On some analytical aspects of Mean Field Games

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February 20, 2019

Contents

Preface

These notes grew out of lectures given during the Fall 2018 course offered at UC Berkeley, in connection with a semester long program at MSRI. The goal here was to introduce the participants to some of the basics on deterministic as well as stochastic games with a large number of players (or agents), including games with infinitely many players. Our focus has been on understanding the three main mechanisms: *Hamiltonian, common noise and individual noises*, which play a role in the so–called *Master Equation* in Mean Field Games (MFG). The theory of optimal transportation provides the appropriate tools to covert some stochastic games into deterministic PDEs. We reveal the geometric aspects of some of the useful operators that have appeared in Mean Field Games and argue why they are "partial Laplacian" on the set of probability measures.

In the games we consider, for simplicity and to avoid considerations on the boundary of domains where the players evolve, we assume the players seat in the space $\mathbb{R}^d$ or in the torus $\mathbb{T}^d$ which we denote as $\mathbb{M}$. This means, the distributions of the players are probability measures on a subset of $\mathbb{R}^d$ or distributions on the torus, which we denote as $\mathcal{P}(\mathbb{M})$. In either case, we rather work on the metric space $\mathcal{P}_2(\mathbb{M})$, the set of Borel probability measures on $\mathbb{M}$, of finite second moments. This space plays an important role in our study. In particular, it facilitates the use the reconciled [30] various notions of differential of functions which appeared in the literature, to formulate problems in Mean Field Games (cf. Corollary 4.22 and Lemma 4.23 in Chapter 4).

It is not our intention to merely repeat or duplicate the emphases amply made in various excellent books [12] [14] [15] [40] or survey articles [11] [38], on the topic. Rather, we will emphasize the connection between geometric objects such as partial traces of Hessian of functions defined on $\mathcal{P}_2(\mathbb{M})$, which stochastic paths which are referred to as *common noise* or *individual noises* in MFG. Our hope is that these paths, which are lifted from $\mathbb{M}$ to $\mathcal{P}_2(\mathbb{M})$ and induce infinitesimal generator on the Wasserstein space, may help to start a theory of Sobolev functions on the

Wasserstein space. They certainly allow for a Fourier analysis on the set of functions defined on the set of probability measures.

A central problem in MFG is the study of the so–called Master Equations, a non–local Hamilton–Jacobi equation, which encodes most of the information needed in games with infinitely many players. When the value function and the running cost are derived from potential functions, we refer to potential Mean Field Games. In this case, the Master Equation can be derived from a local Hamilton–Jacobi equation on the Wasserstein space which induces an infinite dimensional transport equation. We will emphasize the point of view that in absence of common and individual noises, this non–local equation is an infinite dimensional version of a system of conservation laws.

We state the first and the second order Hamilton–Jacobi equations on the Wasserstein space and their link to fluids mechanics. We comment on existence and regularity issues. We show how to produce the Master Equation in Mean Field Games by differentiating the Hamilton–Jacobi equations on the Wasserstein space and state the analogy with conservation laws.

Chapter 1

Introduction

In this course, we are interested in non cooperative games, which means games with no global planner, where each player pursues his own interest. We are mainly interested in games with a large number of players, possibly infinitely many players. For such games, we rely on a foundational concept originally introduced by Nash in his 1951 influential paper [42], and named after him. If each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium. In other words, one must ask what each player would do, taking into account the decision–making of the others. The search of Nash equilibria relies on the study of a system of Hamilton–Jacobi equations in the case of finitely many players or the study of a non–local equation in the case of non–atomic games. The latter equation is the so–called master equation. The goal of these notes is to equip the reader with prerequisites to facilitate the study of this non–local equation, so–called master equation in Mean Field Games.

In Section 7.2 of Chapter 7, we make some interesting remark about monotonicity in the context of Mean Field Games and displacement convexity. For instance, there exists a function $\mathcal{F} : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ which is convex along geodesic paths in $\mathcal{P}_2(\mathbb{M})$, while $-\mathcal{F}$ is concave along traditional paths $t \to (1-t)\mu_0 + t\mu_1$.

Section 2.1 (Chapter 2) gives an overview of the course but the table of content displays a more detailed description of the material we cover.

Throughout these notes $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ and by $f : \mathbb{T}^d \mapsto \mathbb{R}$, we mean $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $f(x+k) = f(x)$ for any $(k, x) \in \mathbb{T}^d \times \mathbb{R}^d$. We say that f is $\mathbb{Z}^d$–periodic (or simply periodic). When $l \geq 0$ is an integer and $f \in C^l(\mathbb{R}^d)$ is periodic, we write $f \in C^l(\mathbb{T}^d)$. Similarly, we denote as $\mathrm{Lip}(\mathbb{T}^d)$ the set of periodic Lipschitz functions $f : \mathbb{R}^d \mapsto \mathbb{R}$.

We consider

$$\mathbb{M} = \mathbb{T}^d,\ \mathbb{R}^d \quad \text{etc...}$$

Let $\mathcal{P}(\mathbb{M})$ denote the set of Borel probability measures on $\mathbb{M}$ and let

$$\mathcal{P}_2(\mathbb{M}) = \Big\{ \mu \in \mathcal{P}(\mathbb{M}) \mid \int_{\mathbb{M}} |q|^2 \mu(dq) < \infty \Big\}.$$

When $D \geq 1$ is an integer, $\mu \in \mathcal{P}_2(\mathbb{M})$ and $\xi : \mathbb{M} \to \mathbb{R}^D$ is a Borel map such that

$$\|\xi\|_\mu := \Big(\int_{\mathbb{M}} |\xi(q)|^2 \mu(dq)\Big)^{\frac{1}{2}} < \infty,$$

we write $\xi \in L^2(\mu, \mathbb{R}^D)$ or simply $\xi \in L^2(\mu)$. We denote the corresponding inner product on the Hilbert space $L^2(\mu)$ as $\langle \cdot ; \cdot \rangle_\mu$.

Setting

$$C_2(\mathbb{M}) := \Big\{ f \in C(\mathbb{M}) \mid C > 0 \quad \text{such that} \quad |f(q)| \leq C(1 + |q|^2) \Big\},$$

note that if $f \in C_2(\mathbb{M})$ and $\mu \in \mathcal{P}_2(\mathbb{M})$ then $\sqrt{|f|} \in L^2(\mu, \mathbb{R})$. If $\mathbb{M} = \mathbb{T}^d$ then $C_2(\mathbb{M}) = C(\mathbb{M})$.

