rangle \right).$$

(iii) *We say that f is in the weak sense a closed (resp. co–closed) differential form if $df = 0$ (resp. $\delta f = 0$) in Ω.*

We consider $k-$forms $\bar{f}_0, \bar{f}_1 \in L^p\left(\Omega; \Lambda^k\right)$ such that, if $1 \leq k \leq n - 1$,

$$(2.2) \qquad \begin{cases} d(\bar{f}_1 - \bar{f}_0) \equiv 0 & in\ the\ weak\ sense\ of\ in\ \Omega \\ (\bar{f}_1 - \bar{f}_0) \wedge \nu = 0 & in\ the\ weak\ sense\ on\ \partial\Omega \end{cases}$$

and, if $k = n$,

$$(2.3) \qquad \int_\Omega (\bar{f}_1 - \bar{f}_0) dx = 0.$$

Definition 2.2. *We say that $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ if*

$$f \in L^p\left((0,1) \times \Omega; \Lambda^k\right), \quad A \in L^p\left((0,1) \times \Omega; \Lambda^{k-1}\right)$$

and

$$(2.4) \qquad \int_0^1 ds \int_\Omega \left(\langle \partial_s h; f \rangle + \langle \delta h; A \rangle \right) dx = \int_\Omega \left(\langle h_1; \bar{f}_1 \rangle - \langle h_0; \bar{f}_0 \rangle \right) dx$$

for all $h \in C^1\left([0,1] \times \bar{\Omega}; \Lambda^k\right)$.

Remark 2.3. *Assume (2.2) holds when $1 \leq k \leq n - 1$ and (2.3) holds when $k = n$.*

(i) *By Theorem 7.2 [13], there exists in the weak sense, $\bar{A} \in W^{1,p}\left(\Omega; \Lambda^{k-1}\right)$ satisfying*

$$\begin{cases} d\bar{A} + \bar{f}_1 - \bar{f}_0 = 0 & \delta\bar{A} = 0 \quad in\ \Omega \\ \nu \wedge \bar{A} = 0 & on\ \partial\Omega \end{cases}$$

and there exists a constant $C = C\left(\Omega, p, k\right)$ such that

$$\|\bar{A}\|_{W^{1,p}\left(\Omega; \Lambda^{k-1}\right)} \leq C\|f\|_{L^p}.$$

(ii) *We have $(\bar{f}_s, \bar{S}_s) := \left((1 - s)\bar{f}_0 + s\bar{f}_1, \bar{A}\right) \in P^p(\bar{f}_0, \bar{f}_1)$.*

Definition 2.4. *We define* $\mathbf{B}^r\big((0,1)\times\Omega;\Lambda^k\big)$ *to be the set of h such that*

$$h,\ \partial_s h \in L^r\big((0,1)\times\Omega;\Lambda^k\big)$$

and there exists

$$B \in L^r\big((0,1)\times\Omega;\Lambda^{k-1}\big)$$

such that

(2.5) $$\int_0^1 ds \int_\Omega \langle h;d\psi\rangle dx = -\int_0^1 ds \int_\Omega \langle B;\psi\rangle dx \quad \forall\,\psi \in C_c^1\big((0,1)\times\Omega;\Lambda^{k-1}\big).$$

Here, $\partial_s h$ is the distributional derivative of h with respect to s.

2.1. **A weak time continuity property for** $P^p(\bar f_0, \bar f_1)$. Let $(f,A) \in P^p(\bar f_0, \bar f_1)$. By Fubini's theorem, the function $s \to \int_\Omega |f(s,x)|^p dx$ is in $L^1(0,1)$ and so, its Lebesgue points are of full measure in $(0,1)$. If $\phi \in C^1(\bar\Omega)$ we set

$$L(s,f,\phi) = \int_\Omega \langle f(s,x);\phi(x)\rangle dx.$$

Using $h(s,x) = \alpha(s)\phi(x)$ in (2.4) for arbitrary $\alpha \in C^1([0,1])$, we obtain that there is a set $\mathcal{N}_\phi$ of null Lebesgue measure such that $L(\cdot,f,\phi)$ coincides on $(0,1)\setminus\mathcal{N}_\phi$ with a function $\overline{L(\cdot,f,\phi)} \in W^{1,p}(0,1)$. More precisely,

$$\overline{L(\cdot,f,\phi)}(s) = \lim_{\delta\to 0^+} \frac{1}{\delta}\int_s^{s+\delta} L(\tau,f,\phi)d\tau.$$

The distributional derivative of $L(\cdot,f,\phi)$ is

(2.6) $$\partial_s L(\cdot,f,\phi) = \int_\Omega \langle A(s,x);\delta\phi(x)\rangle dx$$

We have the following Lemma.

Lemma 2.5. *There exists a function $\tilde f \in L^p\big((0,1)\times\Omega;\Lambda^k\big)$ such that the following hold.*

   (i) *$\tilde f = f$ for almost every $(0,1)\times\Omega$*
   (ii) *For any $\phi \in C_c^1(\Omega;\Lambda^k)$, $\overline{L(\cdot,f,\phi)} = L(\cdot,\tilde f,\phi)$ everywhere on $(0,1)$. In particular, $L(\cdot,\tilde f,\phi) \in W^{1,p}(0,1)$ is continuous.*

Remark 2.6. *Thanks to Lemma 2.5, we will always tacitly assume that given $(f,A) \in P^p(\bar f_0, \bar f_1)$ then for any $\phi \in C_c^1(\Omega;\Lambda^k)$, $L(\cdot,f,\phi) \in W^{1,p}(0,1)$ is continuous.*

2.2. **Properties of** $\mathbf{B}^r\big((0,1)\times\Omega;\Lambda^k\big)$.

Lemma 2.7. *If $h \in \mathbf{B}^r\big((0,1)\times\Omega;\Lambda^k\big)$, then for $\mathcal{L}^1$–almost every $s \in (0,1)$ we have $B(s,\cdot) \in L^r(\Omega)$ and $B(s,\cdot) = \delta h(s,\cdot)$ is the weak sense.*

Proof: Observe first that by Fubini's theorem,

$$B \in L^r\big((0,1)\times\Omega;\Lambda^{k-1}\big) \quad\Longrightarrow\quad B(s,\cdot) \in L^r(\Omega;\Lambda^{k-1}) \quad \mathcal{L}^1 - \text{a. e. on}\quad (0,1).$$

Let $\{g_i\}_{i=1}^\infty \subset C_c^1(\Omega)$ be a dense subset of $L^p(\Omega)$. If for $w \in C_c^1(0,1)$ we set $\psi(s,x) = w(s)g_i(x)$ then (2.5) reads off

$$\int_0^1 w(s)ds \int_\Omega \langle h;dg_i\rangle dx = -\int_0^1 w(s)ds \int_\Omega \langle B(s,\cdot);g_i\rangle dx.$$

Thus, there exists a set $N_i \subset (0,1)$ of $\mathcal{L}^1$–null measure such that

$$(2.7) \qquad \int_\Omega \langle h; dg_i \rangle dx = - \int_\Omega \langle B(s, \cdot); g_i \rangle dx$$

for any $s \in (0,1) \setminus N_i$. Thus, (2.7) hold for all $s \in (0,1) \setminus N$ if N is the union of the N_i's. We conclude that

$$\int_\Omega \langle h; dg \rangle dx = - \int_\Omega \langle B(s, \cdot); g \rangle dx$$

for any $s \in (0,1) \setminus N$ and any $g \in C_c^1(\Omega)$. This concludes the proof of the Lemma. QED.

Remark 2.8. *By standard approximation results, it is enough to assume that Ω is an open bounded contractible set of locally Lipschitz boundary $\partial\Omega$ to obtain that if $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ then (2.4) holds for $h \in W^{1,r}\big((0,1) \times \Omega; \Lambda^k\big)$. The proof of the following Lemma, which extends (2.4) to $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$, can be obtained by standard methods.*

Lemma 2.9. *If $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ and $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$, then(2.4) holds.*

Corollary 2.10 (An invariant). *If $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ and $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$, then*

$$\int_0^1 ds \int_\Omega \Big(\langle \partial_s h; f \rangle + \langle \delta h; A \rangle \Big) dx = \int_0^1 ds \int_\Omega \Big(\langle \partial_s h; \bar{f} \rangle + \langle \delta h; \bar{A} \rangle \Big) dx.$$

Indeed, by Lemma 2.9 these expressions depend only on the initial and final values of h and f.

3. Duality results for smooth superlinear integrands of finite values

Let $p, r \in (1, \infty)$ be such that $rp = r + p$ and let $\bar{f}_0, \bar{f}_1 \in L^p(\Omega; \Lambda^k)$ be two k–forms such that, in the weak sense (2.2) holds when $1 \le k \le n-1$ and (2.3) holds when $k = n$. Let

$$c : \Lambda^k \times \Lambda^{k-1} \to \mathbb{R}, \quad c^* : \Lambda^k \times \Lambda^{k-1} \to (-\infty, \infty]$$

where c is convex and c^* is the Legendre transform of c,

$$(3.1) \qquad\qquad\qquad\qquad \inf c > -\infty$$

and

$$(3.2) \qquad\qquad c^*(b, B) \ge \gamma_1 \big(|b|^r + |B|^r \big) - \gamma_2 =: E(b, B)$$

for any $b \in \Lambda^k$ and $B \in \Lambda^{k-1}$. Here, $\gamma_1, \gamma_2 > 0$ are prescribed constants.

Remark 3.1. *Since the Legendre transform reverses order, the following hold.*

(i) *If c^* satisfies (3.2) then for any $\omega \in \Lambda^k$ and $\xi \in \Lambda^{k-1}$*

$$c(\omega, \xi) \le E^*(\omega, \xi) = \gamma_2 + \gamma_1(r-1) \frac{|\omega|^p + |\xi|^p}{(r\gamma_1)^p}.$$

(ii) *Similarly, assume there are constants $\gamma_6, \gamma_7 > 0$ such that for any $(\omega, \xi) \in \Lambda^k \times \Lambda^{k-1}$ we have*

$$(3.3) \qquad\qquad c(\omega, \xi) \ge \gamma_6(|\omega|^p + |\xi|^p) - \gamma_7.$$

Then for any $b \in \Lambda^k$ and $B \in \Lambda^{k-1}$

$$c^*(b, B) \le \gamma_7 + \gamma_6(p-1) \frac{|b|^r + |B|^r}{(p\gamma_6)^r}.$$

(iii) *If (3.1) holds then $c^*(0,0) = -\inf c$ is a finite real number.*

We define $\mathcal{C} : \Lambda^k \times \Lambda^{k-1} \to (-\infty, \infty]$ by

$$\mathcal{C}(f, A) = \int_{(0,1) \times \Omega} c(f, A) ds dx$$

for

$$(f, A) \in L^p\big((0, 1) \times \Omega; \Lambda^k\big) \times L^p\big((0, 1) \times \Omega; \Lambda^{k-1}\big)$$

The following proposition is obtained using standard techniques of the direct methods of the calculus of variations.

Proposition 3.2. *Suppose $\bar{f}_0, \bar{f}_1 \in L^p\left(\Omega; \Lambda^k\right)$ are $k-$ forms such that (2.2) holds when $1 \leq k \leq n - 1$ and (2.3) holds when $k = n$. Suppose $c : \Lambda^k \times \Lambda^{k-1} \to (-\infty, \infty]$ is convex, lower semicontinuous and satisfies (3.3). Then there exists (f^*, A^*) that minimizes $\mathcal{C}$ over $P^p(\bar{f}_0, \bar{f}_1)$.*

For $h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$ we set

$$\mathcal{D}(h) := \int_\Omega \Big(\big\langle \bar{f}_1; h_1 \big\rangle - \big\langle \bar{f}_0; h_0 \big\rangle\Big) dx - \int_{(0,1) \times \Omega} c^*\left(\partial_s h, \delta h\right) ds dx,$$

and for $s \in [0, 1]$ set

$$\bar{f}_s = (1 - s)\bar{f}_0 + s\bar{f}_1, \quad \bar{A}_s := \bar{A},$$

where $\bar{A}$ is given by Remark 2.3 (i). By Remark 2.3 (ii), $(\bar{f}, \bar{A}) \in P^p(\bar{f}_0, \bar{f}_1)$ and so,

$$(3.4) \qquad \mathcal{D}(h) = \int_{(0,1) \times \Omega} \Big(\big\langle \bar{A}; \delta h \big\rangle + \big\langle \bar{f}; \partial_s h \big\rangle - c^*\left(\partial_s h, \delta h\right)\Big) ds dx.$$

Thus, $\mathcal{D}(h)$ depends only on $\partial_s h$ and δh.

Remark 3.3. *Assume c^* satisfies (3.2). Then*

(i) *There exist constant $\gamma_4, \gamma_5 > 0$ which depends only on Ω, $||\bar{f}_0||_p$, $||\bar{f}_1||_p$ γ_1, γ_2, s and r such that*

$$(3.5) \qquad \mathcal{D}(h) \leq \gamma_5 - \gamma_4\Big(||\delta h||_r^r + ||\partial_s h||_r^r\Big).$$

(ii) *There exists a constant C depending only on Ω, k and r such that for any $h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$ there is $\bar{h} \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$ such that $\mathcal{D}(h) = \mathcal{D}(\bar{h})$ and*

$$||\bar{h}(s, \cdot)||_{L^r(\Omega)} \leq C||\delta \bar{h}||_r + ||\partial_s \bar{h}||_r \quad \mathcal{L}^1 - a.e. \text{ on } (0, 1).$$

(iii) *If c satisfies (3.1) then $\mathcal{D}(0) > -\infty$.*

Proof: (i) Using the expression of $\mathcal{D}$ in (3.4), we have

$$\mathcal{D}(h) \leq ||\bar{A}||_p ||\delta h||_r + ||\bar{f}||_p ||\partial_s h||_r + \gamma_2 + \gamma_1\Big(\mathcal{L}^d(\Omega) - ||\partial_s h||_r^r - ||\delta h||_r^r\Big).$$

This, yields (i).

(ii) By Lemma 2.7 there exists $t_0 \in (0, 1)$ such that

$$(3.6) \qquad \delta h(t_0, \cdot) = B(t_0, \cdot), \quad ||\delta h(t_0, \cdot)||_{L^r(\Omega)}^r \leq \frac{||\delta h||_r^r}{\mathcal{L}^d(\Omega)}.$$

By Theorems 7.2 and 7.4 [13] (written for $r \in [2, \infty)$ but extendable to $r \in (1, 2)$) there is $\bar{h}_{t_0} \in W^{1,r}(\Omega; \Lambda^k)$ such that

$$(3.7) \qquad \begin{cases} \delta \bar{h}_{t_0} = \delta h(t_0, \cdot), \ d\bar{h}_{t_0} = 0 \quad \text{in} \quad \Omega \\ \nu \wedge \bar{h}_{t_0} = 0 \text{ on } \partial\Omega. \end{cases}$$

Furthermore, there is a constant C which depends only on Ω, k and r such that

$$\|\bar{h}_{t_0}\|_{W^{1,r}} \leq C(\mathcal{L}^d(\Omega))^{\frac{1}{r}}\|\delta h(t_0, \cdot)\|_{L^r(\Omega)}.$$

This, together with (3.6) implies

(3.8) $$\|\bar{h}_{t_0}\|_{W^{1,r}} \leq C\|\delta h\|_r.$$

Define

$$\bar{h}(s, x) = h(s, x) - h(t_0, x) + \bar{h}_{t_0}(x).$$

We have

$$\bar{h}(s, \cdot) = \bar{h}_{t_0} + \int_{t_0}^s \partial_s \bar{h}(\tau, \cdot)d\tau = \bar{h}_{t_0} + \int_{t_0}^s \partial_s h(\tau, \cdot)d\tau.$$

Thus,

$$\|\bar{h}(s, \cdot) - \bar{h}_{t_0}\|_{L^r(\Omega)}^r = \int_\Omega \left|\int_{t_0}^s \partial_s h(\tau, x)d\tau\right|^r dx \leq \|\partial_s h\|_r^r.$$

This, together with (3.8) yields

$$\|\bar{h}(s, \cdot)\|_{L^r(\Omega)} \leq \|\bar{h}_{t_0}\|_{L^r(\Omega)} + \|\partial_s h\|_r \leq C\|\delta h\|_r + \|\partial_s h\|_r.$$

Note that $\partial_s h = \partial_s \bar{h}$, $\delta h = \delta \bar{h}$ to conclude the proof of (ii).

(iii) Since $\mathcal{D}(0) = -\mathcal{L}^d(\Omega)c^*(0, 0)$ and by Remark 3.1, $c^*(0, 0)$ is finite we obtain (iii). QED.

We will often refer to the following proposition, which can be obtained using standard techniques of the direct methods of the calculus of variations.

Proposition 3.4. *Assume c satisfies (3.1), c^* satisfies (3.2), $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ and $h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$. Then*

(i) $\mathcal{C}(f, A) \geq \mathcal{D}(h)$.
(ii) $\mathcal{C}(f, A) = \mathcal{D}(h)$ *if and only if $(f, A) \in \partial.c^*(\partial_s h, \delta h)$ for almost every $(s, x) \in (0, 1) \times \Omega$.*

Set

$$c_\epsilon(f, A) := c(f, A) + \frac{\epsilon}{p}(|f|^p + |A|^p), \qquad \forall (f, A) \in \Lambda^k \times \Lambda^{k-1}.$$

Set

(3.9) $$\mathcal{D}_\epsilon(h) := \int_{(0,1)\times\Omega} \Big(\langle \bar{A}; \delta h\rangle + \langle \bar{f}; \partial_s h\rangle - c_\epsilon^*(\partial_s h, \delta h)\Big)dsdx$$

and

$$\mathcal{C}_\epsilon(f, A) := \int_{(0,1)\times\Omega} c_\epsilon(f, A)\Big)dsdx.$$

We now record a remark on convex analysis, which is found in classical literature on the topic.

Remark 3.5. *Suppose c^* satisfies (3.2) and $\epsilon \in (0, 1)$.*

(i) *There exist $\gamma_1^*, \gamma_2^* > 0$ independent of ϵ such that*

$$c_\epsilon^*(b, B) \geq \gamma_1^*\big(|b|^r + |B|^r\big) - \gamma_2^*$$

(ii) *We have that c_ϵ^* is of class C^1 and its domain is $\Lambda^k \times \Lambda^{k-1}$ and*

$$c_\epsilon^* \in C^1\big(\Lambda^k \times \Lambda^{k-1}\big).$$

(iii) *There exists a constant C_ϵ such that*

$$|\nabla c_\epsilon^*(b, B)| \leq C_\epsilon\big(|b|^{r-1} + |B|^{r-1} + 1\big).$$

Lemma 3.6. *(Relying on the smoothness of c_ϵ to compute the differential of the action) Assume c^* satisfies (3.2) and $\epsilon \in (0, 1)$. Let $h^*, h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$ and set $N(u) = \mathcal{D}_\epsilon(h^* + uh)$. Then,*

$$N'(0) = \int_{(0,1)\times\Omega} \big\langle \bar{A} - A_\epsilon; \delta h \big\rangle \, dsdx + \int_{(0,1)\times\Omega} \big\langle \bar{f} - f_\epsilon; \partial_s h \big\rangle \, dsdx$$

where

$$f_\epsilon := \nabla_a c_\epsilon^* \left(\partial_s h^*, \delta h^*\right), \quad A_\epsilon := \nabla_B c_\epsilon^* \left(\partial_s h^*, \delta h^*\right).$$

Proof: The continuity of ∇c^* and Remark 3.5 (iii) allow to directly compute $N'(0)$.   QED.

Proposition 3.7. *(Smoothness of c_ϵ yields a standard duality result) Suppose c is convex, lower semicontinuous, satisfies (3.1) and c^* satisfies (3.2). Then*

(i) *there exists h^* that maximizes $\mathcal{D}$ over $\mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$.*
(ii) *there exists h_ϵ that maximizes $\mathcal{D}_\epsilon$ over $\mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$.*
(iii) *For any $h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$*

$$\int_{(0,1)\times\Omega} \big\langle \bar{A} - A_\epsilon; \delta h \big\rangle \, dsdx + \int_{(0,1)\times\Omega} \big\langle \bar{f} - f_\epsilon; \partial_s h \big\rangle \, dsdx = 0.$$

where

(3.10) $$f_\epsilon := \nabla_a c_\epsilon^* \left(\partial_s h_\epsilon, \delta h_\epsilon\right), \quad A_\epsilon := \nabla_B c_\epsilon^* \left(\partial_s h_\epsilon, \delta h_\epsilon\right).$$

(iv) *We may assume without loss of generality that there is a constant C independent of ϵ such that we can choose h_ϵ such that*

$$\left\| h_\epsilon(s, \cdot) \right\|_{L^r(\Omega)} \le C \|\delta h_\epsilon\|_r + \|\partial h_\epsilon\|_r$$

Proof: (i) Let $\bar{A}$ be given by Remark 2.3 and set

$$\bar{f}(s, x) = (1 - s)\bar{f}_0(x) + s\bar{f}_1(x).$$

We have $(\bar{f}, \bar{A}) \in P^p(\bar{f}_0, \bar{f}_1)$. The bounds in that Remark 2.3 (i) and Remark 3.1 (i) imply

$$\mathcal{C}(\bar{f}, \bar{A}) < \infty.$$

This, together with Proposition 3.4 implies

$$D := \sup_h \Big\{ \mathcal{D}(h) \mid h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big) \Big\} \le \mathcal{C}(\bar{f}, \bar{A}) < \infty.$$

By Remark 3.3 (iii) $D > -\infty$ and by (i) of the same remark, if γ is a real number then the upper level sets of $\mathcal{D}$ satisfy

$$\Big\{ h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big) \mid \mathcal{D}(h) \ge \gamma \Big\} \subset \Big\{ h \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big) \,\Big|\, \|\delta h\|_r^r + \|\partial_s h\|_r^r \le \frac{\gamma_5 - \gamma}{\gamma_4} \Big\}.$$

Combining this with Remark 3.3 (ii) we obtain a maximizing sequence $\{h_i\}_i$ of $\mathcal{D}$ over $\mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$ satisfying

$$\sup_i \|h_i\|_r^r + \|\delta h_i\|_r^r + \|\partial_s h_i\|_r^r < \infty.$$

Hence, we may extract from $\{h_i\}_i$ a subsequence which converges weakly to some h^* in $L^r\big((0, 1) \times \Omega; \Lambda^k\big)$ and such that $\{\delta h_i\}_i$ (resp. $\{\partial_s h_i\}_i$) converges weakly to δh^* (resp. $\partial_s h^*$) in $L^r\big((0, 1) \times \Omega; \Lambda^k\big)$. We have $h^* \in \mathbf{B}^r\big((0, 1) \times \Omega; \Lambda^k\big)$.

Recall that by (3.4), $-\mathcal{D}(h_i)$ can be expressed as a convex function of $\partial_s h_i$ and δh_i. Therefore, by standard results of convex analysis

$$-D = \liminf_{i \to \infty} -\mathcal{D}(h_i) \geq -\mathcal{D}(h^*).$$

This proves that h^* maximizes $\mathcal{D}$ over $\mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$.

(ii) By Remark 3.5 we have all the properties needed to replace c^* by c_ϵ^* in the above proof. The proof of (ii) repeats the arguments used in that of (i) but it is even easier.

(iii) Let $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$. The real valued function $u \in \mathbb{R} \to N_\epsilon(u) = \mathcal{D}_\epsilon(h_\epsilon + uh)$ achieves its minimum at 0. Since by Lemma 3.6 N is differentiable at 0, we have $N_\epsilon'(0) = 0$. This is exactly the identity in (iii).

(iv) Is a direct consequence of Remark 3.3 (ii). QED.

Theorem 3.8. *(A duality result not requiring smoothness of c) Suppose c is convex, lower semicontinuous, it satisfies (3.1) and c^* satisfies (3.2). Further assume there are constants $\gamma_6, \gamma_7 > 0$ such that c satisfies (3.3). Then*

(i) *there exists (f^*, A^*) which minimizes $\mathcal{C}$ over $P^p(\bar{f}_0, \bar{f}_1)$.*

(ii) *For any h^* that maximizes $\mathcal{D}$ over $\mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$ we have $\mathcal{C}(f^*, A^*) = \mathcal{D}(h^*)$.*

(iii) *Let $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$. Then (f, A) minimizes $\mathcal{A}$ over $P^p(\bar{f}_0, \bar{f}_1)$ if and only if there exists $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$ such that $(f, A) \in \partial.c^*(\partial_s h, \delta h)$ for almost every $(s, x) \in (0,1) \times \Omega$.*

Proof: (i) and (ii) Let h_ϵ be a maximizer of $\mathcal{D}_\epsilon$ as provided in Proposition 3.7 and let

$$(f_\epsilon, A_\epsilon) := \nabla c_\epsilon^*(\partial_s h_\epsilon, \delta h_\epsilon).$$

We combine (iii) of the same proposition with the fact that $(\bar{f}, \bar{A}) \in P^p(\bar{f}_0, \bar{f}_1)$ (cf. Remark 2.3 (ii)) to obtain that $(f_\epsilon, A_\epsilon) \in P^p(\bar{f}_0, \bar{f}_1)$. Proposition 3.4 (ii) implies

$$\mathcal{C}(f_\epsilon, A_\epsilon) = \mathcal{D}_\epsilon(h_\epsilon).$$

We then use Proposition 3.4 (i) to conclude that (f_ϵ, A_ϵ) minimizes $\mathcal{C}_\epsilon$ over $P^p(\bar{f}_0, \bar{f}_1)$.

Since for $\epsilon \in (0,1)$

$$\gamma_6 \Big(||f_\epsilon||_p^p + ||A_\epsilon||_p^p - \gamma_7 \mathcal{L}^d(\Omega) \Big) \leq \mathcal{C}_\epsilon(f_\epsilon, A_\epsilon) \leq \mathcal{C}_\epsilon(\bar{f}, \bar{A}) \leq \mathcal{C}_1(\bar{f}, \bar{A}),$$

we have

$$(3.11) \qquad\qquad S := \sup_{\epsilon \in (0,1)} ||f_\epsilon||_p^p + ||A_\epsilon||_p^p < \infty.$$

Also, by Remark 3.5 (i) and the maximality property of h_ϵ

$$\gamma_2^* \mathcal{L}^n(\Omega) \geq -c^*(0,0)\mathcal{L}^n(\Omega) = -\mathcal{D}_\epsilon(0) \geq -\mathcal{D}_\epsilon(h_\epsilon).$$

Thus, using (3.4) we have

$$\gamma_2^* \mathcal{L}^n(\Omega) \geq \int_{(0,1)\times\Omega} \big(c^*(\partial_s h_\epsilon, \delta h_\epsilon) - \langle \bar{A}; \delta h_\epsilon \rangle - \langle \bar{f}; \partial_s h_\epsilon \rangle\big) ds dx.$$

We again use Remark 3.5 (i) to obtain

$$\gamma_2^* \mathcal{L}^n(\Omega) \geq \gamma_1^* \big(||\partial_s h_\epsilon||_r^r + ||\delta h_\epsilon||_r^r \big) - ||\partial_s h_\epsilon||_r ||\bar{f}||_p - ||\delta h_\epsilon||_r ||\bar{A}||_p$$

and so,

$$\sup_{\epsilon \in (0,1)} ||\delta h_\epsilon||_r^r + ||\partial_s h_\epsilon||_r^r < \infty.$$

Thus by Remark 3.3 (ii), we may assume without loss of generality that

$$(3.12) \qquad\qquad \bar{S} := \sup_{\epsilon \in (0,1)} ||h_\epsilon||_r^r + ||\delta h_\epsilon||_r^r + ||\partial_s h_\epsilon||_r^r < \infty$$

By (3.11) there exists a subsequence of $(f_{\epsilon_l}, A_{\epsilon_l})_l$ which converges weakly to some (f^*, A^*) in $L^p\big((0,1) \times \Omega; \Lambda^k\big) \times L^p\big((0,1) \times \Omega; \Lambda^{k-1}\big)$ as l tends to ∞. Passing to another subsequence if necessary, thanks to (3.12), we may assume without loss of generality that $(h_{\epsilon_l})_l$ converges weakly in L^r to some $h^* \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$. Thus, $(\delta h_{\epsilon_l})_l$ converges weakly in L^r to δh^* and $(\partial_s h_{\epsilon_l})_l$ converges weakly in L^r to $\partial_s h^*$. Letting ϵ_l tend to 0 in Proposition 3.7 (iii) we obtain for any $h \in W^{1,r}\big((0,1) \times \Omega; \Lambda^k\big)$

$$\int_{(0,1)\times\Omega} \big\langle \bar{A} - A^*; \delta h \big\rangle \, dsdx + \int_{(0,1)\times\Omega} \big\langle \bar{f} - f^*; \partial_s h \big\rangle \, dsdx = 0.$$

We use the the fact that by Remark 2.3 (ii), $(\bar{f}, \bar{A}) \in P^p(\bar{f}_0, \bar{f}_1)$ to conclude that

$$\int_\Omega \Big(\langle \bar{f}_1; h_1 \rangle - \langle \bar{f}_0; h_0 \rangle \Big) dx - \int_{(0,1)\times\Omega} \Big(\langle A^*; \delta h \rangle - \langle f^*; \partial_s h \rangle \Big) dsdx = 0.$$

and so, $(f^*, A^*) \in P^p(\bar{f}_0, \bar{f}_1)$.

We first use the fact that $(f_\epsilon, A_\epsilon) \in \partial c_\epsilon^*(\partial_s h_\epsilon, \delta h_\epsilon)$ and then use the fact that $c_\epsilon \geq c$ to obtain

$$(3.13) \qquad \big\langle A_\epsilon; \delta h_\epsilon \big\rangle + \big\langle f_\epsilon; \partial_s h_\epsilon \big\rangle = c_\epsilon\big(f_\epsilon, A_\epsilon\big) + c_\epsilon^*\Big(\partial_s h_\epsilon, \delta h_\epsilon\Big) \geq c\big(f_\epsilon, A_\epsilon\big) + c_\epsilon^*\Big(\partial_s h_\epsilon, \delta h_\epsilon\Big).$$

Also

$$c_\epsilon^*\Big(\partial_s h_\epsilon, \delta h_\epsilon\Big) = \big\langle A_\epsilon; \delta h_\epsilon \big\rangle + \big\langle f_\epsilon; \partial_s h_\epsilon \big\rangle - c\big(f_\epsilon, A_\epsilon\big) - \frac{\epsilon}{p}\Big(|f_\epsilon|^p + |A_\epsilon|^p\Big)$$

$$\geq c^*\Big(\partial_s h_\epsilon, \delta h_\epsilon\Big) - \frac{\epsilon}{p}\Big(|f_\epsilon|^p + |A_\epsilon|^p\Big).$$

We combine this with (3.13) to conclude that

$$\int_{(0,1)\times\Omega} \Big(\langle A_\epsilon; \delta h_\epsilon \rangle + \langle f_\epsilon; \partial_s h_\epsilon \rangle \Big) dsdx \geq \int_{(0,1)\times\Omega} \Big(c^*\big(\partial_s h_\epsilon, \delta h_\epsilon\big) + c\big(f_\epsilon, A_\epsilon\big) \Big) dsdx - \frac{\epsilon S}{p}.$$

Since $(f_\epsilon, A_\epsilon) \in P^p(\bar{f}_0, \bar{f}_1)$, we may use Remark 2.10 in the previous inequality to obtain

$$(3.14) \quad \int_{(0,1)\times\Omega} \Big(\langle \bar{A}; \delta h_\epsilon \rangle + \langle \bar{f}; \partial_s h_\epsilon \rangle \Big) dsdx \geq \int_{(0,1)\times\Omega} \Big(c^*\big(\partial_s h_\epsilon, \delta h_\epsilon\big) + c\big(f_\epsilon, A_\epsilon\big) \Big) dsdx - \frac{\epsilon S}{p}$$

One lets ϵ_l tend to 0 to derive the inequality

$$\int_{(0,1)\times\Omega} \Big(\langle \bar{A}; \delta h \rangle + \langle \bar{f}; \partial_s h \rangle \Big) dsdx \geq \int_{(0,1)\times\Omega} \Big(\langle \bar{A}; \delta h \rangle + \langle \bar{f}; \partial_s h \rangle \Big) dsdx.$$

This proves that

$$(3.15) \qquad \int_{(0,1)\times\Omega} \Big(\langle \bar{A}; \delta h \rangle + \langle \bar{f}; \partial_s h \rangle \Big) dsdx = \int_{(0,1)\times\Omega} \Big(c^*\big(\partial_s h^*, \delta h^*\big) + c(f^*, A^*) \Big) dsdx.$$

Rearranging, and using the expression of $\mathcal{D}$ in (3.4), we have $\mathcal{D}(h^*) = \mathcal{C}(f^*, A^*)$. By Proposition 3.4 (i), (f^*, A^*) minimizes $\mathcal{C}$ over $P^p(\bar{f}_0, \bar{f}_1)$ and h^* maximizes $\mathcal{D}$ over $\mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$.

(iii) Let $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ and $h \in \mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$. Since $c(\omega, \xi) \geq \gamma_6(|\omega|^p + |\xi|^p) - \gamma_7$ for all $\omega \in \Lambda^k$ and $\xi \in \Lambda^{k-1}$, there is a constant $\gamma_6^* > 0$ such that $c^*(b, B) \leq \gamma_6^*(|b|^r + |B|^r) + \gamma_7$ for all $b \in \Lambda^k$ and $B \in \Lambda^{k-1}$. This together with the fact that c^* satisfies (3.2) implies $\mathcal{D}(h) < \infty$.

By Proposition 3.4, $(f, A) \in \partial c^*(\partial_s h, \delta h)$ for almost every $(s, x) \in (0,1) \times \Omega$ if and only if h maximizes $\mathcal{D}$ over $\mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$ and (f, A) minimizes $\mathcal{A}$ over $P^p(\bar{f}_0, \bar{f}_1)$.   QED.

4. The set of k-forms; approximations of k–currents

4.1. **Notation.** Throughout this subsection $\mathbb{H}$ is a finite dimensional Hilbert space and $C : \mathbb{H} \to (-\infty, \infty]$ is a proper lower semicontinuous convex function. We fix a non empty open bounded convex set $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$.

We denote by $\mathcal{M}(\Omega)$ the set of signed measure of finite total variations. The upper and lower variations g^+ and g^- are finite measures and the Jordan decomposition $g = g^+ - g^-$ holds (cf. e.g. [19]). The total mass of $|g| := g^+ + g^-$ is

$$(4.1) \qquad ||g||_{\mathcal{M}(\Omega)} = \sup_{f \in C(\bar{\Omega})} \left\{ \int_\Omega f(x) g(dx) \mid |f| \leq 1 \right\} = \sup_{f \in C_c(\Omega)} \left\{ \int_\Omega f(x) g(dx) \mid |f| \leq 1 \right\},$$

$\big(\mathcal{M}(\Omega), || \cdot ||\big)$ is a normed space and by the Banach–Alaoglu Theorem, every bounded subset is pre–compact. Thus, $\big(\mathcal{M}(\Omega), || \cdot ||\big)$ is a complete space.

Let $\mathcal{C}$ be a countable dense subset of $C_c(\Omega)$, contained in $C_c^1(\Omega)$ and which does not contain the null function. If we denote by $\hat{\mathcal{C}}$ the set of $f/||f||_\infty$ such that $f \in \mathcal{C}$ then

$$(4.2) \qquad ||g||_{\mathcal{M}(\Omega)} = \sup_{f \in \hat{\mathcal{C}}} \int_\Omega f(x) g(dx).$$

The set of Borel measures with values into Λ^k, of finite total mass, will be denoted by $\mathcal{M}(\Omega; \Lambda^k)$. This is the set of k–currents of finite mass. For any $F \in \mathcal{M}(\Omega; \Lambda^k)$, we define the norm

$$(4.3) \qquad ||F||_{\mathcal{M}(\Omega)} = \sup_{G \in C(\bar{\Omega}; \Lambda^k)} \left\{ \int_\Omega \langle G(x); F(dx) \rangle \mid |G(x)| \leq 1 \; \forall x \in \Omega \right\}.$$

Definition 4.1. *Given a metric space* $(\mathcal{S}, \mathrm{dist})$ *the total variation of* $h : [0, 1] \to \mathcal{S}$ *is*

$$\mathrm{TV}(h) := \sup_{m \in \mathbb{N}} \sup_{0 \leq t_0 < \cdots < t_m \leq 1} \left\{ \sum_{i=0}^{m-1} \mathrm{dist}\big(h(t_i), h(t_{i+1})\big) \right\}.$$

Definition 4.2. *The following definitions can be found respectively in [22] and [24]. The recession function of* C *is* $\bar{C} : \mathbb{H} \to (-\infty, \infty]$ *given by*

$$\bar{C}(v) = \lim_{t \to \infty} \frac{C(v_0 + tv)}{t} \qquad v \in \mathbb{H} \quad where \quad v_0 \in \mathbb{H} \text{ is arbitrary.}$$

One checks that the definition is independent of v_0.

Set
$$O := (0, 1) \times \Omega, \qquad z := (s, x), \qquad dz := ds dx.$$

Here, we skip the proof of the following elementary Lemma.