The study of dynamical systems on $\mathcal{P}_2(\mathbb{M})$ in Eulerian coordinates, can sometimes be facilitated through a Lagrangian formulation on a Hilbert space. We will sometimes take advantage of working a flat Hilbert space than working on $\mathcal{P}_2(\mathbb{M})$. For instance, consider the set

$$\mathbb{H} := L^2\big((0,1)^d; \mathbb{R}^d\big)$$

which is a Hilbert space when endowed with the L^2 inner product. A function $U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ induces a lift $\hat{U} : \mathbb{H} \mapsto \mathbb{R}$. which associate to $X \in \mathbb{H}$, $\hat{U}(X) := U(\mu)$ when the law of X is $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. The set $\mathbb{H}$ being a flat, its differential structure is simple to describe. When $\nabla_{L^2}\hat{U}(X) \in \mathbb{H}$, the gradient of $\hat{U}$ at X exists, it can be expressed as the composition of a vector field $\nabla_{\omega_2} U(\mu) : \mathbb{R}^d \mapsto \mathbb{R}^d$ and the random variable X such that

$$\nabla_{L^2}\hat{U}(X) = \nabla_{\omega_2} U(\mu) \circ X.$$

In optimal transportation theory, this vector field is known to be the so–called Wasserstein gradient of U at μ (cf [2] and [30]). Under appropriate conditions,

$$\nabla_{\omega_2} U(\mu)(q) = \nabla_q \Big(\frac{\delta U}{\delta \mu}(q)\Big)$$

where $\frac{\delta U}{\delta \mu} : \mathbb{M} \mapsto \mathbb{R}$ is the weak Fréchet derivative of U at μ given in Definition 2.2 (Chapter 2).

Similarly, $\mathrm{Hess}(\hat{U})(X) : \mathbb{H}^2 \mapsto \mathbb{R}$ the hessian of $\hat{U}$ at X is linked to bilinear form

$$\mathrm{Hess}(U)(\mu) : \nabla C_c^\infty(\mathbb{R}^d) \times \nabla C_c^\infty(\mathbb{R}^d) \mapsto \mathbb{R},$$

the Wasserstein hessian of U at μ. The Wasserstein hessian consists of two parts, of which the first one could be termed the individual noises operator. It involves mixed derivatives in the (q, μ) variables:

$$\nabla_q \Big(\nabla_{\omega_2} U(q, \mu)\Big) \equiv \nabla_q \Big(\nabla_{\omega_2} U(\mu)(q)\Big) \quad \text{and} \quad \nabla^2_{\omega_2} U(q, x, \mu) \equiv \nabla^2_{\omega_2} U(\mu)(q, x).$$

These are used to define operators on set of functions of $\mathcal{P}_2(\mathbb{R}^d)$. For instance if $U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ then one of the partial trace operators which we introduce is O such that $O(U) : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is defined as

$$O(U)(\mu) := \int_{\mathbb{R}^d} \nabla_q \cdot \big(\nabla_{\omega_2} U(q, \mu)\big) \mu(dq).$$

The Wasserstein partial Laplacian or common noise operator is the partial trace

$$\sum_{i=1}^{d} \mathrm{Hess}(U)(\mu)(E_i, E_i) =: \triangle_{\omega_2} U(\mu)$$

where $\{E_1, \cdots, E_d\}$ is any orthonormal basis of $\mathbb{R}^d$. One defines operator B which involves only derivatives in μ, as

$$B(U) = \triangle_{\omega_2} U - O(U) = \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla^2_{\omega_2} U(q, x, \mu)\Big)\mu(dq)\mu(dx).$$

While a Fourier analysis shows O and $\triangle_{\omega_2}$ are degenerate elliptic operators, one checks that B is by no mean an elliptic operator.

Given a sufficiently smooth function $w : \mathbb{M} \times \mathbb{R}^d \mapsto \mathbb{R}$, we define

$$\mathcal{L}^{\epsilon_2}_{\epsilon_1}(w) := \mathcal{L}_{\epsilon_1}(w) + \mathcal{L}^{\epsilon_2}(w). \tag{1.1}$$

where

$$\mathcal{L}_{\epsilon_1}(w)(q, \mu) := \epsilon_1 \bigg(\triangle_q w(q, \mu) + \int_{\mathbb{M}} \mathrm{div}_x \Big(\nabla_{\omega_2} w(q, \mu)(x) \Big) \mu(dx) \bigg)$$

and

$$\mathcal{L}^{\epsilon_2}(w)(q, \mu) := \epsilon_2 \bigg(2 \int_{\mathbb{M}} \mathrm{div}_q \Big(\nabla_{\omega_2} u(q, x, \mu) \Big) \mu(dx) + \int_{\mathbb{M}^2} \mathrm{Tr}\Big(\nabla^2_{\omega_2} u(q, x, y, \mu) \Big) \mu(dx)\mu(dy) \bigg).$$

In Mean Field Games, there is an unavoidable non–local operator N_μ which depends on $\mu \in \mathcal{P}_2(\mathbb{M})$ and on a Hamiltonian $H \in C^1(\mathbb{M} \times \mathbb{R}^d)$ such that there is $C > 0$ such that

$$|D_p H(q, p)| \leq C(1 + |p|) \qquad \forall (q, p) \in Mb \times \mathbb{R}^d.$$

It associates to Borel vector fields $\xi, \zeta : \mathbb{M} \mapsto \mathbb{R}^d$ which are μ–square integrable, the real number

$$N_\mu\big[\xi, \zeta\big] := \int_{\mathbb{M}} \big\langle \xi(x), D_p H(x, \zeta(x)) \big\rangle \mu(dx).$$

Given $F, u_* \in C(\mathbb{M} \times \mathcal{P}_2(\mathbb{M})$ an important problem in Mean Field Games consists in finding

$$u : [0, T) \times \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

satisfying the so–called master equation

$$\partial_t u(t, q, \mu) + H\big(q, \nabla_{\omega_2} u(t, q, \mu)\big) + N_\mu\Big[\nabla_{\omega_2} u(t, q, \mu), \nabla_x u(t, \cdot, \mu)\Big] + F(q, \mu) = \mathcal{L}^{\epsilon_2}_{\epsilon_1}(u(t, \cdot, \cdot))(q, \mu)$$

along with the initial condition $u(0, \cdot, \cdot) = u_*$.

Assume there exists $\mathcal{F}, U_*, \beta_* : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ such that

$$F = \frac{\delta \mathcal{F}}{\delta \mu}, \qquad u_* = \frac{\delta U_*}{\delta \mu} + \beta_*.$$

We expect to find a solution of the master equation which will be of the form

$$u = \frac{\delta U}{\delta \mu} + \beta.$$

To achieve this goal, let us start with a function $U : [0,T) \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ viscosity solution to

$$\partial_t U(t,\mu) + \int_{\mathbb{M}} H(\mu, \nabla_{\omega_2} U(t,q,\mu))\mu(dq) + \mathcal{F}(\mu) = \epsilon_1 O(U)(\mu) + \epsilon_2 B(U)(\mu).$$

According to [30] when appropriate conditions are imposed on H, $\mathcal{F}$ and $U(0,\cdot)$ and $\epsilon_1 = \epsilon_2 = 0$, the viscosity solution U is uniquely determined. Smoothness properties of U were established in [41] if we further assume that T is small enough. For $\epsilon_1 > \epsilon_2 \geq 0$ when $U(0,\cdot)$ and F satisfy some monotonicity condition, it is shown in [12] that the Hamilton–Jacobi equation admits a unique smooth solution U. In any case, we identify a linear transport equation satisfied by a function $\beta : [0,T) \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ such that $u := \frac{\delta U}{\delta \mu} + \beta$ is a solution to the master equation.