Lemma 4.3. *Assume* $g \in L^p(O)$ *and* η *be a singular measure. Set* $\eta_* := \eta + \mathcal{L}_O^{n+1}$ *and let* $E \subset O$ *be a Borel set such that*

$$(4.4) \qquad \eta(O \setminus E) = \mathcal{L}^{n+1}(E) = 0.$$

Then for any $\alpha \in \mathbb{R}$, $g_\alpha := g(1 - \chi_E) + \alpha \chi_E \in L^p(O, \eta_*)$ *and* $\mathcal{L}^{n+1}\{g_\alpha \neq g\} = 0$.

Remark 4.4. *Assume* $c : \Lambda^k \times \Lambda^{k-1} \to (-\infty, \infty]$ *is convex, lower semicontinuous and satisfies* (5.3). *We assume the Legendre transform* $c^* : \Lambda^k \times \Lambda^{k-1} \to \mathbb{R}$ *satisfies* (5.4). *Let*

$$b \in \mathcal{M}(O; \Lambda^k), \quad B \in L^r\big(O; \Lambda^{k-1}\big).$$

Let b_s be the singular part of b, set $\eta := |b_s|$ and let $E \subset O$ be a Borel set satisfying (4.4). Consider the Radon–Nikodym derivatives $F := db/d\mathcal{L}^{n+1}$ and $G := db_s/d\eta$. Let

$$\bar{f} \in L^p(O; \Lambda^k), \quad A \in L^p(O; \Lambda^{k-1})$$

be such that

$$\int_O c(\bar{f}, A)dz < \infty.$$

Note $c(\bar{f}, A)$ is finite except may be on a Borel set $F \subset O$ such that $\mathcal{L}^{n+1}(F) = 0$. Let d_* be in $\mathrm{dom}(c)$. According to Lemma 4.3,

$$f := (1 - \chi_E)\bar{f} + d_*\chi_E \in L^p(O; \Lambda^k, \eta_*)$$

where $\eta_* := \eta + \mathcal{L}^{n+1}|_O$. Furthermore, $f = \bar{f} \; \mathcal{L}^{n+1}$–almost everywhere

Assume that $f : O \to \Lambda^k$ is a Borel map which we are free to modify on a set of null $(\mathcal{L}^{n+1} + |b|)$–measure. We have

$$c(f, A) + c^*(F, B) \geq \langle f; F \rangle + \langle A; B \rangle$$

and so, if $c(f, A) + c^*(F, B) \in L^1(O)$ then the positive part of $\langle f; F \rangle + \langle A; B \rangle$ is of finite Lebesgue integral. In that case, in terms of $\bar{C}$, the recession function of $C := c^*$, we have

$$\int_O c(f, A)dz + \int_O c^*(b, B) = \int_O \big(c(f, A) + c^*(F, B)\big)dz + \int_O \overline{c^*}(G, 0)d\eta$$

Since $c(f, A) < \infty \; \eta_*$–a.e., we use Lemma C.1 (i) to infer

$$\int_O c(f, A)dz + \int_O c^*(b, B) \geq \int_O \big(\langle f; F \rangle + \langle A; B \rangle\big)dz + \int_O \langle f; G \rangle d\eta$$

$$= \int_O \langle f; b \rangle + \int_O \langle A; B \rangle dz.$$

Equality holds if and only if

(4.5) $$(f, A) \in \partial.c^*(F, B) \; \mathcal{L}^{n+1} - a.e. \quad and \quad \overline{c^*}(G, 0) = \langle f; G \rangle \quad \eta - a.e.$$

4.2. **Paths of bounded variations on $\mathcal{M}(\Omega; \Lambda^k)$.** Below, we list results on the trace operator of $BV\big((0, 1); \mathcal{M}(\Omega; \Lambda^k)\big)$ functions, needed in the manuscript.

Remark 4.5. *There exists a linear bounded trace (explicitly written below as the left/right limits) operator $T : BV\big((0, 1); \mathcal{M}(\Omega; \Lambda^k)\big) \to L^\infty\big(\{0, 1\}; \mathcal{M}(\Omega; \Lambda^k)\big)$ such that the following hold for any $h \in BV\big((0, 1); \mathcal{M}(\Omega; \Lambda^k)\big)$.*

(i) *If h and $\partial_s h$ are continuous on $[0, 1] \times \bar{\Omega}$ then*

$$Th = h|_{\{0,1\} \times \Omega}$$

(ii) *We have the integration by parts formula*

$$\int_0^1 ds \int_\Omega \langle h(s, dx); \partial_s g(s, x) \rangle + \int_{(0,1) \times \Omega} \langle \partial_s h(ds, dx); g(s, x) \rangle = u$$

for any $g \in C^1\big([0, 1] \times \bar{\Omega}; \Lambda^k\big)$. Here, we have set

$$u := \int_\Omega \langle Th(1, dx); g(1, x) \rangle - \int_\Omega \langle Th(0, dx); g(0, x) \rangle$$

(iii) *If $h \in BV\big((0, 1); \mathcal{M}(\Omega; \Lambda^k)\big)$ is such that $s \to h(s, \cdot)$ is left continuous at 1 and right continuous at 0 then*

$$Th(0, \cdot) = \lim_{s \to 0^+} h(s, \cdot), \quad Th(1, \cdot) = \lim_{s \to 1^-} h(s, \cdot)$$

4.3. **Special paths of bounded variations on** $\mathcal{M}(\Omega; \Lambda^k)$. Let $h \in L^1\big((0,1); \mathcal{M}(\Omega; \Lambda^k)\big)$ be such that there exists $b \in \mathcal{M}\big((0,1) \times \Omega; \Lambda^k\big)$ such that

$$(4.6) \qquad \int_0^1 ds \int_\Omega \langle \partial_s \psi(s,x); h(s,dx)\rangle = -\int_{(0,1)\times\Omega} \langle \psi(s,x); b(ds,dx)\rangle$$

for all $\psi \in C_c^1\big((0,1)\times\Omega; \Lambda^k\big)$. Modifying if necessary, $h(s,\cdot)$ on a subset of $(0,1)$ of null Lebesgue (cf. [23]), we always assume without loss of generality that h satisfies the following Lemma.

Lemma 4.6 (A non smooth variant of Remark 3.3(ii)). *If (4.6) holds, then for any $0 \le t_1 < t_2 < 1$ and $F \in C_c(\Omega; \Lambda^k)$, we have the following.*

(i)

$$\int_\Omega \langle F(x); h(t_2,dx)\rangle - \int_\Omega \langle F(x); h(t_1,dx)\rangle = \int_{(t_1,t_2]\times\Omega} \langle F(x); b(ds,dx)\rangle$$

(ii)

$$\int_\Omega \langle F(x); h(1,dx)\rangle - \int_\Omega \langle F(x); h(t_1,dx)\rangle = \int_{(t_1,1)\times\Omega} \langle F(x); b(ds,dx)\rangle$$

(iii) *Using the definition of* $\mathrm{TV}(h)$ *in Definition 4.1 we have*

$$\mathrm{TV}(h) \le |b|\big((0,1)\times\Omega\big).$$

Lemma 4.7. *Further assume there exists $B \in L^r\big((0,1); L^r(\Omega; \Lambda^{k-1})\big)$ such that*

$$(4.7) \qquad \int_{(0,1)\times\Omega} \langle dg; h\rangle \, dsdx = -\int_{(0,1)\times\Omega} \langle g; B\rangle \, dsdx$$

for all $g \in C_c^1\big((0,1)\times\Omega; \Lambda^{k-1}\big)$, we say $\delta h = B$ in the weak sense and say that $\delta\phi$ belongs to $L^r\big((0,1); L^r(\Omega; \Lambda^{k-1})\big)$. There exists $\bar{h}_{t_0} \in W^{1,r}(\Omega; \Lambda^k)$ such that if we set $[\bar{h}(s,\cdot) := h(s,\cdot) - h(t_0,\cdot) + \bar{h}_{t_0}$ then, the following hold.

(i) *Replacing h by $\bar{h}$, (4.11) holds for any $s \in \mathcal{T}$ and any $H \in C_c^1(\Omega; \Lambda^{k-1})$. In other words, for any $s \in \mathcal{T}$, we have $\delta\bar{h}(s,\cdot) = B(s,\cdot)$.*

(ii) *There exists a constant C_Ω depending only on Ω, r and k such that for all $s \in (0,1)$*

$$||\bar{h}(s,\cdot)|| \le |b|\big((0,1)\times\Omega\big) + C_\Omega \mathcal{L}^n(\Omega)^{\frac{1}{r}} \Big(\int_{(0,1)\times\Omega} |B(\tau,x)|^r d\tau dx\Big)^{\frac{1}{r}}.$$

(iii) *We have $\partial_s \bar{h} = b$ and $\delta\bar{h} = B$ in the sense that we may substitute $\bar{h}$ with h in (4.6) and (4.7).*

Proof: By Lemma 4.6, for each $F \in C^1(\Omega; \Lambda^k)$, the real value function

$$t \to \int_\Omega \langle F(x); h(s,dx)\rangle$$

it is defined everywhere on $[0,1]$, it is in $\mathrm{BV}(0,1)$, right continuous on $[0,1)$ and left continuous at 1. We use (i) of the same Lemma to obtain

$$(4.8) \qquad \int_\Omega |h|(s,dx) \le \int_0^1 ||h(s,\cdot)||_{\mathcal{M}(\Omega)} ds + |b|\big((0,1)\times\Omega\big).$$

Let $\mathcal{T}^1$ be the set of full Lebesgue measure in $(0,1)$ such that for all $s \in \mathcal{T}^1$

$$(4.9) \qquad \int_\Omega |B(s,x)|^r dx < \infty.$$

The set of $\mathcal{T}^0$ which consists of the set of $s \in (0,1)$ such that

$$(4.10) \qquad \int_\Omega |B(s,x)|^r dx \leq \bar{e}^r := \int_0^1 ds \int_\Omega |B(s,x)|^r dx$$

is of positive Lebesgue measure.

We use (4.7) to obtain that for any $H \in C_c^1(\Omega; \Lambda^{k-1})$, the existence of a set $\mathcal{T}^H \subset \mathcal{T}^1$ of full Lebesgue measure in $(0,1)$ such that

$$(4.11) \qquad \int_\Omega \langle dH(x); h(s,dx)\rangle = -\int_\Omega \langle H(x); B(s,x)\rangle dx$$

for any $s \in \mathcal{T}^H$.

Let $\{F_n\}_{n=1}^\infty \subset C_c^1(\Omega; \Lambda^{k-1})$ be a dense of $C_c^1(\Omega; \Lambda^{k-1})$ for the $||\cdot||_{C^1(\Omega)}$–norm. Set

$$\mathcal{T} := \cap_{n=1}^\infty \mathcal{T}^{F_n}.$$

The set $\mathcal{T} \cap \mathcal{T}^0$ has the same measure as $\mathcal{T}^0$. Let $t_0 \in \mathcal{T} \cap \mathcal{T}^0$. By Theorems 7.2 and 7.4 [13] (written for $r \in [2,\infty)$ but extendable to $r \in (1,2)$), there is $\bar{h}_{t_0} \in W^{1,r}(\Omega; \Lambda^k)$ such that

$$\begin{cases} \delta\bar{h}_{t_0} = B(t_0,\cdot), \ d\bar{h}_{t_0} = 0 \quad \text{in} \quad \Omega \\ \nu \wedge \bar{h}_{t_0} = 0 \text{ on } \partial\Omega. \end{cases}$$

Furthermore, there is a constant C which depends only on Ω, k and r such that

$$(4.12) \qquad \|\bar{h}_{t_0}\|_{W^{1,r}} \leq C(\mathcal{L}^d(\Omega))^{\frac{1}{r}}\|B(t_0,\cdot)\|_{L^r(\Omega)}.$$

Set

$$\bar{h}(s,\cdot) := h(s,\cdot) - h(t_0,\cdot) + \bar{h}_{t_0}.$$

(i) Observe that (4.11) holds for any $s \in \mathcal{T}$ and any H which is a point of accumulation on $\{F_n\}$. Using the fact that $\{F_n\}_{n=1}^\infty$ is dense in $C_c^1(\Omega; \Lambda^{k-1})$ we conclude the proof of (i).

(ii) We exploit Corollary 4.6 and to obtain

$$\|\bar{h}(s,\cdot)\| \leq \|h(s,\cdot) - h(t_0,\cdot)\| + \|\bar{h}_{t_0}\| \leq |b|\big((0,1) \times \Omega\big) + \|\bar{h}_{t_0}\|.$$

This, together with (4.12) yields the desired inequality.

(iii) Observe that if $g \in C_c^1\big((0,1) \times \Omega; \Lambda^k\big)$ then

$$\int_0^1 ds \int_\Omega \langle \partial_s g(s,x); \bar{h}_{t_0}(x)\rangle dx - \int_0^1 ds \int_\Omega \langle \partial_s g(s,x); h(t_0,dx)\rangle$$

$$(4.13) \qquad = \int_\Omega \Big\langle \bar{h}_{t_0}(x); \int_0^1 \partial_s g(s,x)ds \Big\rangle dx - \int_\Omega \Big\langle h(t_0,dx); \int_0^1 \partial_s g(s,x)ds \Big\rangle dx = 0.$$

That all is needed to conclude that we may substitute $\bar{h}$ with h in (4.6). By (i) $\delta h(t_0,\cdot) = B(t_0,\cdot)$. Using the definition of $\bar{h}_{t_0}$ we conclude that we may substitute $\bar{h}$ with h in (4.7).   QED.

Definition 4.8. *We define $BV_*^r(0,1;\Omega)$ to be the set of $h \in L^1\big((0,1); \mathcal{M}(\Omega; \Lambda^k)\big)$ such that $\delta h \in L^r\big((0,1); L^r(\Omega; \Lambda^{k-1})\big)$, and there exists $b \in \mathcal{M}\big((0,1) \times \Omega; \Lambda^k\big)$ such that (4.6) holds. We write $b = \partial_s h$.*

Lemma 4.9. *Let $(h^\epsilon)_{\epsilon \in (0,1)} \subset BV_*^r(0,1;\Omega)$ such that that*

$$(4.14) \qquad \sup_{\epsilon \in (0,1)} \|\partial_s h^\epsilon\|_1 + \|\delta h^\epsilon\|_r^r < \infty$$

and

$$(4.15) \qquad m_0 := \sup_{\epsilon \in (0,1)} \sup_{s \in (0,1)} \|h^\epsilon(s,\cdot)\|_1 < \infty.$$

Then there exists $h^0 \in BV_*^r(0, 1; \Omega)$ such that up to a subsequence the following hold.

   (i) *$(\delta h^\epsilon)_\epsilon$ converges to δh^0 weakly in $L^r((0, 1) \times \Omega; \Lambda^{k-1})$.*

   (ii) *$(\partial_s h^\epsilon)_\epsilon$ converges weak $*$ to $\partial_s h^0$ on $(0, 1) \times \Omega$.*

   (iii) *Except for countably many $s \in (0, 1)$, $(h^\epsilon(s, \cdot))_\epsilon$ converges weak $*$ to $h^0(s, \cdot)$ on Ω*

Proof: There are

$$b \in \mathcal{M}((0, 1) \times \Omega; \Lambda^k), \quad B \in L^r((0, 1); L^r(\Omega; \Lambda^{k-1})), \quad \beta \in \mathcal{M}((0, 1) \times \Omega), \ \beta \geq 0$$

and a sequence $\{\epsilon_m\}_m$ decreasing to 0 such that the following hold:

   (a) $(\delta h^\epsilon)_\epsilon$ converges to B weakly in $L^r((0, 1) \times \Omega; \Lambda^{k-1})$

   (b) $(\partial_s h^\epsilon)_\epsilon$ converges weak $*$ to b on $(0, 1) \times \Omega$

   (c) $(|\partial_s h^\epsilon|)_\epsilon$ converges weak $*$ to β on $\mathbb{R} \times \mathbb{R}^n$.

Write $(0, 1) \cap \mathbb{Q} = \{t_i\}_{i=1}^\infty$. Since

$$||h^{\epsilon_m}(t_i, \cdot)|| \leq m_0$$

we use a diagonal sequence argument to obtain a subsequence of $(\epsilon_m)_m$, which we continuous to label $(\epsilon_m)_m$, such that for each $i \in \mathbb{N}$ there exists $\bar{h}_i \in \mathcal{M}(\Omega; \Lambda^k)$ such that $(h^{\epsilon_m}(t_i, \cdot))_m$ converges weak $*$ to $\bar{h}_i$ on Ω.

Let D be the set of $s \in (0, 1)$ such that $\beta(\{s\} \times \mathbb{R}^n) > 0$. Since b is a finite measure, D is at most countable. Let $s \in (0, 1) \setminus D$ and let $(t_{i_j})_j$ be a subsequence of $(t_i)_i$ that converges to s. By Lemma 4.6

$$(4.16) \qquad ||h^{\epsilon_m}(s, \cdot) - h^{\epsilon_m}(t_{i_j}, \cdot)|| \leq |\partial_s h^{\epsilon_m}|([\min\{s, t_{i_j}\}, \max\{s, t_{i_j}\}] \times \Omega).$$

Because $||h^{\epsilon_m}(s, \cdot)|| \leq m_0$, the set $\{h^{\epsilon_m}(s, \cdot)\}_m$ admits points of accumulation for the weak $*$ topology. Let $h^0(s, \cdot)$ be one of these points of accumulation. Letting m tend to ∞ in (4.16) we have

$$||h^0(s, \cdot) - h_{i_j}|| \leq \beta([\min\{s, t_{i_j}\}, \max\{s, t_{i_j}\}] \times \Omega)$$

and so,

$$\limsup_{j \to \infty} ||h^0(s, \cdot) - h_{i_j}|| \leq \beta(\{s\} \times \bar{\Omega}) = 0$$

Thus, $\{h^{\epsilon_m}(s, \cdot)\}_m$ admits only one points of accumulation and $(h_{i_j})_j$ converges weak $*$ to $h^0(s, \cdot)$. We extend $s \to h^0(s, \cdot)$ to $(0, 1)$ by setting $h^0(s, \cdot) \equiv 0$ for $s \in D$.

Let $g \in C_c^1((0, 1) \times \Omega; \Lambda^k)$. Since

$$\lim_{m \to \infty} \int_\Omega \langle h^{\epsilon_m}(s, x); \partial_s g(s, x) \rangle dx = \int_\Omega \langle h^0(s, dx); \partial_s g(s, x) \rangle$$

and

$$\left| \int_\Omega \langle h^{\epsilon_m}(s, dx); g(s, x) \rangle \right| \leq m_0 ||\partial_s g||_\infty$$

for every $s \in (0, 1) \setminus D$, we use the dominated convergence theorem to conclude that

$$\int_{(0,1) \times \Omega} \langle b(ds, dx); \partial_s g(s, x) \rangle = - \lim_{m \to \infty} \int_{(0,1) \times \Omega} \langle h^{\epsilon_m}(s, x); \partial_s g(s, x) \rangle ds dx$$

$$= - \int_0^1 ds \int_\Omega \langle h^0(s, dx); \partial_s g(s, x) \rangle.$$

Thus, $b = \partial h^0$. Similarly, we show that $\delta h^0 = B$ and so, modifying h^0 on a subset of $(0, 1)$ of null Lebesgue measure $h^0 \in BV_*^r(0, 1; \Omega)$.   QED.

5. Finsler type metrics

Assume $\Omega \subset \mathbb{R}^n$ is an open bounded convex set, $p \in (1, \infty)$ and $rp = r + p$. Motivated by examples of cost functions c such as the one in Subsection B.1, we relax the condition imposed on the lower bound of c^* in Section 3 (cf. 3.2). This allows to extend Theorem 3.8 to cost functions which take on infinite values. Throughout this section,

$$c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty]$$

is lower semicontinuous convex function. We assume that when $c(\omega, \xi) < \infty$ then

$$(5.1) \qquad\qquad c(\omega, \xi) = 0 \quad \text{if and only if} \quad \xi = 0$$

and for any $\lambda > 0$ we have

$$(5.2) \qquad\qquad c(\omega, \lambda\xi) = \lambda^p c(\omega, \xi).$$

We assume that there are constants $\gamma_1, \gamma_2, \gamma_6, \gamma_7 > 0$ such that

$$(5.3) \qquad\qquad c(\omega, \xi) \geq \gamma_6 \big(|\omega|^p + |\xi|^p \big) - \gamma_7$$

and

$$(5.4) \qquad\qquad \infty > c^*(b, B) \geq \gamma_1 \big(|b| + |B|^r \big) - \gamma_2$$

for any $\omega, b \in \Lambda^k$ and $\xi, B \in \Lambda^{k-1}$. Note that we may have

$$(5.5) \qquad\qquad \{(\omega, \xi) \in \Lambda^k \times \Lambda^{k-1} \mid c(\omega, \xi) = \infty\} \neq \emptyset.$$

Let $|| \cdot ||_f$ and $M_p(\cdot, \cdot)$ be defined as in (1.7) and (1.8).

Remark 5.1. *Observe the following.*

   (i) *In case (5.5) does not hold, then by Lemma B.4 there exists a norm $|| \cdot ||_{norm}$ such that $c(\omega, A) \equiv \|A\|_{norm}^p$ is independent of ω. According to [17] the solutions of (1.10) are minimizers of (1.8) and the only minimizers if we further impose that $\| \cdot \|_{norm}^p$ is strictly convex.*

   (ii) *When $k = n$, which is the case of volume forms, in the current literature, most work studying geodesics of length, deal with either the case when c assumes only finite values (as in Section 3) or the case when $c^*(b, B) \in \{0, \infty\}$ for all $(b, B) \in \Lambda^k \times \Lambda^{k-1}$. It seems obvious that when $c^*(b, B) \in \{0, \infty\}$ (see Remark B.1 for such an example when $k = 2$), the study of geodesics of optimal length in the set of k–form will only mimic the well–known theory of n–forms. Therefore, in the current manuscript, we keep or focus on the case where (5.4) is satisfied (cf. Subsection B.1 for an example).*

For any Borel map $f : \Omega \to \Lambda^k$, we define

$$c^\infty(f) := \operatorname*{ess\,sup}_{x, \xi} \Big\{ |c\big(f(x), \xi\big)| \mid \xi \in \Lambda^{k-1}), \ |\xi| \leq 1, x \in \Omega \Big\}$$

Let

$$(5.6) \qquad\qquad \bar{f}_0, \bar{f}_1 : \Omega \to \Lambda^k$$

be Borel maps. When $1 \leq k \leq n - 1$, we assume that

$$(5.7) \qquad\qquad \begin{cases} c^\infty(\bar{f}_0), \ c^\infty(\bar{f}_1) < \infty \\ \quad d\bar{f}_0 = d\bar{f}_1 \equiv 0 & \text{in the weak sense in } \Omega \\ \quad (\bar{f}_1 - \bar{f}_0) \wedge \nu = 0 & \text{in the weak sense on } \partial\Omega. \end{cases}$$

However when $k = n$, we assume that

$$(5.8) \qquad c^\infty(\bar{f}_0), \ c^\infty(\bar{f}_1) < \infty \quad \text{and} \quad \int_\Omega (\bar{f}_0(x) - \bar{f}_1(x)) dx = 0$$

By (5.2) and (5.3)

$$\lambda_6 |f|^p \le (\lambda_7 + c^\infty(f)),$$

and so, (5.7) implies that $|\bar{f}_0|, |\bar{f}_1|$ are bounded functions. Let $(\bar{f}, \bar{A})$ be as in Remark 2.3. The same Remark provides us with a constant $C_{p,\Omega}$ independent of $\bar{f}_0, \bar{f}_1$ such that

$$(5.9) \qquad ||\bar{A}||_{W^{1,p}} \le C_{p,\Omega} ||\bar{f}_1 - \bar{f}_0||_{L^p}.$$

By the convexity property of c,

$$c\big((1-s)\bar{f}_0 + s\bar{f}_1, \bar{A}\big) \le (1-s)c(\bar{f}_0, \bar{A}) + sc(\bar{f}_1, \bar{A})$$

and so, by the homogeneity with respect to the second variables

$$(5.10) \qquad c\big((1-s)\bar{f}_0 + s\bar{f}_1, \bar{A}\big) \le \big((1-s)c^\infty(\bar{f}_0) + sc^\infty(\bar{f}_1)\big)|\bar{A}|^p$$

Recall that $P^p(\bar{f}_0, \bar{f}_1)$ is a set of paths connecting $\bar{f}_0$ to $\bar{f}_1$ as given in Definition 2.2. In other words, if $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ then in the weak sense

$$(5.11) \qquad \begin{cases} \partial_s f + dA \equiv 0 & \text{in} \quad (0,1) \times \Omega \\ A \wedge \nu = 0 & \text{on} \quad (0,1) \times \partial\Omega \\ f(0, \cdot) = \bar{f}_0, \ f(1, \cdot) = \bar{f}_1 & \text{on} \ \partial\Omega \end{cases}$$

5.1. **A metric on a subset of the set of differential forms.**

Lemma 5.2. *(Reparametrization by arc lengths) Suppose $c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty]$ is a lower semicontinuous convex function that satisfies (5.1) and (5.2). If $\bar{f}_0$ and $\bar{f}_1$ are such that (5.6) and (5.7 – 5.8) hold then*

$$M_p^p(\bar{f}_0, \bar{f}_1) = \inf_{(f,A)} \Big\{ \mathcal{C}(f, A) \mid (f, A) \in P^p(\bar{f}_0, \bar{f}_1) \Big\}.$$

Proof: For any $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ we use Jensen's inequality to conclude that

$$\Big(\int_0^1 ds ||A_s||_{f_s} ds \Big)^p \le \mathcal{C}(f, A).$$

Thus,

$$M_p^p(\bar{f}_0, \bar{f}_1) \le \inf_{(f,A)} \Big\{ \mathcal{C}(f, A) \mid (f, A) \in P^p(\bar{f}_0, \bar{f}_1) \Big\}.$$

It remains to prove the reverse inequality. Assume without loss of generality that $M_p^p(\bar{f}_0, \bar{f}_1) < \infty$ otherwise, there will be nothing to prove. Let $\epsilon > 0$ and let $(f^\epsilon, A^\epsilon) \in P^p(\bar{f}_0, \bar{f}_1)$ be such that

$$(5.12) \qquad \Big(\int_0^1 ds \int_\Omega ||A_s^\epsilon||_{f_s^\epsilon} ds \Big)^p < M_p^p(\bar{f}_0, \bar{f}_1) + \epsilon.$$

Define

$$L_\epsilon := \int_0^1 (\epsilon + ||A_s^\epsilon||_{f_s^\epsilon}) ds, \quad S_\epsilon(s) := \frac{1}{L_\epsilon} \int_0^s (\epsilon + ||A_l^\epsilon||_{f_l^\epsilon}) dl.$$

Observe that $S_\epsilon : [0,1] \to [0,1]$ is a bijection and so has an inverse $T_\epsilon : [0,1] \to [0,1]$ such that

$$(5.13) \qquad \dot{T}_\epsilon = \frac{1}{\dot{S}_\epsilon \circ T_\epsilon} = \frac{L_\epsilon}{\epsilon + ||A_{T_\epsilon}^\epsilon||_{f_{T_\epsilon}}}.$$

Define

$$\tilde{f}(\tau, x) = f^\epsilon(T_\epsilon(\tau), x), \quad \tilde{A}(\tau, x) = \dot{T}_\epsilon(\tau) A^\epsilon(T_\epsilon(\tau), x).$$

We have $(\tilde{f}, \tilde{A}) \in P^p(\bar{f}_0, \bar{f}_1)$ and

$$||\tilde{A}_\tau||^p_{\tilde{f}_\tau} = \int_\Omega c(\tilde{f}(\tau, x), \tilde{A}(\tau, x))dx = |\dot{T}_\epsilon(\tau)|^p \int_\Omega c\Big(f^\epsilon(T_\epsilon(\tau), x), A^\epsilon(T_\epsilon(\tau), x)\Big)dx.$$

Thus, using (5.13) we obtain that

$$||\tilde{A}_\tau||^p_{\tilde{f}_\tau} = |\dot{T}_\epsilon(\tau)|^p \, ||A^\epsilon_{T_\epsilon(\tau)}||^p_{f^\epsilon_{T_\epsilon(\tau)}} = \frac{L^p_\epsilon||A^\epsilon_{T_\epsilon(\tau)}||^p_{f^\epsilon_{T_\epsilon(\tau)}}}{(\epsilon + ||A^\epsilon_{T_\epsilon(\tau)}||_{f^\epsilon_{T_\epsilon(\tau)}})^p} \leq L^p_\epsilon.$$

After an integration over $(0, 1)$ we use (5.12) to conclude that

$$\inf_{(f,A)}\Big\{\mathcal{C}(f, A) \mid (f, A) \in P^p(\bar{f}_0, \bar{f}_1)\Big\} \leq \Big(\int_0^1 (\epsilon + ||A^\epsilon_s||_{f^\epsilon_s})ds\Big)^p \leq \Big((M^p_p(\bar{f}_0, \bar{f}_1) + \epsilon)^{\frac{1}{p}} + \epsilon\Big)^p$$

Letting ϵ tend to 0 we have

$$\inf_{(f,A)}\Big\{\mathcal{C}(f, A) \mid (f, A) \in P^p(\bar{f}_0, \bar{f}_1)\Big\} \leq M^p_p(\bar{f}_0, \bar{f}_1).$$

<div align="right">QED.</div>

Lemma 5.3. *Suppose $c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty]$ is a lower semicontinuous convex function that satisfies (5.1) and (5.2). There exists a constant $\bar{C}_\Omega$ which depends only on Ω and s such that if $\bar{f}_0$ and $\bar{f}_1$ are such that (5.6) and (5.7 – 5.8) hold then*

$$M^p_p(\bar{f}_0, \bar{f}_1) \leq \bar{C}_\Omega||\bar{f}_1 - \bar{f}_0||^p_p.$$

Proof: Define $(\bar{f}, \bar{A})$ is as in Remark 2.3 (i) and recall that by (ii) of the same Remark, $(\bar{f}, \bar{A}) \in P^p(\bar{f}_0, \bar{f}_1)$. We integrate the expressions in (5.10) to obtain

$$\mathcal{C}(\bar{f}, \bar{A}) \leq \frac{c^\infty(\bar{f}_0) + c^\infty(\bar{f}_1)}{2}||\bar{A}||^p_p$$

We first use Lemma 5.2 and then (5.9) to conclude that

$$M^p_p(\bar{f}_0, \bar{f}_1) \leq \mathcal{C}(\bar{f}, \bar{A}) \leq \bar{C}_\Omega||\bar{f}_1 - \bar{f}_0||^p_p,$$

which completes the proof.   QED.

Denote by $\mathcal{H}_p$ the set of $k-$forms $f \in L^p(\Omega; \Lambda^k)$ such that

$$\begin{cases} df \equiv 0 & \text{in the weak sense on } \bar{\Omega} \\ (f - \bar{f}_0) \wedge \nu = 0 & \text{in the weak sense on } \partial\Omega \end{cases} \qquad \text{if } 1 \leq k \leq n-1$$

and

$$\begin{cases} df \equiv 0 & \text{in the weak sense on } \bar{\Omega} \\ \int_\Omega (f - \bar{f}_0)dx = 0 \end{cases} \qquad \text{if } k = n$$

Theorem 5.4. *Suppose $c : \Lambda^k \times \Lambda^{k-1} \to [0, \infty]$ is a lower semicontinuous convex function that satisfies (5.1), (5.2) and (5.3). Then the following hold.*

(i) *If $\bar{f}_0$ and $\bar{f}_1$ satisfy (5.6) and (5.7) then there exists (f^*, A^*) that minimizes $\mathcal{C}$ and $\int_0^1 ||A_s||_{f_s}ds$ over $P^p(\bar{f}_0, \bar{f}_1)$.*

(ii) *The function M_p in (1.8) is a metric on the set $\{f \in \mathcal{H}_p \mid c^\infty(f) < \infty\}$.*

Proof: (i) follows from Proposition 3.2 and Lemma 5.2.

(ii) Let $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2 \in \mathcal{H}_p$. By (i) and Lemma 5.2 there are $(f^0, A^0) \in P^p(\bar{\omega}_0, \bar{\omega}_1)$ and $(f^1, A^1) \in P^p(\bar{\omega}_1, \bar{\omega}_2)$ such that

$$(5.14) \qquad \int_0^1 ||A_s^0||_{f_s^0}^p ds = \mathcal{C}(f^0, A^0) = M_p^p(\tilde{f}_0, \tilde{f}_1) = \left(\int_0^1 ||A_s^0||_{f_s} ds \right)^p$$

and

$$(5.15) \qquad \int_0^1 ||A_s^1||_{f_s^1}^p ds = \mathcal{C}(f^1, A^1) = M_p^p(\tilde{f}_1, \tilde{f}_2) = \left(\int_0^1 ||A_s^1||_{f_s} ds \right)^p$$

By Lemma 5.3, if $\tilde{f}_0 = \tilde{f}_1$ then $M_p(\tilde{f}_0, \tilde{f}_1) = 0$. Conversely, $M_p(\tilde{f}_0, \tilde{f}_1) = 0$ means

$$\int_0^1 ds \int_\Omega c(f_s^0(x), A_s^0(x)) dx = 0,$$

and so, $c(f^0, A^0) = 0$ almost everywhere on $(0, 1) \times \Omega$. By (5.1) $A_0 = 0$ almost everywhere on $(0, 1) \times \Omega$. This means $(f^0, 0) \in P^p(\tilde{f}_0, \tilde{f}_1)$ and so, $\tilde{f}_1 = \tilde{f}_0$.

Setting

$$\tilde{f}(s, x) = f^0(1 - s, x), \quad \tilde{A}(s, x) = -A^0(1 - s, x),$$

we have $(\tilde{f}, \tilde{A}) \in P^p(\tilde{f}_1, \tilde{f}_0)$ and so,

$$M_p^p(\tilde{f}_1, \tilde{f}_0) \leq \mathcal{C}(\tilde{f}, \tilde{A}) = \mathcal{C}(f^0, A^0) = M_p^p(\tilde{f}_0, \tilde{f}_1)$$

By symmetry, the reverse inequality holds and so, $M_p^p(\tilde{f}_1, \tilde{f}_0) = M_p^p(\tilde{f}_0, \tilde{f}_1)$.

Set

$$f(s, x) = \begin{cases} f^0(2s, x) & \text{if } 0 \leq s \leq \frac{1}{2} \\ f^1(2s - 1, x) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \qquad A(s, x) = \begin{cases} 2A^0(2s, x) & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2A^1(2s - 1, x) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

We have $(f, A) \in P^p(\tilde{f}_0, \tilde{f}_2)$ and

$$||A_s||_{f_s} = \begin{cases} 2||A_{2s}^0||_{f_{2s}^0} & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2||A_{2s-1}^1||_{f_{2s-1}^1} & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence

$$M_p(\tilde{f}_0, \tilde{f}_2) \leq \int_0^1 ||A_s||_{f_s} ds = \int_0^{\frac{1}{2}} 2||A_{2s}^0||_{f_{2s}^0} ds + \int_{\frac{1}{2}}^1 2||A_{2s-1}^1||_{f_{2s-1}^1} ds = M_p(\tilde{f}_0, \tilde{f}_1) + M_p(\tilde{f}_1, \tilde{f}_2)$$

This concludes the proof of (ii). QED.

5.2. **A duality result for non–finite cost function.**

Remark 5.5. *The following hold.*

(i) *By the convexity and lower semicontinuity properties of*

$$(b, B) \to \underline{c}_\epsilon(b, B) := c^*(b, B) + \frac{\epsilon}{p}(|b|^p + |B|^p),$$

setting $c_\epsilon := (\underline{c}_\epsilon)^*$, *we have* $c_\epsilon^* = \underline{c}_\epsilon$.