Chapter 2

Start up

2.1 Lecture 1: Overview (Aug 23)

-Definition of Nash equilibria in games with a finite number of players.

- Examples of games with arbitrarily many players, including games with infinitely many players. The role of a system of PDEs in the study of Nash equilibria; lack of the right concept of solution. Limiting PDEs obtained as the number of players tend to ∞; Role of the space

$$\mathbb{H} := \lim_{n\to\infty} \mathbb{M}^n = L^2\big((0,1)^d, \mathbb{M}\big)$$

and explain how to circumvent the difficulty in working with such a space by working with

$$\mathcal{P}_2(\mathbb{M}) = \lim_{n\to\infty} \mathbb{M}^n / P_n$$

where P_n is the set of permutations of n letters. In fact, $\mathcal{P}_2(\mathbb{M})$ is the quotient of $\mathbb{H}$ by a relation we later specify.

-Review on the differential structure on the Wasserstein space and compare it to the differential structure on a quotient space of $\mathbb{H}$. List various concepts of differential of functions and their relations. As a by product, study the Hessian of functions.

-Review on measure theory, stochastic differential equations. Introduce the individual and common noises; Introduce the corresponding operators which involve what we term the partial Laplacian operator and the operator O. State the Feynman–Kac for both stochastic paths. Observe why the theory stands even for finitely many particles. Allude to polynomial functions on $\mathcal{P}_2(\mathbb{M})$ which combined with the partial Laplacian operator yield to Sobolev spaces on the set $\mathcal{P}_2(\mathbb{M})$.

- Lay down arguments favoring the fact that the master equation yields an infinite dimensional system of conservation law.

- Derive the master equation for a local Hamilton–Jacobi equation on the Wasserstein space when the running cost and the initial value function are weak Fréchet differential of functions defined on the set of measures.

- List existence results for the master equation when only one of the three important mechanisms are in force and also when combinations of them are in forces. For instance, we study the analogue of the linear heat equations involving the partial Wasserstein Laplacian and its perturbations.

2.2 Lecture 2: A differentiation on the set of measures (Aug 28)

We shall learn two related concepts of differential for functions $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ when $\mathbb{M} = \mathbb{T}^d, \mathbb{R}^d$. This will later be needed to be able to write the so–called *Master Equation*

2.2.1 A topology on the set of probability measures

In order to achieve the main goals of the next two lectures, we need to endow $\mathcal{P}_2(\mathbb{M})$ with a topology and introduce a concepts of differential of functions defined on $\mathcal{P}_2(\mathbb{M})$. We say that $(\mu_n)_n \subset \mathcal{P}_2(\mathbb{M})$ W_2–converges to $\mu \in \mathcal{P}_2(\mathbb{M})$ if

$$\lim_{n\to\infty} \int_{\mathbb{M}} f(q)\mu_n(dq) = \int_{\mathbb{M}} f(q)\mu(dq), \qquad \forall f \in C_2(\mathbb{M}).$$

We will later learn this is the Wasserstein topology which is induced by the so–called Wasserstein metric W_2.

Exercise 2.1. Let $E := [0,T] \times [0,T] \times [0,1]$. Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{M})$ and let $S : [0,T] \times \mathbb{M} \mapsto \mathbb{M}$ be a continuous maps such that there exists $C > 0$ such that

$$|S(t,q)| \leq C(1 + |q|), \qquad \forall (t,q) \in [0,T] \times \mathbb{M}. \tag{2.1}$$

(i) Show that
$$(t_1, t_2, s) \to (1-s)S(t_2,\cdot)_\sharp\mu_0 + sS(t_1,\cdot)_\sharp\mu_0 =: \sigma^{s,t_1,t_2}$$
is (W_2–)continuous on E.

(ii) Show and set $\mathcal{E} =: \{\sigma^{(t_1,t_2,s)} \mid (t_1,t_2,s) \in E\}$ is a compact subset of $\mathcal{P}_2(\mathbb{M})$.

(iii) Show that $s \to (1-s)\mu_0 + s\mu_1$ is continuous.

Proof: (i) We are going to use the formulation of the Wasserstein convergence as defined above (cf. [2]). Let $g \in C_2(\mathbb{M})$. We have

$$\int_{\mathbb{M}} g(z)\sigma^{(s,t_1,t_2)}(dz) = \int_{\mathbb{M}} \Big((1-s)g(S(t_2,z)) + sg\big(S(t_1,z)\big)\Big)\mu_0(dx). \tag{2.2}$$

Since $g \in C_2(\mathbb{M})$ there exists a constant C_0 such that if we use (2.1) we obtian

$$|g\big(S(t,z)\big)| \leq C_0(1 + |S(t,z)|)^2 \leq C_0\big(1 + C(1+|q|)\big)^2 \quad \forall (t,z) \in [0,T] \times \mathbb{M}.$$

We may apply the dominated convergence theorem to the expression at the right handside of (2.2) to conclude that

$$\lim_{(t_1,t_2,s)\to(\bar t_1,\bar t_2,\bar s)} \int_{\mathbb{M}} g(z)\sigma^{(s,t_1,t_2)}((dz) = \int_{\mathbb{M}}\Big((1-\bar s)g(S(\bar t_2,z))+sg\big(S(\bar t_1,z)\big)\Big)\mu_0(dz) = \int_{\mathbb{M}} g(z)\sigma^{(\bar t_1,\bar t_2,\bar s)}(dz)$$

for any $(\bar t_1, \bar t_2, \bar s) \in E$. This proves (i).

(ii) By (i) $\mathcal{E}$ is the image of the compact set E by a continuous function. Thus, $\mathcal{E}$ is itself a compact.

(iii) The proof of (ii) follows the same lines of arguments as that of (i). QED.

2.2.2 Frechet differential and intrinsic gradient on $\mathcal{P}_2(\mathbb{M})$

Definition 2.2. Let $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$.