(ii) *Observe that since* c^* *is convex,* c_ϵ^* *is strictly convex. Furthermore,*

$$c_\epsilon^* \geq c^* \quad \text{and} \quad c_\epsilon \leq c.$$

(iii) *By (5.4) there is a constant* $\gamma_3^\epsilon > 0$ *depending on* $\epsilon > 0$ *such that*

$$-\gamma_2 + \frac{\epsilon}{p}(|b|^p + |B|^p) \leq c_\epsilon^*(b, B) \leq \gamma_7 + \gamma_3^\epsilon \big(|b| + |B|^r\big)$$

(iv) *By (5.3) there are constants $\gamma_6^* > 0$ and $\gamma_7^* \geq 0$ independent of $\epsilon \in (0,1)$ such that*

$$c_\epsilon(\omega, \xi) \geq \gamma_6^* \big(|\omega|^p + |\xi|^p\big) - \gamma_7^*.$$

(v) *Using the notation of Section 3, since c^* satisfies (iii), Proposition 3.7 asserts the existence of h_ϵ that maximizes $\mathcal{D}_\epsilon$ over $\mathbf{B}^r\big((0,1) \times \Omega; \Lambda^k\big)$. By Theorem 3.8 there exists (f^ϵ, A^ϵ) which minimizes $\mathcal{C}_\epsilon$ over $P^p(\bar{f}_0, \bar{f}_1)$. Furthermore, $\mathcal{D}_\epsilon(h_\epsilon) = \mathcal{C}_\epsilon(f^\epsilon, A^\epsilon)$. Since c_ϵ^* is strictly convex, c_ϵ is continuously differentiable and so, Theorem 3.8 gives*

$$(f^\epsilon, A^\epsilon) \in \partial c_\epsilon^*(\partial_s h_\epsilon, \delta h_\epsilon) \quad i.e. \quad (\partial_s h_\epsilon, \delta h_\epsilon) = \nabla c_\epsilon(f^\epsilon, A^\epsilon).$$

Theorem 5.6. *Assume c satisfies (5.1), (5.2) and (5.3) and c^* satisfies (5.4). We assume that $\bar{f}_0, \bar{f}_1 \in C_0(\Omega, \Lambda^k)$ are such that (5.6), (5.7) hold and there exists $\epsilon_0 > 0$ such that $|\bar{f}_0|, |\bar{f}_1| \leq \gamma_1 - \epsilon_0$. Then*

$$(5.16) \qquad \max_{h \in BV_*^r(0,1;\Omega)} \mathcal{D}(h) = \min_{(f,A) \in P^p(\bar{f}_0, \bar{f}_1)} \mathcal{C}(f, A).$$

Proof: 1. Let (f^ϵ, A^ϵ) and h^ϵ be the optima in Remark 5.5. We first use the minimality property of (f^ϵ, A^ϵ) and then use Remark 5.5 (ii) to conclude that

$$\mathcal{C}_\epsilon(f^\epsilon, A^\epsilon) \leq \mathcal{C}_\epsilon(\bar{f}, \bar{A}) \leq \mathcal{C}(\bar{f}, \bar{A}) < \infty.$$

This, together with Remark 5.5 (iv) implies

$$\sup_{\epsilon \in (0,1)} ||f^\epsilon||_p + ||A^\epsilon||_p < \infty.$$

Thus, up to a subsequence $(f^\epsilon)_\epsilon$ converges weakly in $L^p((0,1) \times \Omega; \Lambda^k)$ to some f^0 and $(A^\epsilon)_\epsilon$ converges weakly in $L^p((0,1) \times \Omega; \Lambda^{k-1})$ to some A^0. For any $b \in C_0((0,1) \times \Omega; \Lambda^k)$ and $B \in C_0((0,1) \times \Omega; \Lambda^{k-1})$ we have

$$\liminf_{\epsilon \to 0^+} \mathcal{C}_\epsilon(f^\epsilon, A^\epsilon) \geq \liminf_{\epsilon \to 0^+} \int_{(0,1) \times \Omega} \big(\langle f^\epsilon; b \rangle + \langle A^\epsilon; B \rangle - c_\epsilon^*(b, B)\big) ds dx$$

$$= \int_{(0,1) \times \Omega} \big(\langle f^0; b \rangle + \langle A^0; B \rangle - c^*(b, B)\big) ds dx.$$

Thus, since c^* takes on only finite values, maximizing over (b, B), we can use Proposition C.5 (iii) to conclude that

$$(5.17) \qquad \liminf_{\epsilon \to 0^+} \mathcal{C}_\epsilon(f^\epsilon, A^\epsilon) \geq \mathcal{C}(f^0, A^0).$$

Recall the expression of $\mathcal{D}_\epsilon$ in (3.9), use the maximality property of h^ϵ and (5.4) to obtain that

$$\gamma_2 \mathcal{L}^n(\Omega) \geq -\mathcal{D}_\epsilon(0) \geq -\mathcal{D}_\epsilon(h^\epsilon) \geq \epsilon_0 ||\partial_s h^\epsilon||_1 + \gamma_1 ||\delta h^\epsilon||_r^r - ||\bar{A}||_p \, ||\delta h^\epsilon||_r - \gamma_2 \mathcal{L}^d(\Omega).$$

Thus, (4.14) holds. Thanks to Lemma 4.7, we may assume without loss of generality that (4.14) holds. We use Lemma 4.9 to conclude that there exists $h^0 \in BV_*^r(0,1;\Omega)$ such that up to a subsequence

(i) $(\delta h^{\epsilon_m})_m$ converges to δh^0 weakly in $L^r\big((0,1) \times \Omega; \Lambda^{k-1}\big)$.

(ii) $(\partial_s h^{\epsilon_m})_m$ converges weak $*$ to $\partial_s h^0$ on $(0,1) \times \Omega)$.

(iii) For $\mathcal{L}^1-$ almost every $s \in (0,1)$, $(h^{\epsilon_m}(s, \cdot))_m$ converges weak $*$ to $h^0(s, \cdot)$ on Ω

Since $c_\epsilon^* \geq c^*$,

$$(5.18) \qquad \liminf_{\epsilon \to 0^+} \int_{(0,1) \times \Omega} c_\epsilon^*(\partial_s h^\epsilon, \delta h^\epsilon) ds dx \geq \liminf_{\epsilon \to 0^+} \int_{(0,1) \times \Omega} c^*(\partial_s h^\epsilon, \delta h^\epsilon) dx.$$

By Theorem 3.3.1 [11] and the convergence in (i) and (ii), we have

$$(5.19) \qquad \liminf_{\epsilon \to 0^+} \int_{(0,1) \times \Omega} c^*(\partial_s h^\epsilon, \delta h^\epsilon) ds dx \geq \int_{(0,1) \times \Omega} c^*(\partial_s h^0, \delta h^0)$$

The integral of $c^*(\partial_s h^0, \delta h^0)$ needs to be interpreted as in Definition 4.2 which involves the recession function $\bar{c}^*$. Combining (5.18) and (5.19) we obtain

$$(5.20) \qquad \liminf_{\epsilon \to 0^+} \int_{(0,1) \times \Omega} c_\epsilon^*(\partial_s h^\epsilon, \delta h^\epsilon) ds dx \geq \int_{(0,1) \times \Omega} c^*(\partial_s h^0, \delta h^0).$$

Recall that we can assume without loss of generality that $s \to h^0(s, \cdot)$ is left continuous at 1 and right continuous at 0. We use the trace operator in Subsection 4.2, and combine (4.14) with (4.15) to obtain that

$$\lim_{m \to \infty} \int_\Omega \langle \bar{f}_1(x); h^{\epsilon_m}(1, x) \rangle dx - \int_\Omega \langle \bar{f}_0(x); h^{\epsilon_m}(0, x) \rangle dx$$

$$(5.21) \qquad = \int_\Omega \langle \bar{f}_1(x); h^0(1, dx) \rangle - \int_\Omega \langle \bar{f}_0(x); h^0(0, dx) \rangle.$$

Rearranging the expressions in the identify $\mathcal{C}_\epsilon(f^\epsilon, A^\epsilon) = \mathcal{D}_\epsilon(h^\epsilon)$ we have

$$(5.22) \qquad \int_{(0,1) \times \Omega} \left(c_\epsilon(f^\epsilon, A^\epsilon) + c_\epsilon^*(\partial_s h^\epsilon, \delta h^\epsilon) \right) ds dx = \int_\Omega \langle \bar{f}_1(x); h^\epsilon(1, dx) \rangle - \int_\Omega \langle \bar{f}_0(x); h^\epsilon(0, dx) \rangle$$

Thus, using (5.17), (5.20) and (5.21), together with the fact that

$$\liminf_{\epsilon \to 0^+} \int_0^1 ds \int_\Omega \left(c_\epsilon(f^\epsilon, A^\epsilon) dx + \liminf_{\epsilon \to 0^+} \int_0^1 ds \int_\Omega c_\epsilon^*(\partial_s h^\epsilon, \delta h^\epsilon) \right) dx$$

$$\leq \liminf_{\epsilon \to 0^+} \left(\int_0^1 ds \int_\Omega \left(c_\epsilon(f^\epsilon, A^\epsilon) dx + \int_0^1 ds \int_\Omega c_\epsilon^*(\partial_s h^\epsilon, \delta h^\epsilon) \right) dx \right),$$

we obtain

$$(5.23) \qquad \mathcal{C}(f^0, A^0) + \int_{(0,1) \times \bar{\Omega}} c^*(\partial_s h^0, \delta h^0) \leq \int_\Omega \langle \bar{f}_1(x); h^0(1, dx) \rangle - \int_\Omega \langle \bar{f}_0(x); h^0(0, dx) \rangle.$$

This means

$$(5.24) \qquad \mathcal{C}(f^0, A^0) \leq \mathcal{D}(h^0).$$

2. We claim that $\mathcal{C}(f, A) \geq \mathcal{D}(h)$ for any $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$ and any $h \in BV_*^r(0, 1; \Omega)$. Observe that (5.3) and (5.4) imply that $C := c^*$ satisfies (C.1) and (C.2). By the assumption on c, we have $C^* \geq C^*(0) = 0$. Let $h_l^\epsilon \in C^\infty(\Omega; \Lambda^k)$ and $h_l \in BV_*^r(0, 1; \Omega_l)$ be the approximations of h as defined by (D.2) in Section D. Here, Ω_l is the l–neighborhood of Ω. We have

$$\mathcal{C}(f, A) \geq \int_O \left(\langle f; \partial_s h_l^\epsilon \rangle + \langle A; \delta h_l^\epsilon \rangle dx \right) - \int_O c^*(\partial_s h_l^\epsilon, \delta h_l^\epsilon) ds dx$$

$$= \int_\Omega \left(\langle f_1(x); h_l^\epsilon(1, x) \rangle - \langle f_0(x); h_l^\epsilon(0, x) \rangle \right) dx - \int_O c^*(\partial_s h_l^\epsilon, \delta h_l^\epsilon) ds dx.$$

Letting ϵ tend to 0 in Lemmas D.3 and D.4 we obtain

$$\mathcal{C}(f, A) \geq \int_\Omega \left(\langle f_1(x); h_l(1, x) \rangle - \langle f_0(x); h_l(0, x) \rangle \right) dx - \int_{O_l} c^*(\partial_s h_l, \delta h_l).$$

Letting l tend to 0 in Lemmas D.3 and D.4 we obtain $\mathcal{C}(f, A) \geq \mathcal{D}(h)$. This, together with (5.24) concludes the proof of the Theorem. QED.

Appendix A. Open problems

Throughout this section, we use the same notation as in Subsection 5.2. To alleviate the notation, we denote by (f, A) the pair (f^0, A^0) in Remark 4.4 and write h instead of h^0. Let $(\partial_s h)_a$ denote the absolutely continuous part of $\partial_s h$. By abuse of notation, we don't distinguish between $(\partial_s h)_a$ and its Radon Nikodym derivative with respect to $\mathcal{L}^{n+1}$.

Remark A.1. *According Remark 4.4*

$$\mathcal{C}(f, A) \geq \int_O \big(\langle f; \partial_s h(ds, dx)\rangle + \langle A; \delta h\rangle ds dx\big) - \int_O c^*(\partial_s h, \delta h)$$

and if equality holds

(A.1) $$(f, A) \in \partial c^*\big((\partial_s h)_a, \delta h\big) \quad \mathcal{L}^{n+1} \quad a.e..$$

We next list few open problems, sources of future investigations.

Problem A.2. *These problems are stated under the hypotheses of Section 5.*

(i) *What are the regularity properties of the minimizing geodesics in* (5.16), *or equivalently, thanks to* (A.1), *what are the regularity properties of the maximizer h in* (5.16)?

(ii) *For the sake of illustration, let c be given by* (1.11) *so that*

$$c^*(b, B) = \sqrt{|b|^2 + \frac{|B|^{2r}}{r^2}}.$$

Hence, formally at least, using (A.1) *and expressing the fact that $(f, A) \in P^p(\bar{f}_0, \bar{f}_1)$, we have*

(A.2) $$\partial_s\left(\frac{(\partial_s h)_a}{\sqrt{\big|(\partial_s h)_a\big|^2 + r^{-2}|\delta h|^{2r}}}\right) + d\left(\frac{\delta h |\delta h|^{2(r-1)}}{r\sqrt{\big|(\partial_t h)_a\big|^2 + r^{-2}|\delta h|^{2r}}}\right) = 0$$

in the sense of distribution in the interior of $\mathbb{U}$ where

$$\mathbb{U} := \left\{\big|(\partial_s h)_a\big|^2 + r^{-2}|\delta h|^{2r} > 0\right\}$$

Observe (A.2) *is a type of system of elliptic PDEs. What can we show about the set $\mathbb{U}$?*

(iii) *Continuing with c given by* (1.11), *what are the regularity properties of $\big((\partial_t h)_a, \delta h\big)$ or equivalently, since the regularity properties of h transfer to those of (f, A) through* (A.1), *what are the regularity properties of (f, A)?*

Appendix B. Convex functions

Throughout this section, we assume that $\Omega \subset \mathbb{R}^n$ is an open bounded convex set, $p, r \in (1, \infty)$ and $rp = r + p$.

B.1. **Examples.** A prototype cost is

(B.1) $$c(\omega, A) = U\big(-\theta(\omega), A\big)$$

where

(B.2) $$U(\rho, A) = \begin{cases} \frac{|A|^p}{p\,\rho^{p-1}} & \text{if } \rho \in (0, \infty) \\ 0 & \text{if } (A = \vec{0} \text{ and } \rho = 0) \text{ or } (\rho = \infty) \\ \infty & \text{if } (A \neq \vec{0} \text{ and } \rho = 0) \text{ or } (\rho \in [-\infty, 0)) \end{cases}$$

and

(B.3) $$\theta(w) = \begin{cases} -\sqrt{1 - |w|^2} & \text{if } |w| \leq 1 \\ \infty & \text{if } |w| > 1. \end{cases}$$

In this case, the Legendre transform of θ is the strictly convex function $\theta^* : \Lambda^k \to [1, \infty)$ of class C^1 given by

$$\theta^*(z) = \sqrt{1 + |z|^2}, \quad z \in \Lambda^k.$$

The Legendre transform of U is U^* and

(B.4)
$$U^*(-\lambda, B) = \begin{cases} 0 & \text{if } \frac{|B|^{r'}}{r'} \leq \lambda, \\ \infty & \text{if } \frac{|B|^{r'}}{r'} > \lambda \end{cases}$$

If $b \in \Lambda^k$ and $B \in \Lambda^{k-1}$ then

$$c^*(b, B) := \min_{\alpha > 0} \left[\alpha \theta^* \left(\frac{b}{\alpha} \right) + U^*(-\alpha, B) \right] = \sqrt{|b|^2 + \frac{|B|^{2r}}{r^2}}.$$

For $B \neq 0$, the minimum is achieved at $0 < \alpha_0 := |B|^r / r$.

Remark B.1. *As mentioned in Remark 5.1, we have chosen not to include cases such as*

$$c(\omega, \xi) := U \left(\sum_{i < j} \omega_{ij}, \xi \right),$$

which satisfy $c(\lambda \omega, \lambda \xi) = \lambda c(\omega, \xi)$ for any $\lambda \in \mathbb{R}$. Indeed, in this case, $c^*(b, B) \in \{0, \infty\}$ for all $(b, B) \in \Lambda^2 \times \Lambda^1$.

B.2. **Bounds on gradients of convex functions.** Let $\mathbb{H}$ be a finite dimensional Hilbert space and assume $c, c^* : \mathbb{H} \to (-\infty, \infty]$ Legendre transform of each other and $\gamma_6, \gamma_7, \gamma_8 > 0$.

Remark B.2. *The following hold.*

(i) *Suppose $c(w) \geq -\gamma_8$ and $c^*(z) \geq \gamma_6 |z|^r - \gamma_7$ for any $w, z \in \mathbb{H}$. Then there exists a constant $\bar{C}_\gamma$ depending only on s, γ_6, γ_7 and γ_8 such that*

(B.5)
$$\sup_{z \in \partial . c(w)} |z| \leq \bar{C}_\gamma (|w|^{p-1} + 1), \qquad \forall \, w \in \mathbb{H}.$$

(ii) *Similarly, suppose $c(w) \geq \gamma_6 |w|^p - \gamma_7$ and $c^*(z) \geq -\gamma_8$ for any $w, z \in \mathbb{H}$. Then there exists a constant $\tilde{C}_\gamma$ depending only on r, γ_6, γ_7 and γ_8 such that*

(B.6)
$$\sup_{w \in \partial . c^*(z)} |w| \leq \tilde{C}_\gamma (|z|^{r-1} + 1), \qquad \forall \, p \in \mathbb{H}.$$

B.3. **A class of convex functions.** Assume that $c : \Lambda^k \times \Lambda^{k-1} \to (-\infty, \infty]$ is lower semicontinuous and for each if $\omega \in \Lambda^k$ and $A \in \Lambda^{k-1}$ are such that $c(\omega, A) < \infty$ then

(B.7)
$$c(\omega, \lambda A) = |\lambda|^p c(\omega, A)$$

and

(B.8)
$$c(\omega, A) = 0 \quad \text{if and only if} \quad A = 0.$$

For $\xi \in \Lambda^k$ we define

$$G_c(\xi) = \inf_\omega c(\omega, \xi).$$

We set

$$\lambda_c := \inf_\xi \left\{ G_c(\xi), \, | \, |\xi| = 1 \right\}.$$

Denote by $\Pi : \Lambda^k \times \Lambda^{k-1} \to \Lambda^k$ the projection operator. We assume that

(B.9)
$$\forall \, b \in \Lambda^k \setminus \{0\} \; \exists \; (\omega_m)_m \subset \Pi(\text{dom}c) \; | \; \lim_{m \to \infty} \langle \omega_m, b \rangle = \infty.$$

Obviously, if c takes on only finite values, then (B.9) holds.

Lemma B.3. *Suppose c satisfies (B.9).*

(i) *We have*

$$c^*(b, B) = \sup_\omega \left\{ \langle b; \omega \rangle + (c_\omega)^*(B) \right\} = \begin{cases} \infty & \text{if } b \neq 0 \\ \sup_{\omega \in \Lambda^k} (c_\omega)^*(B) & \text{if } b = 0. \end{cases}$$

(ii) *For any $(\omega, \xi) \in \Lambda^k \times \Lambda^{k-1}$ we have*

$$c^{**}(\omega, \xi) \leq \inf_{w \in \Lambda^k} (c_w)^{**}(\xi) \leq G_c(\xi).$$

(iii) *Because c is lower semicontinuous and (B.8) holds, we have $\lambda_c > 0$. Defining $s > 0$ by $s^p = p\lambda_c$ we have*

$$c^*(0, B) \leq \frac{|B|^r}{rs^r}.$$

Proof: We only comment on the proof of (iii).

Let $(\omega, \xi), (b, B) \in \Lambda^k \times \Lambda^{k-1}$. If $\xi \neq 0$ then by Young's inequality

$$\langle B; \xi \rangle \leq \frac{s^p |\xi|^p}{p} + \frac{|B|^r}{rs^r} \leq |\xi|^p c\left(\omega, \frac{\xi}{|\xi|}\right) + \frac{|B|^r}{rs^r} = c(\omega, \xi) + \frac{|B|^r}{rs^r}.$$

Rearranging and maximizing the subsequent inequality over ξ we obtain

$$(c_\omega)^*(B) = \sup_\xi \left\{ \langle B; \xi \rangle - c_\omega(\xi) \mid \xi \in \Lambda^{k-1} \right\} \leq \frac{|B|^r}{rs^r},$$

which, together with (i) implies (iii).   QED.

Lemma B.4. *Suppose c takes on only finite values.*

(i) *If c is upper semicontinuous, so is G_c.*

(ii) *If G_c is lower semicontinuous and convex then for any we have*

$$c^{**}(\omega, \xi) = G_c(\xi) \quad \forall (\omega, \xi) \in \Lambda^k \times \Lambda^{k-1}.$$

(iii) *If c is convex so is G_c. If in addition c is bounded below, then G_c is locally Lipschitz and $c(\omega, \xi) \equiv G_c(\xi)$.*

Proof: We shall only comment on the last statement of (iii) and leave it to the reader to show (i), (ii) and that if c is convex so is G_c. Assume we know G_c is convex. Since it takes on only finite values, it is locally Lipschitz. By the convexity of c and (ii), $c(\omega, \xi) = c^{**}(\omega, \xi) \equiv G_c(\xi)$. QED.

APPENDIX C. REPRESENTATION FORMULAS FOR $\int_O C(F)$ WHEN F IS A MEASURE

Let $\mathbb{H}$ be either the Hilbert space $\Lambda^k \times \Lambda^{k-1}$ or $\mathbb{R}^N$. We assume that

(C.1) $$C : \mathbb{H} \to (-\infty, \infty).$$

and denote by C^* the Legendre transform of C. Set $D := \text{dom}\,(C^*)$ and let $\text{int}(D)$ be the interior of D. Note that if $0 \in D$ then $C(u) \geq \langle 0; u \rangle - C^*(0) = -C^*(0)$ and so, C is bounded below. We sometimes make the stronger assumption that

(C.2) $$0 \in \text{int}(D).$$

C.1. **Basic properties of recession function.** Consider the Minkowski function of $\bar{D}$ and its polar

$$(C.3) \qquad \varrho_{\bar{D}}(u) := \inf_{t>0}\left\{t \mid \frac{u}{t} \in \bar{D}\right\}, \qquad \varrho_{\bar{D}}^0(v) := \sup_{u\in\mathbb{H}}\left\{\langle u; v\rangle \mid u \in \bar{D}\right\}.$$

Recall that $\bar{C}$ is the recession function of C as given by Definition 4.2. We recall the following two Lemmas from Convex Analysis.

Lemma C.1. *Assume C is convex and* (C.1) *holds. Then,*

   (ii) $\bar{C} = \varrho_{\bar{D}}^0$.

   (iii) $(\bar{C})^*(u) = \begin{cases} 0 & if\ u \in \bar{D} \\ \infty & if\ u \notin \bar{D} \end{cases}$

Lemma C.2. *Assume C is convex and* (C.1) *and* (C.2) *hold. Then,*

   (i) $\varrho_{\bar{D}}$ *is Lipschitz and there exists $\epsilon > 0$ such that $\varrho_{\bar{D}}^0 \geq \epsilon\|\cdot\|$.*

   (ii)

$$(C.4) \qquad\qquad \bar{D} = \{\varrho_{\bar{D}} \leq 1\}, \quad \text{int}(D) = \{\varrho_{\bar{D}} < 1\}.$$

C.2. **Integral of functions of measures in terms of Legendre transform.** Let $O \subset \mathbb{R}^{n+1}$ be a bounded open set and assume that C is convex and (C.1) holds and $0 \in D$. For each $l > 0$ we define C_l^*

$$C_l^*(w) = \begin{cases} C^*(w) & if\ |w| \leq l \\ \infty & if\ |w| > l \end{cases} \qquad w \in \mathbb{H}.$$

Because is lower semicontinuous and does not achieve the value $-\infty$, C_l, the Legendre transform of C_l^* is l–Lipschitz and convex. Furthermore, by the fact that $C = (C^*)^*$ and $C_l^* \geq C^*$,

$$(C.5) \qquad\qquad \lim_{l\to\infty} C_l(v) = C(v), \quad \text{and} \quad C_l(v) \leq C(v)$$

for all $v \in \mathbb{H}$. Note that

$$(C.6) \qquad\qquad C_l \geq \langle v, 0\rangle - C^*(0) = -C^*(0).$$

Identify $\mathbb{H}$ with $\mathbb{R}^N$. For N signed Borel measures $F_1, \cdots, F_N$ on O of finite total mass, we write Radon–Nikodym decomposition

$$F = F_a\mathcal{L}^N + F_s\eta.$$

Here, η is a finite Borel measure on $\mathbb{R}^N$ such that $\mathcal{L}^N$ and η are mutually singular, $F_a \in L^1(O)$ is a Borel map and $F_s \in L^1(\mathbb{R}^N, \eta)$ is a Borel map. We set

$$\mathcal{K}_1(F) := \int_O C(F_a)dx + \int_O \bar{C}(F_s)d\eta,$$

where $\bar{C}$ is the recession function of C. By Lemma C.1, since $0 \in D$, $0 = \bar{C}(0) \leq \bar{C}$ and $-C^*(0) \leq C$. Hence, $\int_O C(F_a)dx$ and $\int_O \bar{C}(F_s)d\eta$ exist although they may be ∞. Observe that because $\bar{C}$ is 1–homogeneous, if g_1 and g_2 are two finite Borel measures on O which are absolutely continuous with respect to each other then

$$\int_O \bar{C}\left(\frac{dF}{dg_1}\right)dg_1 = \int_O \bar{C}\left(\frac{dF}{dg_2}\right)dg_2.$$

Hence, even if the pair (F_s, η) is not uniquely determined, by the fact that the product $F_s\eta$ is uniquely determined, it is well–understood that $\int_O \bar{C}(F_s)d\eta$ is well defined. Similarly, (C.6)

implies that $\int_O C_l(F)dz$ makes sense. Thanks to (C.5) and (C.6) we can apply Fatou's Lemma to obtain that

$$\text{(C.7)} \qquad \lim_{l \to \infty} \int_O C_l(F_a)dx = \int_O C(F_a)dx$$

Remark C.3. *Assume $G_1 : O \to \mathbb{H}$ is a bounded Borel map. If $|G_1| \leq l$, then*

$$\int_0 \big(\langle F_a; G_1 \rangle - C_l^*(G_1)\big)dx = \int_0 \big(\langle F_a; G_1 \rangle - C^*(G)\big)d\nu$$

makes sense although it may be $-\infty$. It is not $-\infty$ if and only if $C^*(G_1), C_l^*(G_1) \in L^1(O)$.

Proof: Since $F_a \in L^1(O)$, if $|G_1| \leq l$, using the definition of C_l^*, Young's inequality and eventually (C.5), we have

$$\langle F_a; G_1 \rangle - C^*(G) = \langle F_a; G_1 \rangle - C_l^*(G_1) \leq C_l(F_a) \leq l|F_a| + C_l(0) \leq l|F_a| + C(0) \in L^1(O).$$

This allows to conclude the proof of the remark.   QED.

Thanks to Remark C.3 it makes sense to define

$$J(G_1, G_2) := \int_O \big(\langle F_a; G_1 \rangle - C^*(G_1)\big)dx + \int_O \langle F_s; G_2 \rangle d\eta$$

$$\mathcal{K}_2(F) = \sup_{G_1, G_2} \Big\{ J(G_1, G_2) \mid (G_1, G_2) \in \mathcal{B}_2 \Big\}.$$

Here, $\mathcal{B}_2$ is the set of pairs (G_1, G_2) such that $G_1, G_2 : O \to \mathbb{H}$ are Borel and bounded, $G_1 \in \text{dom}(C^*)$ $\mathcal{L}^{n+1}$–a.e. and $G_2 \in \text{dom}(C^*)$ η–a.e.

Lemma C.4. *Assume C is convex, (C.1) holds and $0 \in D$. Then $\mathcal{K}_1(F) = \mathcal{K}_2(F)$.*

Proof: If $(G_1, G_2) \in \mathcal{B}_2$ then except on a set of $\mathcal{L}^{n+1}$–null measure

$$\text{(C.8)} \qquad \langle F_a; G_1 \rangle - C^*(G_1) \leq C(F_a)$$

By Lemma C.1 (i), except on a set of η–null measure

$$\text{(C.9)} \qquad \langle F_s; G_2 \rangle \leq \bar{C}(F_s)$$

We integrate the two terms in (C.8) with respect to $\mathcal{L}^{n+1}$ and those in (C.9) with respect to η and add up the subsequent inequalities to conclude that $\mathcal{K}_1(F) \geq \mathcal{K}_2(F)$.

Suppose first that $\mathcal{K}_1(G) < \infty$ and fix $\epsilon > 0$ arbitrary. Since C_l assumes only finite values, for any $v \in \mathbb{H}$, the subdifferential $\partial C_l(v)$ is not empty. Because, C_l is l–Lipschitz, $\partial C_l(v)$ is a compact set contained in the ball of radius l. The theory of multifunctions [12] ensures existence of a Borel map $M_l : \mathbb{H} \to \mathbb{H}$ such that for any $v \in \mathbb{H}$, $M_l(v) \in \partial C_l(v)$ and $|M_l(v)| \leq l$. The map $G_l := M_l \circ F_a$ is a Borel map such that $|G_l| \leq l$ and

$$\text{(C.10)} \qquad C_l^*(G_l(z)) + C_l\big(F(z)\big) = \langle F(z); G_l(z) \rangle$$

for any $z \in O$. Thus,

$$\text{(C.11)} \qquad \int_O C_l(F_a)dz = \int_O \big(\langle F_a; G_l \rangle - C_l^*(G_l)\big)dz.$$

In light of (C.7) there exists l such that

$$\int_O C_l(F_a)dz \geq \int_O C(F_a)dz - \epsilon.$$

We combine this, together with (C.11) and use the fact that $C_l^* \geq C^*$ to conclude that

$$\text{(C.12)} \qquad \int_O \big(\langle F_a; G_l \rangle - C^*(G_l)\big)dz \geq \int_O C(F_a)dz - \epsilon.$$

Since $\int_O C(F_a)dx$ is finite and C^* is bounded below, (C.12) implies

(C.13) $\qquad \int_O \left(\langle F_a; G_l \rangle - C^*(G_l) \right) dz \geq \int_O C(F_a)dz - \epsilon, \quad G_l(z) \in D = \mathrm{dom}\,(C^*) \ \ \mathcal{L}^{n+1} - \text{a.e.}$

Observe that $\bar{C} : \mathbb{H} \to (-\infty, \infty)$ is a convex function that is bounded below. We apply (C.12) after replacing C by $\bar{C}$, $\mathcal{L}^{n+1}$ by η and F_a by F_s to conclude that we can choose l large enough so that there exists a Borel map $\bar{G}_l : O \to \mathbb{H}$ such that $|\bar{G}_l| \leq l$ and

(C.14) $\qquad \int_O \left(\langle F_s; \bar{G}_l \rangle - (\bar{C})^*(\bar{G}_l) \right) d\eta \geq \int_O \bar{C}(F_s)d\nu - \epsilon.$

By Lemma C.1 (ii), $(\bar{C})^*$ takes the value 0 in $\bar{D}$ otherwise takes the value ∞. Since we have assumed that $\int_O \bar{C}(F_s)d\nu$ is finite, (C.14) is equivalent to

(C.15) $\qquad \int_O \langle F_s; G_l \rangle d\eta \geq -\epsilon, \qquad \bar{G}_l(z) \in \bar{D} \ \ \eta - \text{a.e.}$

We combine (C.13) and (C.15) to obtain that $\mathcal{K}_1(F) \leq \mathcal{K}_2(F) + 2\epsilon$ and then use the fact that $\epsilon > 0$ is arbitrary to that $\mathcal{K}_1(F) \leq \mathcal{K}_2(F)$.

Assume that $\mathcal{K}_1(G) = \infty$ and so, for instance, $\int_O C(F_a)dx = \infty$. We are to show that for every $\epsilon > 0$, $\mathcal{K}_2(G) \geq \epsilon^{-1}$. In light of (C.7) there exists l such that

$$\int_O C_l(F_a)dx \geq \epsilon^{-1}.$$

We use (C.11) to obtain a bounded Borel map $G_l : O \to \mathbb{H}$ such that

$$\epsilon^{-1} \leq \int_O C_l(F_a)dz = \int_O \left(\langle F_a; G_l \rangle - C_l^*(G_l) \right) dz \leq \int_O \left(\langle F_a; G_l \rangle - C^*(G_l) \right) dz.$$

Observe that since C_l is l–Lipschitz and $F_a \in L^1(O)$. Hence, $C^*(G_l) \in L^1(O)$ and so, $G_l(z) \in D$ $\mathcal{L}^{n+1}$–a.e. This proves that $\mathcal{K}_2(F) \geq \epsilon^{-1}$ and so, $\mathcal{K}_2(F) = \infty$. $\hspace{2cm}$ QED.

Define

$$\mathcal{K}_3(F) = \sup_{G_1, G_2} \left\{ J(G_1, G_2) \mid (G_1, G_2) \in \mathcal{B}_3 \right\}.$$

Here, $\mathcal{B}_3$ is the set of pairs (G_1, G_2) in $\mathcal{B}_2$ such that there exists a compact set $K \subset O$ such that $G_1 \in K$ $\mathcal{L}^{n+1}$–a.e. and $G_2 \in K$ η–a.e. Define

$$\mathcal{K}_4(F) = \sup_{G_1, G_2} \left\{ J(G_1, G_2) \mid (G_1, G_2) \in \mathcal{B}_4 \right\}.$$

Here, $\mathcal{B}_4$ is the set of pairs (G_1, G_2) in $\mathcal{B}_3$ that are continuous and of compact supports such that $G_1 = G_2$ and the range of G_1 is contained in a compact subset of the interior of D.

Proposition C.5. *As in Lemma C.4, we suppose that C is convex, (C.1) holds and $0 \in D$. Then*

$$\mathcal{K}_1(F) = \mathcal{K}_2(F) = \mathcal{K}_3(F).$$

If we further assume that (C.2) holds then

$$\mathcal{K}_1(F) = \mathcal{K}_2(F) = \mathcal{K}_4(F).$$

Proof: Replacing C^* by $C^* - C^*(0)$ if necessary, let us assume without loss of generality that $C^*(0) = 0$. What is obvious is

$$\mathcal{K}_1(F) \geq \mathcal{K}_2(F) \geq \mathcal{K}_3(F) \geq \mathcal{K}_4(F),$$

and by Lemma C.4 $\mathcal{K}_1(F) = \mathcal{K}_2(F)$. It remains to show the reverse inequalities.

Part 1. To show that $\mathcal{K}_2(F) \leq \mathcal{K}_3(F)$ it suffices to show that for any $\epsilon > 0$ and $(G_1.G_2) \in \mathcal{B}_2$,

$$J(G_1, G_2) \leq 2\epsilon + \mathcal{K}_3(F).$$

We can assume that $C^*(G_1) \in L^1(O)$ and $C^*(G_2) \in L^1(\eta)$ otherwise, there is nothing to prove. For each m positive integer, we define

$$S_m := \left\{ x \in O \mid \operatorname{dist}(z, \partial O) > \frac{1}{m} \right\}$$

and let χ_{S_m} be the indicator function of S_m. The dominated convergence theorem allows to choose m large enough so that

(C.16) $$\int_O (1 - \chi_{S_m})\left(\left|\langle F_a; G_1\rangle\right| + \left|C^*(G_1)\right|\right) dz < \epsilon.$$

We have

(C.17) $$\operatorname{spt}(\chi_{S_m} G_1) \subset \left\{ x \in O \mid \operatorname{dist}(z, \partial O) \geq \frac{1}{m} \right\} \subset O.$$

Since $C^*(0) = 0$, by convexity

$$C^*\left(\chi_{S_m} G_1\right) = C^*\left(\chi_{S_m} G_1 + (1 - \chi_{S_m})0\right) \leq \chi_{S_m} C^*(G_1)$$

Thus

$$\langle \chi_{S_m} G_1; F_a\rangle - C^*\left(\chi_{S_m} G_1\right) \geq \chi_{S_m}\left(\langle F_a; G_1\rangle - C^*(G_1)\right)$$
$$= \langle F_a; G_1\rangle - C^*(G_1) + (1 - \chi_{S_m})\left(\langle F_a; G_1\rangle - C^*(G_1)\right)$$

and so,

$$\langle F_a; G_1\rangle - C^*(G_1) \leq -(1 - \chi_{S_m})\left(\langle F_a; G_1\rangle - C^*(G_1)\right) + \langle \chi_{S_m} G_1; F_a\rangle - C^*\left(\chi_{S_m} G_1\right)$$

Integrating over O and using (C.16), we have

(C.18) $$\int_O \left(\langle F_a; G_1\rangle - C^*(G_1)\right) dx \leq \epsilon + \int_O \left(\langle F_a; \chi_{S_m} G_1\rangle - C^*(\chi_{S_m} G_1)\right) dx,$$

and so, $C^*(\chi_{S_m} G_1) \in L^1(O)$. Replace $(F_a, G_1, \mathcal{L}^{n+1}, C)$ by $(F_s, G_2, \eta, \bar{C})$ in (C.18) to obtain for m large enough,

$$\int_O \left(\langle F_s; G_2\rangle - (\bar{C})^*(G_2)\right) d\eta \leq \epsilon + \int_O \left(\langle F_s; \chi_{S_m} G_2\rangle - (\bar{C})^*(\chi_{S_m} G_2)\right) d\eta$$

and so $(\bar{C})^*(\chi_{S_m} G_2) \in L^1(\eta)$. As in the proof of Lemma C.4, use the fact that $(\bar{C})^*$ takes the value 0 in $\bar{D}$ otherwise it takes the value ∞ to conclude that

(C.19) $$\int_O \langle F_s; G_2\rangle dx \leq \epsilon + \int_O \langle F_s; \chi_{S_m} G_2\rangle dx.$$

Note that $(\bar{C})^*(\chi_{S_m} G_2) \in L^1(\eta)$ is equivalent to $(\bar{C})^*(\chi_{S_m} G_2) = 0$ η–a.e., which means that $\chi_{S_m} G_2 \in \bar{D}$ η–a.e. By (C.18) and (C.19)

(C.20) $$J(G_1, G_2) \leq 2\epsilon + J\left(\chi_{S_m} G_1, \chi_{S_m} G_2\right).$$

Since $(\chi_{S_m} G_1, \chi_{S_m} G_2) \in \mathcal{B}_3$, we obtain

$$J(G_1, G_2) \leq 2\epsilon + \mathcal{K}_3(F).$$

We use the fact that $(G_1, G_2) \in \mathcal{B}_2$ and $\epsilon > 0$ are arbitrary to conclude that $\mathcal{K}_2(F) \leq \mathcal{K}_3(F)$, and so, $\mathcal{K}_2(F) = \mathcal{K}_3(F)$.