(i) We say that U is weakly Fréchet continuously differentiable if there exists a continuous map $\omega : \mathbb{M}\times\mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ such that $\omega(\cdot,\mu) \in C_2(\mathbb{M})$ and

$$\lim_{s\to 0^+} \frac{U\big((1-s)\mu + s\mu'\big) - U(\mu)}{s} = \int_{\mathbb{M}} \omega(q,\mu)(\mu'-\mu)(dq) \qquad \forall \mu,\mu' \in \mathcal{P}_2(\mathbb{M}).$$

We set

$$\frac{\delta U}{\delta\mu}(q,\mu) = \omega(q,\mu) - \int_{\mathbb{M}} \omega(q',\mu)\mu(dq')$$

so that

$$\int_{\mathbb{M}} \frac{\delta U}{\delta\mu}(q,\mu)\mu(dq) = 0.$$

We call $\frac{\delta U}{\delta \mu}$ the weak Fréchet differential of U at μ.

(ii) If $q \to \frac{\delta U}{\delta\mu}(q,\mu)$ is differentiable and belongs to $L^2(\mu,\mathbb{R}^d)$, we define the intrinsic gradient of U and μ to be the map

$$q \to D_\mu U(q,\mu) := \nabla_q\bigg(\frac{\delta U}{\delta\mu}(q,\mu)\bigg).$$

(iii) Further assume there is a neighborhood $\mathcal{O}$ of μ such that $D_\mu U$ exists and is continuous on $\mathbb{M}\times\mathcal{O}$. Then we say U is intrinsically continuously differentiable near μ.

Exercise 2.3. Assume $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is weakly Fréchet continuously differentiable. Show that $\frac{\delta U}{\delta\mu}(\cdot,\mu)$ is uniquely determined.

Proof: It suffices to show that if

$$0 = \int_{\mathbb{M}} \omega(q,\mu)(\mu'-\mu)(dq) \qquad \forall \mu,\mu' \in \mathcal{P}_2(\mathbb{M}) \tag{2.3}$$

and $\int_{\mathbb{M}} \omega(q',\mu)\mu(dq') = 0$ then $\omega \equiv 0$. But, setting $\mu' = \delta_{q_0}$ in (2.3) we obtain

$$0 = \int_{\mathbb{M}} \omega(q,\mu)(\mu'-\mu)(dq) = \int_{\mathbb{M}} \omega(q,\mu)\mu'(dq) = \omega(q_0,\mu)$$

Since q_0 is arbitrary, the desired conclusions hold. QED.

Lemma 2.4. *Let $S \in C([0,1] \times \mathbb{M}; \mathbb{M})$ be such that $S(0,\cdot) = Id$, $\partial_t S$ exists, is continuous and $|\partial_t S|$ is bounded by a constant C_1 (note this implies there is a constant C such that (2.1) holds). Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{M})$ and set*

$$\mu_s := (1-s)\mu_0 + s\mu_1, \qquad \bar{\mu}_t := S(t,\cdot)_\sharp \mu_0 \qquad \forall (s,t) \in [0,1] \times [0,T].$$

let O be an open subset of $\mathcal{P}_2(\mathbb{M})$ that contains μ_s for any $s \in [0,1]$. Assume $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is weakly Fréchet continuously differentiable in $\mathcal{O}$ (cf. Definition 2.2 in Chapter 2). Assume for any compact set $\mathcal{K} \subset \mathcal{O}$ there is a constant $c_0(\mathcal{K}) > 0$ such that

$$\Big|\frac{\delta U}{\delta \mu}(q,\nu)\Big| \le c_0(\mathcal{K})(1+|q|^2) \qquad \forall \nu \in \mathcal{K}. \tag{2.4}$$

(i) We have

$$\lim_{h\to 0} \frac{U(\mu_{s+h}) - U(\mu_s)}{h} = \int_{\mathbb{M}} \frac{\delta U}{\delta \mu}(q,\mu_s)(\mu_1 - \mu_0)(dq)$$

(ii) As a consequence

$$U(\mu_1) - U(\mu_0) = \int_0^1 ds \int_{\mathbb{M}} \frac{\delta U}{\delta \mu}(q,\mu_s)(\mu_1 - \mu_0)(dq)$$

(iii) Further assume T is small enough, U is intrinsically continuously differentiable and for each compact set $\mathcal{K} \subset \mathcal{O}$ there is a constant $c(\mathcal{K})$ such that

$$\big|D_\mu U(q,\nu)\big| \le c(\mathcal{K})(1+|q|) \qquad \forall (q,\nu) \in \mathbb{M} \times \mathcal{K}. \tag{2.5}$$

Then $t \to U\big(\bar{\mu}_t\big)$ is differential on $[0,1]$ and

$$\frac{d}{dt} U\big(\bar{\mu}_t\big) = \int_{\mathbb{M}} \Big\langle D_\mu U\big(S(t,q), \bar{\mu}_t\big)\big); \partial_t S(t,q)\Big\rangle \mu_0(dq).$$

Proof: (i) Let $s \in [0,1)$ and assume $h > 0$ is small enough. Setting $t := h/(1-s)$, since $\mu_{s+h} = (1-t)\mu_s + t\mu_1$ we have

$$\lim_{h\to 0} \frac{U(\mu_{s+h} - U(\mu_s)}{h} = \frac{1}{1-s} \lim_{t\to 0} \frac{U\Big((1-t)\mu_s + t\mu_1\Big) - U(\mu_s)}{t} = \frac{1}{1-s} \int_{\mathbb{M}} \frac{\delta U}{\delta \mu}(q,\mu_s)(\mu_1 - \mu_s)(dq).$$

Since $\mu_1 - \mu_s = (1-s)(\mu_1 - \mu_0)$, this verifies (i).

(ii) By Exercise 2.1 $s \to \mu_s$ is a continuous map and so, the image of the compact set $[0,1]$ by the application $s \to \mu_s$ is a compact subset which we denote as $\mathcal{K}$. By (i), $s \to U\big(\mu_s\big)$ is differential on $(0,1)$ and its derivative is

$$\alpha(s) := \int_{\mathbb{M}} \frac{\delta U}{\delta \mu}(q,\mu_s)(\mu_1 - \mu_0)(dq).$$

Since $\frac{\delta U}{\delta \mu}$ is continuous, Exercise 2.1 implies $(s,q) \to \frac{\delta U}{\delta \mu}(q,\mu_s)$ is continuous. Thanks to (2.4), we may apply the dominated convergence theorem to conclude that α is continuous and for a constant $\bar{c}$

$$|\alpha| \le \bar{c}\Big(2 + \int_{\mathbb{M}} |q|^2 (\mu_0 + \mu_1)(dq)\Big) < \infty.$$

Thus, $t \to U\big(\bar\mu_t\big)$ is continuously differentiable and belongs to $W^{1,\infty}(0,1)$. By the fundamental theorem of calculus

$$U(\mu_1) - U(\mu_0) = \int_0^1 \alpha(s)ds,$$

which verifies (ii).

(iii) For $t \in [0,T]$ set

$$\Lambda(t) = U\big(\bar\mu_t\big).$$

Let σ^{s,t_1,t_2} and $\mathcal{E}$ be as in Exercise 2.1 and set

$$\mu_h^{st} := (1-s)\mu_t + s\mu_{t+h} = \sigma^{s,t,t+h}, \qquad t, t+h \in [0,T],\ s \in [0,1].$$

Since $W_2^2(\cdot,\mu_0)$ is convex (cf. [2]) and $S(t,\cdot)_\sharp\mu_0 = \bar\mu_t$ for $t \in \{t_1,t_2\}$ we use the fact that

$$|q - S(t,q)|^2 = |S(0,q) - S(t,q)|^2 \le C_1^2 t^2$$

to conclude

$$W_2^2\big(\sigma^{s,t_1,t_2},\mu_0\big) \le (1-s)W_2^2\big(\mu_{t_2},\mu_0\big) + sW_2^2\big(\mu_{t_1},\mu_0\big) \le (1-s)C_1^2T^2 + sC_1^2T^2 = C_1^2T^2.$$

Hence, for T is small enough we have

$$\mathcal{E} \subset \mathcal{O}. \tag{2.6}$$

Thus, (2.4) holds when $\mathcal{K}$ is replaced by $\mathcal{E}$. We apply (ii) to obtain

$$\begin{aligned}\Lambda(t+h) - \Lambda(t) &= \int_0^1 ds \int_{\mathbb{M}} \frac{\delta U}{\delta\mu}\Big(q,\mu_h^{st}\Big)\big)(\mu_{t+h} - \mu_t)(dq)\\ &= \int_0^1 ds \int_{\mathbb{M}} \bigg(\frac{\delta U}{\delta\mu}\Big(S(t+h,q),\mu_h^{st}\Big) - \frac{\delta U}{\delta\mu}\Big(S(t,q),\mu_h^{st}\Big)\bigg)\mu_0(dq)\end{aligned}$$

We use the mean value theorem to obtain $\theta \equiv \theta(t,s,q,h) \in [0,1]$ such that

$$\begin{aligned}&\Lambda(t+h) - \Lambda(t)\\ &= \int_0^1 ds \int_{\mathbb{M}} \Big\langle \nabla_q\Big(\frac{\delta U}{\delta\mu}\Big)\Big((1-\theta)S(t+h,q) + \theta S(t,q),\mu_h^{st}\Big); S(t+h,q) - S(t,q)\Big\rangle\mu_0(dq)\end{aligned} \tag{2.7}$$

By (2.5) (with $\mathcal{K} = \mathcal{E}$) and (2.6)

$$\Big|D_\mu U\Big((1-\theta)S(t+h,q) + \theta S(t,q),\mu_h^{st}\Big)\Big| \le c(\mathcal{E})\Big(1 + \Big|(1-\theta)S(t+h,q) + \theta S(t,q)\Big|\Big)$$

This, together with (2.1) implies

$$\Big|D_\mu U\Big((1-\theta)S(t+h,q) + \theta S(t,q),\mu_h^{st}\Big)\Big| \le c(\mathcal{E})\Big(1 + C(1+|q|)\Big)$$

We apply the Lebesgue dominated convergence theorem to the integral in (2.7) to obtain

$$\lim_{h\to 0}\frac{\Lambda(t+h) - \Lambda(t)}{h} = \int_0^1 ds \int_{\mathbb{M}} \Big\langle D_\mu\Big(S(t,q),\bar\mu_t\Big); \partial_t S(t,q)\Big\rangle\mu_0(dq).$$

Since the expression in the latter integral is independent of s, this concludes the proof. QED.

Remark 2.5. The conclusions in Lemma 2.4 continue to hold under the following weaker assumptions: U is continuously intrinsically differentiable, (2.4) holds and there is a constant C_0 such that

$$|\partial_t S(t,q)| \leq C_0(1+|q|) \qquad \forall (t,q) \in [0,T] \times \mathbb{M}.$$

Exercise 2.6. Let $\mathcal{O} \subset \mathcal{P}_2(\mathbb{M})$ be an open set and let $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ be an intrinsically continuously differentiable function on $\mathcal{O}$. Show that if (2.5) holds on for a compact set $\mathcal{K} \subset \mathcal{O}$ the there is a constant $\bar{c}(\mathcal{K})$ such that (2.4) holds.

Proof: Note

$$\left|\frac{\delta U}{\delta \mu}(q,\nu)\right| \leq \left|\frac{\delta U}{\delta \mu}(0,\nu)\right| + |q| c(\mathcal{K})(1+|q|)$$

Since $\mathcal{K}$ is a compact set and $\frac{\delta U}{\delta \mu}(0,\cdot)$ is a continuous function on $\mathcal{K}$, it is bounded by a constant which depends on $\mathcal{K}$ but is independent of $\nu \in \mathcal{K}$. This is enough to establish the desired result. QED.

2.3 Lecture 3, 4: PDEs methods for Nash equilibria (Aug 30, Sep 04).

The goal of this lecture is two folds.

(i) First, we plan to show how partial differential equations (PDEs) can be used to find Nash equilibria for players whose trajectories lie in $\mathbb{M}$.

(ii) Second, we show how our approach favors the prediction of Robert Aumann that in games formulated in certain contexts, Nash equilibria should not be expected to exist in deterministic games comprising only finitely many players.

The distribution of our players will be Borel probability measures which we require to satisfy some additional conditions such as having finite second moments.

2.3.1 Description of a dynamical deterministic game

The goal of this subsection is to make formal calculations which enlighten us on how to use PDEs methods to find Nash equilibria. The aim is to develop our intuition making often assumptions much stronger than what is needed. The computations in this section are done for a special Hamiltonian and will later be extended to a larger class of Hamiltonians.

Let

$$F, G : \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

be bounded continuous functions such that $\nabla_q F$ and $\nabla_q G$ exists, are continuous and uniformly bounded. The function G represents an initial cost function in a deterministic game and F represents a running cost.