Part 2. Further assume that 0 is in the interior of D. To show that $\mathcal{K}_3(F) \leq \mathcal{K}_4(F)$, it suffices to show that for any arbitrary $\epsilon > 0$, if $(G_1, G_2) \in \mathcal{B}_3$ then

(C.21) $$J(G_1, G_2) \leq \epsilon + \mathcal{K}_4(F).$$

Fix such a (G_1, G_2) and assume without loss of generality that $C^*(G_1) \in L^1(O)$ and $(\bar{C})^*(G_2) \in L^1(\eta)$. Extend G_1 by setting it to be null outside O. Let m_0 be such that $\mathcal{L}^{n+1}$–a.e, G_1 is supported by S_{m_0} and η–a.e, G_2 is supported by S_{m_0} and

(C.22) $$(\eta + \mathcal{L}^{n+1})(O \setminus S_{m_0}) \leq \frac{\epsilon}{16\big((\eta + \mathcal{L}^{n+1})(O)\big)}.$$

Consider a standard mollifier $\varrho \in C_c^\infty(\mathbb{R}^{n+1})$ which is a probability density supported by the unit ball centered at the origin. Define

$$G_l = \varrho_l * G_l; \quad \varrho_l(z) = \frac{1}{l^{n+1}} \varrho\Big(\frac{z}{l}\Big).$$

By Jensen's inequality

(C.23) $$C^*(G_l) \leq \varrho_l * C^*(G_1).$$

Similarly, since G_1 is supported by $\bar{D}$ $\mathcal{L}^{n+1}$–a.e. and the latter is a convex set, by Jensen's inequality, the range of G_l is contained in $\bar{D}$. By the fact that both G_1 and $C^*(G_1)$ are in $L^1(O)$, standard arguments show that

(C.24) $$\lim_{l \to 0^+} ||G_l - G_1||_{L^1(O)} = \lim_{l \to 0^+} ||\varrho_l * C^*(G_1) - C^*(G_1)||_{L^1(O)} = 0.$$

Thus, for l small enough

$$\int_O \big(\langle F_a; G_1 \rangle - C^*(G)\big)dz \leq \frac{\epsilon}{8} + \int_O \big(\langle F_a; G_l \rangle - \varrho_l * C^*(G_1)\big)dz.$$

This, together with (C.23) implies

(C.25) $$\int_O \big(\langle F_a; G_1 \rangle - C^*(G_1)\big)dz \leq \frac{\epsilon}{8} + \int_O \big(\langle F_a; G_l \rangle - C^*(G_l)\big)dz.$$

By Lusin's theorem theorem there exists for each positive l, there exists a continuous function $\bar{G}^l$ such that

(C.26) $$\eta\Big\{\bar{G}^l(z) \neq G_2\Big\} < l.$$

Since G_2 is bounded and has its support in O, we may assume without loss of generality that

(C.27) $$|\bar{G}^l| \leq ||G_2||_{L^\infty(\eta)}.$$

Furthermore, we may assume without loss of generality that $\bar{G}^l$ is supported by S_{2m_0}. Consider the function $A \in C(\mathbb{R})$ defined

$$A(t) := \begin{cases} 1 & \text{if } t \leq 1 \\ \frac{1}{t} & \text{if } t > 1. \end{cases}$$

The function $G_0^l := \bar{G}^l A(\varrho_{\bar{D}} \circ \bar{G}^l)$ is continuous, supported by S_{2m_0} and its range is contained in $\bar{D}$. We have

$$\big\{G_0^l \neq G_2\big\} \subset \big\{\bar{G}^l \neq G_2\big\}$$

and so, by (C.26)

(C.28) $$\eta\Big\{G_0^l \neq G_2\Big\} < l.$$

By (C.27)

(C.29) $$|G_0^l| \leq ||G_2||_{L^\infty(\eta)}.$$

Since $F_s \in L^1(\eta)$ there exists $e > 0$ such that if $S \subset O$ and $\eta(S) \le e$ then

$$\int_O |F_s| d\eta < \frac{\epsilon}{16(\|G_2\|_{L^\infty(\eta))+1})}.$$

Thus, if $l \in (0, e)$, using (C.27) we have

$$\left| \int_O \langle F_s; G_2 - G_0^l \rangle d\eta \right| = \left| \int_{\{G_2 \ne G_0^l\}} \langle F_s; G_2 - G_0^l \rangle d\eta \right| < \frac{\epsilon}{8}.$$

If $\bar{e} \in (0, 1)$ is closed enough to 1 we conclude that setting $G^l := \bar{e} G_0^l$ then

(C.30) $$\int_O \langle F_s; G_2 \rangle d\eta \le \int_O \langle F_s; G^l \rangle d\eta < \frac{\epsilon}{8}.$$

Observe that $\varrho_{\bar{D}}(G^l) = \bar{e} \varrho_{\bar{D}}(G_0^l) \le \bar{e} < 1$ and so, by Lemma C.2, G^l belongs to $\text{int}(D)$. We combine (C.25) and (C.30) to conclude that

(C.31) $$J(G_1, G_2) \le \frac{\epsilon}{2} + J(G_l, G^l).$$

Let $E \subset O$ be a Borel set such that

$$\mathcal{L}^{n+1}(E) = \eta(O \setminus E) = 0.$$

By Lusin's theorem, we may find a sequence $\{\chi_j\}_j \subset C(\bar{O}, [0, 1])$ that converges $(\mathcal{L}^{n+1} + \eta)$–a.e. to χ_E. Set

$$g_j := (1 - \chi_j) G_l + \chi_j G^l.$$

We have that $g_j \in C_c(O, \mathbb{H})$ is bounded and so, since the ranges of both G_l and G^l are contained in $\bar{D}$ we have

(C.32) $$\varrho_{\bar{D}}(g_j) \le (1 - \chi_j) \varrho_{\bar{D}}(G_l) + \chi_j \varrho_{\bar{D}}(G^l) \le \bar{e}.$$

By Lemma C.2, $\{\varrho_{\bar{D}} \le \bar{e}\}$ is a compact set contained in $\text{int}(D)$, while (C.32) ensures that the range of g_j is contained in $\{\varrho_{\bar{D}} \le \bar{e}\}$. We use the convexity of C^* to conclude that

(C.33) $$\chi_j \big(C^*(G_l) - C^*(G^l) \big) \le C^*(G_l) - C^*(g_j)$$

We have

$$J(g_j, g_j) = J(G_l, G^l)$$
$$+ \int_O (1 - \chi_j) \langle F_s; G_l - G^l \rangle d\eta + \int_O \chi_j \langle F_a; G^l - G_l \rangle dx$$
$$+ \int_O (C^*(G_l) - C^*(g_j)) dx.$$

and so, by (C.33)

$$J(g_j, g_j) \ge J(G_l, G^l)$$
$$+ \int_O (1 - \chi_j) \langle F_s; G_l - G^l \rangle d\eta + \int_O \chi_j \langle F_a; G^l - G_l \rangle dx$$
(C.34) $$+ \int_O \chi_j \big(C^*(G_l) - C^*(G^l) \big) dx.$$

Since C^* is continuous in $\text{int}(D)$ and $G^l \in C_c(O, \mathbb{H})$ has its range contained in $\text{int}(D)$, we conclude that $C^*(G^l) \in C(\bar{O})$. In fact, since we have assumed that $C^*(0) = 0$, $C^*(G^l) \in C_c(O)$. What matters is the conclusion that $C^*(G^l) \in L^1(O)$ and so,

$$C^*(G_l) - C^*(G^l) \in L^1(O), \quad \langle F_s; G_l - G^l \rangle \in L^1(\eta), \quad \langle F_a; G^l - G_l \rangle \in L^1(O).$$

We apply the dominated convergence theorem to conclude that since $\eta(O \setminus E) = 0$ then

$$(C.35) \qquad \lim_{j \to \infty} \int_O (1 - \chi_j)\langle F_s; G_l - G^l \rangle d\eta = \int_O (1 - \chi_E)\langle F_s; G_l - G^l \rangle d\eta = 0.$$

Similarly, since $\mathcal{L}^{n+1}(E) = 0$ then

$$(C.36) \qquad \lim_{j \to \infty} \int_O \chi_j \langle F_a; G^l - G_l \rangle dx = \int_O \chi_E \langle F_a; G^l - G_l \rangle dx = 0.$$

Finally, since $\mathcal{L}^{n+1}(E) = 0$ then

$$(C.37) \qquad \lim_{j \to \infty} \int_O \chi_j \big(C^*(G_l) - C^*(G^l)\big) dx = \int_O \chi_E \big(C^*(G_l) - C^*(G^l)\big) dx = 0.$$

We combine (C.34 -C.37) to conclude that for j large enough

$$(C.38) \qquad J(g_j, g_j) \geq J(G_l, G^l) - \frac{\epsilon}{2}.$$

This, together with (C.31) and the fact that $(g_j, g_j) \in \mathcal{B}_4$ yields that

$$J(G_1, G_2) \leq \epsilon + J(g_j, g_j) \leq \epsilon + \mathcal{K}_4(F).$$

Since (G_1, G_2) is an arbitrary element of $\mathcal{B}_3$ and $\epsilon > 0$ is arbitrary, we conclude that $\mathcal{K}_3(F) \leq \mathcal{K}_4(F)$ and so, $\mathcal{K}_3(F) = \mathcal{K}_4(F)$.   QED.

APPENDIX D. APPROXIMATION OF k–CURRENTS BY SMOOTH CURRENTS

Throughout this section, we assume that $\Omega \subset \mathbb{R}^n$ is an open bounded convex set. We assume without loss of generality that Ω contains the origin and denote by $\varrho_{\bar\Omega}$ the Minkowski function of Ω (cf. (C.3)). Recall that (cf. Lemma C.2)

$$(D.1) \qquad \bar\Omega = \{\varrho_{\bar\Omega} \leq 1\}, \quad \text{int}(\Omega) = \{\varrho_{\bar\Omega} < 1\}.$$

For $0 < l < 1$, we use the notation

$$O := (0,1) \times \Omega, \quad \Omega_l := \{\varrho_{\bar\Omega} < 1 + l\}, \quad I_l := (-\frac{l}{2}, 1 + \frac{l}{2}) \quad O_l = I_l \times \Omega_l.$$

and

$$z := (s, x) \in O, \quad w := (\tau, y) \in O_l.$$

We define $T_l, S_l : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$T_l(\tau, y) = \Big(\frac{\tau + \frac{l}{2}}{1 + l}, \frac{y}{1 + l}\Big), \quad S_l(s, x) = \Big((1+l)s - \frac{l}{2}, (1+l)x\Big).$$

Fix $h \in BV_*^r(0, 1; \Omega)$ (cf. Definition 4.8) and define $h_l \in BV_*^r(I_l; \Omega_l)$ by

$$(D.2) \qquad \int_{O_l} \langle h_l(\tau, dy); \phi(\tau, y)\rangle d\tau := \int_O \langle h(s, dx); \phi\big(S_l(s, x)\big)\rangle ds \qquad \forall\, \phi \in C(\bar O_l; \Lambda^k).$$

For instance if $H \in C(\bar O)$ and $h(z) = H(z)e^1 \wedge \cdots \wedge e^k$ then $h_l(w) = H_l(w)e^1 \wedge \cdots \wedge e^k$ where

$$H_l(w) = \det \nabla_w T_l(w) H(T_l w).$$

Reminder D.1. *By Lemma 4.6 (iii), $t \to h(t, \cdot) \in \mathcal{M}(\Omega, \Lambda^k)$ is of bounded variations and so, it is continuous except may be at countably many t. Furthermore, by (ii) of the same Remark, we may tacitly choose an appropriate representative such that $t \to h(t, \cdot) \in \mathcal{M}(\Omega, \Lambda^k)$ is right continuous at any $t \in [0, 1)$. (iii) of the Remark will ensure left continuity at 1 and so, $h(t, \cdot)$ is well–defined for every $t \in [0, 1]$. Since $|\partial_s h|$ is a finite measure, $\|h(t, \cdot)\|$ is bounded by a constant independent of t.*

Remark D.2. *For any $\phi \in C(\bar{O}_l; \Lambda^k)$, the following hold.*

(i)
$$\int_{O_l} \langle \partial_\tau h_l(dw); \phi(w) \rangle = \frac{1}{1+l} \int_O \langle \partial_s h(dz); \phi(S_l z) \rangle.$$

(ii)
$$\int_{O_l} \langle \delta h_l(w); \phi(w) \rangle dw = \frac{1}{1+l} \int_\Omega \langle \delta h(z); \phi(S_l z) \rangle dz.$$

(iii) *For any $\psi \in C(\bar{\Omega}_l; \Lambda^k)$*
$$\int_{\Omega_l} \langle h_l(\tau, dy); \psi(y) \rangle = \frac{1}{1+l} \int_\Omega \left\langle h\left(\frac{\tau + \frac{l}{2}}{1+l}, dx\right); \psi((1+l)x) \right\rangle, \quad \forall\, \tau \in I_l.$$

(iv) *Since $|\partial_s h|\big((0,1) \times \Omega\big) < \infty$, the set of $t \in (0,1)$ such that $|\partial_s h|(\{t\} \times \Omega) > 0$ is at most countable. This implies, the set $\mathcal{T}$ of $l \in (0,1)$ such that*
$$|\partial_s h|\left(\left\{\frac{1 + \frac{l}{2}}{1+l}\right\} \times \Omega\right) > 0$$

is at most countable.

Proof: Thanks to (D.2), the proof of (i) and (ii) is direct. To prove (iii), we need to show that for any $\beta \in C_c(I_l)$ and $\psi \in C_c(\bar{\Omega}_l; \Lambda^k)$ we have
$$\int_{O_l} \langle h_l(\tau, dy); \psi(y) \rangle \beta(\tau) d\tau = \frac{1}{1+l} \int_{I_l} \beta(\tau) d\tau \int_\Omega \left\langle h\left(\frac{\tau + \frac{l}{2}}{1+l}, dx\right); \psi((1+l)x) \right\rangle.$$
We first use the change of variables $s(1+l) - l/2 = \tau$ in the second integral and then use (D.2) with $\phi(\tau, y) = \beta(t)\psi(y)$ to conclude.   QED.

Lemma D.3. *Suppose that C is a convex function on $\mathbb{H} := \Lambda^k, \times \Lambda^{k-1}$ such that (C.1) and (C.2) hold. Let $F_l := (\partial_\tau h_l, \delta h_l)$ and $F = (\partial_s h, \delta h)$ let $\mathcal{K}_1$ and $\mathcal{K}_4$ be as in Section C. We have $(1+l)\mathcal{K}_1(F_l) \le \mathcal{K}_1(F)$.*

Proof: Replacing C by $C(0)$ if necessary, we assume without loss of generality that $C(0) = 0$, which yields $C^* \ge 0$. By Proposition C.5, $\mathcal{K}_1(F_l) = \mathcal{K}_4(F_l)$. Hence, for any $\epsilon > 0$, there exist a compact set S contained in the interior of D and
$$g \in C_c(O; \Lambda^k), \quad g_* \in C_c(O; \Lambda^{k-1})$$
such that the range of $(g, g_*)(O_l)$ is contained in S and
$$\mathcal{K}_1(F_l) \le \epsilon + \int_{O_l} \langle \partial_\tau h_l(dw); g(w) \rangle + \langle \delta h_l(w); g_*(w) \rangle dw - \int_{O_l} C^*(g(w), g_*(w)) dw.$$
We use the change of variables provided by Remark D.2 to infer
$$\mathcal{K}_1(F_l) \le \epsilon + \frac{1}{1+l} \int_O \langle \partial_s h(dz); g(S_l z) \rangle + \langle \delta \bar{h}(z); g(S_l z) \rangle dz - \int_O C^*(g(S_l z), g_*(S_l z))(1+l)^{n+1} dz.$$
Using that $C^* \ge 0$ we conclude that , we obtain
$$\mathcal{K}_1(F_l) \le \epsilon + \frac{1}{1+l}\left(\int_O \langle \partial_s h(dz); g(S_l z) \rangle + \langle \delta \bar{h}(z); g(S_l z) \rangle dz - \int_O C^*(g(S_l z), g_*(S_l z)) dz\right).$$
Thus,
$$\mathcal{K}_1(F_l) \le \epsilon + \frac{1}{1+l}\mathcal{K}_4(F),$$
which, together with Proposition C.5, proves the Lemma.   QED.

Lemma D.4. *Suppose that C is a convex function on $\mathbb{H} := \Lambda^k, \times \Lambda^{k-1}$ such that (C.1) and (C.2) hold. Suppose C achieves its minimum at 0. Assume $f_0, f_1 \in C_0(\bar{\Omega})$ in the sense that their restriction to the boundary is the null function. Then,*

(i)

$$\lim_{l \to 0^+} \int_\Omega \langle h_l(1, x); f_1(x) \rangle dx = \int_\Omega \langle h(1, x); f_1(x) \rangle dx$$

(ii)

$$\lim_{l \to 0^+} \int_\Omega \langle h_l(0, x); f_0(x) \rangle dx = \int_\Omega \langle h(0, x); f_0(x) \rangle dx$$

Proof: By Lemma 4.6 (ii) and Remark D.2 (iii)

(D.3) $$\int_\Omega \langle h_l(1, x); f_1(x) \rangle dx = \frac{1}{1+l} \int_\Omega \chi_\Omega \big((1+l)x \big) \Big\langle f_1 \big((1+l)x \big); h \Big(\frac{1 + \frac{l}{2}}{1+l}, dx \Big) \Big\rangle = \frac{a_l + b_l}{1+l}$$

where

$$a_l := \int_\Omega \chi_\Omega \big((1+l)x \big) \Big\langle f_1 \big((1+l)x \big); h \Big(\frac{1 + \frac{l}{2}}{1+l}, dx \Big) - h(1, dx) \Big\rangle$$

and

$$b_l := \int_\Omega \chi_\Omega \big((1+l)x \big) \big\langle f_1 \big((1+l)x \big); h(1, dx) \big\rangle$$

Since $|h|(1, \cdot)$ is a finite measure, he Lebesgue dominated convergence theorem ensures that

(D.4) $$\lim_{l \to 0^+} b_l = \int_\Omega \big\langle f_1(x); h(1, dx) \big\rangle.$$

By Lemma 4.6 (ii)

$$|a_l| \leq ||f_1||_\infty |\partial_s h| \Big(\Big(\frac{1 + \frac{l}{2}}{1+l}, 1 \Big) \times \Omega \Big).$$

Hence,

$$\limsup_{l \to 0^+} |a_l| \leq ||f_1||_\infty |b| \big(\emptyset \times \Omega \big) = 0.$$

This with (D.3) and (D.4) proves (i). The proof of (ii) is obtained in a similar way.   QED.