Fix $T > 0$ and let $\mu \in \mathcal{P}_2(\mathbb{M})$ represents the distributions of a group of players at time T. Each player is identified with its position $q \in \mathbb{M}$ at time T. A collective control is

$$\alpha \in L^2\big((0,T) \times \mathbb{M}; \mathbb{R}^d\big)$$

so that $\alpha(\cdot, q) : [0,T] \mapsto \mathbb{R}^d$ is the control of player q. The control set for player q is

$$\Lambda_q := L^2(0, s; \mathbb{R}^d),$$

The rule of the game is to determine the trajectory $Q : [0,T] \mapsto \mathbb{M}$ as the unique solution to the equation

$$\alpha(t,q) = \dot{Q}, \quad Q(T) = q. \tag{2.8}$$

The trajectories of all players can be recorded in a function

$$S : [0,T] \times \mathbb{M} \to \mathbb{M}, \qquad S(T,\cdot) = Id, \qquad \partial_t S = \alpha.$$

By (2.8), there is a one–to–one correspondence between the set of α and the set of S. Indeed, S is uniquely given as

$$S(t,q) = q + \int_0^t \alpha(\tau, q) d\tau. \tag{2.9}$$

The cost function for player q is

$$J_q(T; \alpha_q, \alpha) := \int_0^T \left(\frac{|\alpha_q|^2}{2} - F\big(Q(t), S(t,\cdot)_\sharp \mu\big) \right) dt + F\big(Q(0), S(0,\cdot)_\sharp \mu\big). \tag{2.10}$$

Although the expressions in (2.8) still mathematically makes sense even if α_q and α are not related, let us recall that in our context, we require them to satisfy the consistency condition

$$\alpha_q = \alpha(\cdot, q) \qquad \forall q \in \mathbb{M}.$$

Assume all the players keep their collective control and only player q changes his strategy into $A \in L^2(0, s; \mathbb{R}^d)$. Then the new collective strategies of the players will be

$$\alpha^A(t,q') = \begin{cases} \alpha(t,q') & \text{if} \quad q' \neq q \\ A(t) & \text{if} \quad q' \neq q. \end{cases} \tag{2.11}$$

We say that $\alpha \in L^2\big((0,T) \times \mathbb{M}; \mathbb{R}^d\big)$ is a Nash equilibrium for the family of pay off functions $(J_q)_{q \in \mathbb{M}}$ if for any $q \in \mathbb{M}$ and any $A \in \Lambda_q$

$$J_q\Big(T; \alpha(\cdot, q), \alpha\Big) \leq J_q\Big(T; A, \alpha^A\Big), \qquad \forall A \in \Lambda_q.$$

2.3.2 Sufficient conditions for Nash equilibria in a dynamic deterministic game

Definition 2.7. Let $H(q,p) \equiv |p|^2/2$. We say that $u : [0,T] \times \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is a continuously differentiable solution of the master equation for the game (H,F), with bounded first order derivatives if the following hold:

(i) $\partial_t u$ and $\nabla_q u$ exist and are both continuous and bounded.

(ii) u is intrinsically continuously differential for each $(t,q) \in [0,T] \times \mathbb{M}$ and $D_\mu u$ is bounded.

(iii)

$$\partial_t u(t,q,\mu) + \Big\langle D_\mu u(t,q,\mu); \nabla_q u(t,\cdot,\mu) \Big\rangle_\mu + \frac{1}{2} \Big| \nabla_q u(t,q,\mu) \Big| + F(q,\mu) = 0$$

for any $t \in (0,T)$, $q \in \mathbb{M}$ and $\mu \in \mathcal{P}_2(\mathbb{M})$.

We need the following definition which characterizes measures for which we can verify R. Aumann predictions.

Definition 2.8. Given a topological space $\mathcal{S}$ and a Borel measure μ on $\mathcal{S}$, a Borel set B is called an atom for μ if, $\mu(B) > 0$ and given any Borel set $C \subset B$, we have either $\mu(C) = 0$ or $\mu(B \setminus C) = 0$.

For example, if $x \in \mathcal{S}$ is such that $\mu(\{x\}) > 0$ and N is a Borel set such that $\mu(N) = 0$ then $B = N \cup \{x\}$ is an atom. According to Lemma 14 page 408 [46] if $\mathcal{S}$ is a separable metric space, μ is a Borel measure and B is an atom for μ then there exists $x \in B$ such that $\mu(B \setminus \{x\}) = 0$. In particular if $\mu \in \mathcal{P}_2(\mathbb{M})$ is such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{M}$ then μ has no atoms.

Lemma 2.9. *Let $H(q,p) \equiv |p|^2/2$ and let $u : [0,T] \times \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ be a continuously differentiable function. Assume μ has no atoms and u is a continuously differentiable solution of the master equation for the game (H,F). Assume the following differential equation*

$$\partial_t S(t,q) = \nabla_q u\Big(t, S(t,q), S(t,\cdot)_\sharp \mu\Big), \qquad S(T,\cdot) = Id \tag{2.12}$$

admits a solution (a sufficient condition for that it that $\nabla_q u(t,q,\cdot)$ be Lipschitz (we refer the reader to Section 3 of [26] for an analogous result on the Wasserstein space). Then the control

$$\alpha(t,q) := \nabla_q u\Big(t, S(t,q), S(t,\cdot)_\sharp \mu\Big)$$

is a Nash equilibrium for the family of pay off functions $(J_q)_{q\in\mathbb{M}}$. In fact

$$J_q\big(T, \alpha(\cdot,q), \alpha\big) < J_q\big(T, \dot{Q}, \alpha^{\dot{Q}}\big)$$

if $Q(T) = q$, unless $\dot{Q}(t) = \nabla_q u\big(t, Q(t), \sigma_t\big)$.

Proof: Note α is an admissible control since $\alpha \in L^2\big((0,T)\times\mathbb{M};\mathbb{R}^d\big)$.

Fix $q \in \mathbb{M}$ and let $Q : [0,T] \to \mathbb{M}$ be a continuous path such that $Q(T) = q$ and $A := \dot{Q} \in L^2\big((0,T);\mathbb{R}^d\big)$, which means A is a control for player q. To alleviate notation, let us set

$$\sigma_t := S(t,\cdot)_\sharp \mu.$$

The collective strategies when only q changes its control are given by

$$S^A(t,q') = \begin{cases} S(t,q') & \text{if} \quad q' \neq q \\ Q(t) & \text{if} \quad q' \neq q. \end{cases}$$

Let $\sigma_t^A := S^A(t,\cdot)_\sharp \mu$ denote the new distribution of the players. For any $\varphi \in C_2(\mathbb{M})$, since μ has no atoms, then $\mu(\{q\}) = 0$ and so,

$$\int_{\mathbb{M}} \varphi\big(S^A(t,q')\big)\mu(dq') = \int_{\mathbb{M}\setminus\{q\}} \varphi\big(S^A(t,q')\big)\mu(dq') = \int_{\mathbb{M}\setminus\{q\}} \varphi\big(S(t,q')\big)\mu(dq') = \int_{\mathbb{M}} \varphi\big(S(t,q')\big)\mu(dq').$$

In other words, for any $t \in [0,T]$ we have $\sigma_t = \sigma_t^A$. Thus,

$$\frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t^A\big) - J_q\big(t,A,\alpha^A\big)\Big) = \frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t\big) - J_q\big(t,A,\alpha\big)\Big).$$

Claim 1. We claim that

$$\frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t^A\big) - J_q\big(t,A,\alpha^A\big)\Big) < 0 \tag{2.13}$$

unless $\dot{Q}(t) = \nabla_q u\big(t,Q(t),\sigma_t\big)$ in which case equality holds.

Proof of Claim 1. We use Lemma 2.4 (ii) to infer

$$\begin{aligned}\frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t^A\big) - J_q\big(t,A,\alpha^A\big)\Big) &= \frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t\big) - J_q\big(t,A,\alpha\big)\Big)\\ &= \partial_t u\big(t,Q(t),\sigma_t\big) + \langle \nabla_q u\big(t,Q(t),\sigma_t\big); \dot{Q}\rangle\\ &+ \int_{\mathbb{M}} \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big)\big(S(t,q')\big); \partial_t S(t,q')\Big\rangle \mu_0(dq')\\ &- \frac{|\dot{Q}(t)|^2}{2} + F\Big(Q(t),\sigma_t\Big)\end{aligned} \tag{2.14}$$

But by (2.12)

$$\begin{aligned}&\int_{\mathbb{M}} \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big)\big(S(t,q')\big); \partial_t S(t,q')\Big\rangle \mu_0(dq')\\ &= \int_{\mathbb{M}} \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big)\big(S(t,q')\big); \nabla_q u\big(t,S(t,q'),\sigma_t\big)\Big\rangle \mu_0(dq')\\ &= \int_{\mathbb{M}} \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big)(x); \nabla_q u\big(t,x,\sigma_t\big)\Big\rangle \sigma_t(dx)\\ &= \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big); \nabla_q u\big(t,\cdot,\sigma_t\big)\Big\rangle_{\sigma_t}.\end{aligned}$$

This, together with (2.14) implies, after completing the square,

$$\begin{aligned}\frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t^A\big) - J_q\big(t,A,\alpha^A\big)\Big) &= \partial_t u\big(t,Q(t),\sigma_t\big) + \frac{\Big|\nabla_q u\big(t,Q(t),\sigma_t\big)\Big|^2}{2} +\\ &+ \Big\langle D_\mu u\big(t,Q(t),\sigma_t)\big); \nabla_q u\Big(t,\cdot,\sigma_t\Big)\Big\rangle_{\sigma_t}\\ &- \frac{\Big|\dot{Q}(t) - \nabla_q u\big(t,Q(t),\sigma_t\big)\Big|^2}{2} + F\Big(Q(t),\sigma_t\Big).\end{aligned}$$

Recall that u satisfies the master equation and so,

$$\frac{d}{dt}\Big(u\big(t,Q(t),\sigma_t^A\big) - J_q\big(t,A,\alpha^A\big)\Big) = -\frac{\Big|\dot{Q}(t) - \nabla_q u\big(t,Q(t),\sigma_t\big)\Big|^2}{2}.$$

This verifies (2.13).

Concluding step of the proof of Lemma 2.9. By (2.13)

$$u\big(T, S(T,q), \sigma_T\big) - u\big(0, S(0,q), \sigma_0\big) - J_q\big(T, \alpha(\cdot, q), \alpha\big) + J_q\big(0, \alpha(\cdot, q), \alpha\big) = 0, \tag{2.15}$$

and if $\dot{Q}(t) \not\equiv \nabla_q u\big(t, Q(t), \sigma_t\big)$ we have

$$u\big(T, Q(T), \sigma_T^A\big) - u\big(0, Q(0), \sigma_0^A\big) - J_q\big(T, A, \alpha^A\big) + J_q\big(0, A, \alpha^A\big) < 0. \tag{2.16}$$

We use the facts that

$$\sigma_T = \mu, \quad \sigma_t = \sigma_t^A, \quad Q(T) = q = S(T,q) \quad \text{and} \quad J(0, A, \alpha^A) = G(Q(0), \sigma_0^A)$$

in (2.15)-(2.16) to obtain

$$u\big(T, q, \mu\big) = u\big(0, S(0,q), \sigma_0\big) + J_q\big(T, \alpha(\cdot, q), \alpha\big) - J_q\big(0, \alpha(\cdot, q), \alpha\big) = J_q\big(T, \alpha(\cdot, q), \alpha\big) \tag{2.17}$$

and if $\dot{Q}(t) \not\equiv \nabla_q u\big(t, Q(t), \sigma_t\big)$ then

$$u\big(T, q, \mu\big) < u\big(0, Q(0), \sigma_0\big) + J_q\big(T, A, \alpha\big) - J_q\big(0, A, \alpha\big) = J_q\big(T, A, \alpha\big). \tag{2.18}$$

We use (2.17) and (2.18) to verify the remaining claims of the Lemma.

Chapter 3

Preliminary: tools from optimal transport theory

3.1 Lecture 5, 6: Probability spaces; Wasserstein metric (Sep 08, 11)

The games we have been describing so far have been very restrictive from two points of view. Not only we have been dealing with a very special Hamiltonian, in our games, the trajectories of the players have been deterministic. To remedy the later point, we need to briefly introduce some concepts from probability theory and stochastic analysis. We also need to endow the set of probability measures with a metric, the so–called Wasserstein distance.

Throughout this section, $\mathbb{B}_b$ denote the open ball in $\mathbb{R}^d$, of volume 1, centered at the origin and I_d is the identity matrix on $\mathbb{R}^d$.

If X is a non–empty set then 2^X is a σ–algebra on X. The smallest σ–algebra containing a subset $\mathcal{A}$ of X is called the σ–algebra generated by a set $\mathcal{A}$. A triple $(\mathcal{S}, \Sigma, \mu)$ is a probability space if Σ be a σ–algebra on $\mathcal{S}$ and μ is a positive countably additive function on Σ such that $\mu(\mathcal{S}) = 1$.

Let $\mathcal{S}$ and $\mathcal{S}_*$ be topological spaces. The σ–algebra generated by the Borel subsets of $\mathcal{S}$ is called the Borel σ–algebra and is denoted as $\mathcal{B}_{\mathcal{S}}$. The set of Borel probability measure on $\mathcal{S}$ is denoted as $\mathcal{P}(\mathcal{S})$. If $\mu \in \mathcal{P}(\mathcal{S})$, any Borel map $X : \mathcal{S} \mapsto \mathcal{S}_*$ induces a Borel measure $X_\sharp \mu$ on $\mathcal{S}_*$ defined as

$$\big(X_\sharp \mu\big)(B) = \mu\big(X^{-1}(B)\big)$$

for any B Borel subset of $\mathcal{S}_*$ (Exercise 3.1.3).