Let $\varrho_1 \in C_c^\infty(\mathbb{R})$ and $\varrho_n \in C_c^\infty(\mathbb{R}^n)$ be nonnegative symmetric probability density functions. Suppose ϱ_n is positive on the open ball of radius 1 and null outside the closed ball of radius 1 and ϱ_1 satisfies the analogous condition. We set

$$\varrho_1^\epsilon(s) = \frac{1}{\epsilon} \varrho_1 \Big(\frac{s}{\epsilon} \Big), \quad \varrho_n^\epsilon(x) = \frac{1}{\epsilon^n} \varrho_n \Big(\frac{x}{\epsilon} \Big), \quad \varrho^\epsilon(s, x) := \varrho_1^\epsilon(s) \varrho_n^\epsilon(x).$$

For

$$\psi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \psi_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \in C(\bar{\Omega}; \Lambda^k)$$

we define

$$\psi^\epsilon(y) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} e^{i_1} \wedge \cdots \wedge e^{i_k} \int_\Omega \psi_{i_1 \cdots i_k}(x) \varrho_n^\epsilon(x - y) dx.$$

Similarly, for

$$\phi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \psi_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \in C(\bar{O}; \Lambda^k)$$

we define

$$\phi^\epsilon(w) = \sum_{1 \le i_1 < \cdots < i_k \le n} e^{i_1} \wedge \cdots \wedge e^{i_k} \int_O \psi_{i_1 \cdots i_k}(z) \varrho_n^\epsilon(z - w) dz.$$

In the remaining of this section, we fix $f_0, f_1 \in C(\bar\Omega; \Lambda^k)$.

$$f_0^\epsilon(y) := \int_\Omega \varrho_n^\epsilon(x - y) f_0(x) dx, \qquad f_1^\epsilon(y) := \int_\Omega \varrho_n^\epsilon(x - y) f_1(x) dx.$$

Since $h_l \in BV_*^r(I_l; \Omega_l)$ and for $\epsilon \in (0, l/2)$ we define can $h_l^\epsilon \in BV_*^r(0, 1; \Omega)$ by

(D.5) $$\int_O \langle h_l^\epsilon(s, dx); \phi(s, x) \rangle ds := \int_{O_\epsilon} \langle h_l(\tau, dy); \phi^\epsilon(w) \rangle d\tau \qquad \forall\, \phi \in C_c(O; \Lambda^k).$$

For instance, if

$$h_l(s, \cdot) = \sum_{1 \le i_1 < \cdots < i_k \le n} h_{i_1 \cdots i_k}(s, dx) e^{i_1} \wedge \cdots \wedge e^{i_k}$$

then

$$h_l^\epsilon(z) = \sum_{1 \le i_1 < \cdots < i_k \le n} e^{i_1} \wedge \cdots \wedge e^{i_k} \int_{O_\epsilon} \varrho^\epsilon(z - w) h_{i_1 \cdots i_k}(s, dy) ds \quad \forall\, z \in O_\epsilon.$$

Thus, $h_l^\epsilon \in C^\infty(\bar O; \Lambda^k)$.

Remark D.5. *Let $\phi \in C(\bar O; \Lambda^k)$ and $\psi \in C(\bar\Omega; \Lambda^k)$. Then the following hold.*

(i)
$$\int_O \langle \partial_s h_l^\epsilon(dz); \phi(z) \rangle = \int_{O_\epsilon} \langle \partial_s h_l(dw); \varrho^\epsilon * \phi(w) \rangle.$$

(ii)
$$\int_O \langle \delta h_l^\epsilon(z); \phi(z) \rangle dz = \int_{O_\epsilon} \langle \delta h_l(w); \varrho^\epsilon * \phi(w) \rangle dw.$$

Proof: The proof of the Remark is straightforward to obtain.   QED.

Lemma D.6. *Suppose that C is a convex function on $\mathbb{H} := \Lambda^k, \times \Lambda^{k-1}$ such that (C.1) and (C.2) hold. Suppose C achieves its minimum at 0. Let $F_l^\epsilon := (\partial_s h_l^\epsilon, \delta h_l^\epsilon)$ and $F_l = (\partial_s h_l, \delta h_l)$ let $\mathcal{K}_1$ and $\mathcal{K}_4$ be as in Section C. We have $\mathcal{K}_1(F_l^\epsilon) \le \mathcal{K}_1(F_l)$.*

Proof: Replacing C by $C(0)$ if necessary, we assume without loss of generality that $0 = C(0) \le C$, which yields $C^* \ge 0 = C^*(0)$. By Proposition C.5 $\mathcal{K}_1(F_l^\epsilon) = \mathcal{K}_4(F_l^\epsilon)$. Hence, for any $\bar\epsilon > 0$, there exist a compact set S contained in the interior of D and

$$g \in C_c(O; \Lambda^k), \quad g_* \in C_c(O; \Lambda^{k-1})$$

such that the range of $(g, g_*)(O)$ is contained in S and

$$\mathcal{K}_1(F_l^\epsilon) \le \bar\epsilon + \int_O \Big(\langle \partial_s h_l^\epsilon(z); g(z) \rangle + \langle \delta h_l^\epsilon(z); g_*(z) \rangle \Big) dz - \int_O C^*(g(z), g_*(z)) dw.$$

We use the change of variables provided by Remark D.5 to infer

(D.6) $$\mathcal{K}_1(F_l^\epsilon) \le \bar\epsilon + \int_{O^\epsilon} \Big(\langle \partial_s h_l(dz); \varrho^\epsilon * g(z) \rangle + \langle \delta h_l(z); \varrho^\epsilon * g(z) \rangle dz \Big) - \int_O C^*(g(z), g_*(z)) dz.$$

By Jensen's inequality

$$\int_{\mathbb{R}^{n+1}} C^*\big(\varrho^\epsilon * g(z), \varrho^\epsilon * g_*(z)\big)dz \leq \int_{\mathbb{R}^{n+1}} dz \int_{\mathbb{R}^{n+1}} \varrho^\epsilon(z-w)C^*\big(g(w), g_*(w)\big)dw$$

$$= \int_{\mathbb{R}^{n+1}} C^*\big(g(w), g_*(w)\big)dw..$$

Since $C^* \geq 0 = C^*(0)$ and (g, g_*) is supported by O, we obtain

$$\int_{O^\epsilon} C^*\big(\varrho^\epsilon * g(z), \varrho^\epsilon * g_*(z)\big)dz \leq \int_O C^*\big(g(w), g_*(w)\big)dw,$$

which together with (D.6) yields

$$\mathcal{K}_1(F_l^\epsilon) \leq \bar{\epsilon} + \int_{O^\epsilon} \Big(\langle \partial_s h_l(dz); \varrho^\epsilon * g(z)\rangle + \langle \delta h(z); \varrho^\epsilon * g(z)\rangle dz \Big) - \int_{O^\epsilon} C^*\big(\varrho^\epsilon * g(z), \varrho^\epsilon * g_*(z)\big)dz.$$

Thus, $\mathcal{K}_1(F_l^\epsilon) \leq \bar{\epsilon} + \mathcal{K}_4(F_l)$. Since $\bar{\epsilon} > 0$ is arbitrary, we use Proposition C.5, to conclude the proof.   QED.

Lemma D.7. *Suppose that C is a convex function on $\mathbb{H} := \Lambda^k, \times\Lambda^{k-1}$ such that (C.1) and (C.2) hold. Suppose C achieves its minimum at 0. Assume $f_0, f_1 \in C_0(\bar{\Omega})$ in the sense that their restriction to the boundary is the null function. Then, for almost every $l \in (0,1)$*

(i)

$$\lim_{\epsilon \to 0^+} \int_\Omega \langle h_l^\epsilon(1, x); f_1(x)\rangle dx = \int_\Omega \langle h_l(1, x); f_1(x)\rangle dx$$

(ii)

$$\lim_{\epsilon \to 0^+} \int_\Omega \langle h_l^\epsilon(0, x); f_0(x)\rangle dx = \int_\Omega \langle h_l(0, x); f_0(x)\rangle dx$$

Proof: We shall only show (i) as the proof of (ii) follows the same lines of arguments. We have

$$(D.7) \qquad \int_\Omega \langle h_l^\epsilon(1, x); f_1(x)\rangle dx = \int_\Omega dx \Big\langle f_1(x); \int_{I_\epsilon} a_l^\epsilon(\tau, x)d\tau \Big\rangle dx + \int_{\Omega_\epsilon} \langle h_l(1, dy); f_1^\epsilon(y)\rangle,$$

where,

$$a_l^\epsilon(\tau, x) := \int_{\Omega^\epsilon} \varrho^\epsilon(1-\tau, x-y)\big(h_l(\tau, dy) - h_l(1, dy)\big).$$

Part 1. Since f_1 vanishes on $\partial\Omega$, we can extend it by setting its value to be 0 outside $\bar{\Omega}$, and obtain a function $C_c(\mathbb{R}^n)$. Consequently, $(f_1^\epsilon)_\epsilon$ converges uniformly to f_1 on $\mathbb{R}^n$. Since $(\chi_{\Omega_\epsilon})_\epsilon$ converges pointwise to $\chi_{\bar{\Omega}}$, $(f_1^\epsilon \chi_{\Omega_\epsilon})_\epsilon$ converges pointwise to $f_1\chi_{\bar{\Omega}} = f_1\chi_\Omega$. Thus,

$$(D.8) \qquad \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon} \langle h_l(1, dy); f_1^\epsilon(y)\rangle = \int_\Omega \langle h_l(1, dy); f_1(y)\rangle.$$

Part 2. We use Remark D.2 (iii) to obtain

$$a_l^\epsilon(\tau, x) = \begin{cases} \frac{-1}{1+l} \int_{\left(\frac{\tau+\frac{l}{2}}{1+l}, \frac{1+\frac{l}{2}}{1+l}\right] \times \Omega} \chi_{\Omega^\epsilon}\big((1+l)y\big)\varrho^\epsilon\big(1-\tau, x-(1+l)y\big)\partial_s h(ds, dy) & \text{if } \tau \leq 1 \\ \frac{1}{1+l} \int_{\left(\frac{1+\frac{l}{2}}{1+l}, \frac{\tau+\frac{l}{2}}{1+l}\right] \times \Omega} \chi_{\Omega^\epsilon}\big((1+l)y\big)\varrho^\epsilon\big(1-\tau, x-(1+l)y\big)\partial_s h(ds, dy) & \text{if } \tau > 1 \end{cases} \qquad w \in \mathbb{H}.$$

Since

$$\varrho^\epsilon\big(1-\tau, x-(1+l)y\big) = \varrho_1^\epsilon(1-\tau)\varrho_n^\epsilon\big(x-(1+l)y\big)$$

vanishes outside $[1 - \epsilon, 1 + \epsilon]$, we conclude that

$$(D.9) \qquad \left| \int_\Omega \left\langle f_1(x); \int_{I_\epsilon} a_l^\epsilon(\tau, x) d\tau \right\rangle dx \right| \leq \frac{||f_1||_\infty}{1 + l} \int_{I_\epsilon \times \Omega} e_l^\epsilon(\tau, x) d\tau dx$$

with

$$e_l^\epsilon(\tau, x) = \varrho_1^\epsilon(1 - \tau) \int_{\left(\frac{1 - \epsilon + \frac{l}{2}}{1 + l}, \frac{1 + \epsilon + \frac{l}{2}}{1 + l} \right) \times \Omega} \varrho_n^\epsilon \big(x - (1 + l)y) \big) |\partial_s h|(ds, dy)$$

We have

$$\int_{I_\epsilon \times \Omega} e_l^\epsilon(\tau, x) d\tau dx = \int_{I_\epsilon} \varrho_1^\epsilon(1 - \tau) d\tau \int_{\left(\frac{1 - \epsilon + \frac{l}{2}}{1 + l}, \frac{1 + \epsilon + \frac{l}{2}}{1 + l} \right) \times \Omega} |\partial_s h|(ds, dy) \int_\Omega \varrho_n^\epsilon \big(x - (1 + l)z) \big) dx$$

$$\leq \int_{I_\epsilon} \varrho_1^\epsilon(1 - \tau) |\partial_s h| \left(\left(\frac{1 - \epsilon + \frac{l}{2}}{1 + l}, \frac{1 + \epsilon + \frac{l}{2}}{1 + l} \right) \times \Omega \right) d\tau$$

$$(D.10) \qquad = |\partial_s h| \left(\left(\frac{1 - \epsilon + \frac{l}{2}}{1 + l}, \frac{1 + \epsilon + \frac{l}{2}}{1 + l} \right) \times \Omega \right)$$

We combine (D.9) and (D.10) to conclude that for any $l \in (0, 1) \setminus \mathcal{T}$ (cf. Remark D.2 (iv))

$$(D.11) \qquad \lim_{\epsilon \to 0^+} \left| \int_\Omega \left\langle f_1(x); \int_{I_\epsilon} a_l^\epsilon(\tau, x) d\tau \right\rangle dx \right| = 0.$$

We combine (D.8) and (D.11) to conclude the proof of (i). $\hfill$ QED.

APPENDIX E. CLOSED DIFFERENTIAL 2–FORMS AND ELECTROMAGNETISM

The search of optimal k–forms can be put in the context of electro–magnetism. Indeed, consider a contractible open bounded convex set $O \subset \mathbb{R}^3$, denote by $\mathbf{n}$ the unit outward vector to ∂O and suppose

$$\Omega := (0, T) \times O.$$

Define $\mathcal{S}$ to be the set of pairs of magnetic/electric vector fields

$$(B, E) : \Omega \to \mathbb{R}^6$$

which are integrable and satisfy (in the weak sense) Gauss's law for magnetism

$$(E.1) \qquad\qquad \nabla_\mathbf{x} \cdot B = 0,$$

and the Maxwell–Faraday induction equations

$$(E.2) \qquad\qquad \partial_t B + \nabla_\mathbf{x} \times E = \vec{0}.$$

The well–known correspondence between $\mathcal{S}$ and the set of closed differential 2–forms on Ω is given by the isometry M which associates to (B, E) the 2–differential form $M(B, E)$ defined by

$$M(B, E) := -E_1 dt \wedge dx_1 - E_2 dt \wedge dx_2 - E_3 dt \wedge dx_3 + B_1 dx_2 \wedge dx_3 - B_2 dx_1 \wedge dx_3 + B_3 dx_1 \wedge dx_2.$$

One identifies the differential form $M(B, E)$ with the skew–symmetric matrix

$$(E.3) \qquad\qquad M(B, E) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}.$$

If we set $f = M(B, E)$, direct computations reveal that

$$df = (\nabla \cdot B)dx_1 \wedge dx_2 \wedge dx_3 + \Big(\partial_t B_3 + (\nabla \times E)_3\Big)dt \wedge dx_1 \wedge dx_2$$

(E.4) $$+ \Big(\partial_t B_2 + (\nabla \times E)_2\Big)dt \wedge dx_1 \wedge dx_3 + \Big(\partial_t B_1 + (\nabla \times E)_1\Big)dt \wedge dx_2 \wedge dx_3$$

and so, f is closed if and only if both (E.1) and (E.2) hold. Furthermore,

(E.5) $$\det M(B, E) = (E \cdot B)^2$$

and so, f is symplectic if and only if $(E \cdot B)^2 > 0$.

Let $w = (w^0, w^1, w^2, w^3) = (w^0, \mathbf{w}) : (0, T) \times O \to \mathbb{R}^4$ be a vector field. Let $A \in \Lambda^1(\mathbb{R}^4)$ be written as

$$A = A^0 dt + \mathbf{A}dx = A^0 dt + A^1 dx_1 + A^2 dx_2 + A^3 dx_3.$$

When we write $A = w_\lrcorner f$ we mean that

(E.6) $$A = \begin{bmatrix} A^0 \\ \mathbf{A} \end{bmatrix} = \begin{bmatrix} E \cdot \mathbf{w} \\ -\mathbf{F} \end{bmatrix}$$

where

$$\mathbf{F} = w^0 E + \mathbf{w} \times B.$$

Therefore in terms of (B, E, w) the system of equations $\partial_s f + dA = 0$ is equivalent to

(E.7) $$\partial_s B = \nabla_{\mathbf{x}} \times \mathbf{A}; \qquad \partial_s E = \nabla_{\mathbf{x}} A_0 - \partial_t \mathbf{A}$$

This means

(E.8) $$\partial_s B = -\nabla_{\mathbf{x}} \times (w^0 E + \mathbf{w} \times B); \qquad \partial_s E = \nabla_{\mathbf{x}}(E \cdot \mathbf{w}) + \partial_t(w^0 E + \mathbf{w} \times B)$$

Using the identity

$$\nabla_{\mathbf{x}} \cdot (B \otimes \mathbf{w} - \mathbf{w} \otimes B) = \nabla_{\mathbf{x}} \times B \times \mathbf{w}$$

we equivalently write

(E.9) $\partial_s B + \nabla_{\mathbf{x}} \times (w^0 E) = \nabla_{\mathbf{x}} \cdot (B \otimes \mathbf{w} - \mathbf{w} \otimes B); \qquad \partial_s E = \nabla_{\mathbf{x}}(E \cdot \mathbf{w}) + \partial_t(w^0 E + \mathbf{w} \times B)$

Therefore, considering action of the form

$$\int_0^1 ds \int_\Omega c(f, A)d\mathbf{x}dt$$

under the conditions that

$$\partial_s f + dA = 0, \quad df = 0$$

amounts to considering actions of the form

$$\int_0^1 ds \int_\Omega c(B, E, E \cdot \mathbf{v}, w^0 E + \mathbf{w} \times B)d\mathbf{x}dt$$

under the conditions that (E.1), (E.2) and (E.9) hold. The boundary condition $A \wedge \nu = 0$ is equivalent to

(E.10) $E \cdot \mathbf{w} = (w^0 E + \mathbf{w} \times B) \times \mathbf{n} = 0$ on $(0, T) \times \partial O$ and $w^0 E + \mathbf{w} \times B = 0$ on $\{0, T\} \times O$.

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