Let $\mathcal{S}$ and $\mathcal{T}$ be a topological spaces, let $\pi^{\mathcal{S}}$ be the projection of $\mathcal{S} \times \mathcal{T}$ onto $\mathcal{S}$ and let $\pi^{\mathcal{T}}$ be the projection of $\mathcal{S} \times \mathcal{T}$ onto $\mathcal{T}$. Given $\gamma \in \mathcal{P}(\mathcal{S} \times \mathcal{T})$, we call $\pi^{\mathcal{S}}_\sharp \gamma$, the first marginal of γ and called $\pi^{\mathcal{T}}_\sharp \gamma$ the second marginal of γ. If $\mu \in \mathcal{P}(\mathcal{S})$ and $\nu \in \mathcal{P}(\mathcal{T})$, we denote as $\Gamma(\mu, \nu)$ the set of $\gamma \in \mathcal{P}(\mathcal{S} \times \mathcal{T})$ which have μ as first marginal and ν as second marginal. Note $\gamma \in \Gamma(\mu, \nu)$ is equivalent to

$$\gamma(A \times \mathcal{T}) = \mu(A), \quad \gamma(\mathcal{S} \times B) = \nu(B) \qquad \forall (A, B) \in \mathcal{B}_{\mathcal{S}} \times \mathcal{B}_{\mathcal{T}}. \tag{3.1}$$

Let $(\mathcal{S}, \mathrm{dist}\,)$ be a metric space, let $s_0 \in \mathcal{S}$ and let $p \in [1, \infty)$. The following set is independent of s_0 :

$$\mathcal{P}_p(\mathcal{S}) := \left\{ \mu \in \mathcal{P}(\mathcal{S}) \;\Big|\; \int_{\mathcal{S}} \operatorname{dist}^p(s, s_0)\mu(ds) < \infty \right\}$$

Let Σ be a symmetric and non–negative real $d \times d$ matrix and let $u \in \mathbb{R}^d$. The Gaussian d–dimensional Gaussian (or normal distribution) measure on $\mathbb{R}^d$, with mean u and covariance matrix U denoted as $N_d(u, U)$ is defined as

$$N_d(u, U)(B) = \frac{1}{\sqrt{(2\pi)^d \det U}} \int_B e^{-\frac{1}{2}\left\langle x-u, U^{-1}(x-u)\right\rangle} dx$$

Given a positive number b and denoting as $\vec{0}$ the null vector in $\mathbb{R}^d$, we write $N_d(0, b)$ in place of $N_d(\vec{0}, bI_d)$.

3.1.1 Basic results in measure theory

Theorem 3.1. *Let $G_{\mathbb{B}_d}$ be the set of Borel maps $S : \mathbb{B}_d \to \mathbb{B}_d$ which have a Borel inverse S^{-1} and such that both S and S^{-1} preserving Lebesgue measure. Let $p \in [1, \infty)$ and let $X, Y \in \mathbb{H}_p$. Then $X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = Y_\sharp \mathcal{L}^d_{\mathbb{B}_d}$ if and only if there exists a sequence $(S_n)_n \subset G_{\mathbb{B}_d}$ such that*

$$\lim_{n\to\infty} \int_{\mathbb{B}_d} \|X(\omega) - Y(\omega) \circ S_n\|^p d\omega = 0.$$

Proof: Cf. e.g. Lemma 6.4 [11]. QED.

Definition 3.2. Let $\mathcal{S}$ be a separable metric space and let $\mathcal{P}(\mathcal{S})$ denote the set of Borel probability measures on $\mathcal{S}$. Let $\mu \in \mathcal{P}(\mathcal{S})$. We say that a sequence $(\mu_n)_n \subset \mathcal{P}(\mathcal{S})$ converges narrowly to μ if for every bounded continuous function $f : \mathcal{S} \mapsto \mathbb{R}$ we have

$$\lim_{n\to\infty} \int_{\mathcal{S}} f(s)\mu_n(ds) = \int_{\mathcal{S}} f(s)\mu(ds)$$

The next classical theorem characterizes relatively compact sets of $\mathcal{P}(\mathcal{S})$ with respect to the narrow convergence topology.

Theorem 3.3 (Prokhorov)**.** *Let $\mathcal{S}$ be a complete separable metric space and let $\mathcal{K} \subset \mathcal{P}(\mathcal{S})$. The following are equivalent:*

(i) $\mathcal{K}$ is relatively compact for the narrow topology.

(ii) There exists $\varphi : \mathcal{S} \mapsto [0, \infty]$ such that for each $c \geq 0$ the sublevel set $\{s \in \mathcal{S} \mid s \in \mathcal{S}\}$ is compact and

$$\sup_{\mu\in\mathcal{K}} \int_{\mathcal{S}} \varphi(s)\mu(ds) < \infty$$

(iii) $\mathcal{K}$ is tight, meaning for every $\epsilon > 0$ there exists $K_\epsilon \subset \mathcal{S}$ compact such that

$$\mu(\mathcal{S} \setminus K_\epsilon) \leq \epsilon \qquad \forall \mu \in \mathcal{K}.$$

Proof: Cf. e.g. [20]. QED.

Corollary 3.4. *If $\mathcal{S}$ and $\mathcal{S}_*$ are complete separable metric spaces and $\mu \in \mathcal{P}(\mathcal{S})$ and $\nu \in \mathcal{P}(\mathcal{S}_*)$, then $\Gamma(\mu, \nu)$ is compact for the narrow convergence.*

Proof: We are to show that $\Gamma(\mu, \nu)$ is pre–compact and close for the narrow convergence topology.

(i) Given $\epsilon > 0$, by Theorem 3.3, there are two compact sets $K_\epsilon \subset \mathcal{S}$ and $L_\epsilon \subset \mathcal{S}_*$ such that

$$\mu(\mathcal{S} \setminus K_\epsilon) \le \epsilon, \quad \nu(\mathcal{S}_* \setminus L_\epsilon) \le \epsilon.$$

Note that $K_\epsilon \times L_\epsilon$ is a compact subset of $\mathcal{S} \times \mathcal{S}_*$ and

$$\mathcal{S} \times \mathcal{S}_* \setminus (K_\epsilon \times L_\epsilon) \subset \big(\mathcal{S} \times (\mathcal{S}_* \setminus L_\epsilon)\big) \cup \big(\mathcal{S} \setminus K_\epsilon \times \mathcal{S}_*\big).$$

Thus for any $\gamma \in \Gamma(\mu, \nu)$, using (3.1), we obtain

$$\gamma\Big(\mathcal{S} \times \mathcal{S}_* \setminus (K_\epsilon \times L_\epsilon)\Big) \le \gamma\big(\mathcal{S} \times (\mathcal{S}_* \setminus L_\epsilon)\big) + \gamma\big(\mathcal{S} \setminus K_\epsilon \times \mathcal{S}_*\big) = \nu(\mathcal{S}_* \setminus L_\epsilon) + \mu(\mathcal{S} \setminus K_\epsilon) \le 2\epsilon.$$

Thanks to Theorem 3.3, we verify that $\Gamma(\mu, \nu)$ is pre–compact.

(ii) Let $(\gamma_n)_n \subset \Gamma(\mu, \nu)$ be a sequence that converges narrowly to γ. Since the projection $\pi^{\mathcal{S}} : \mathcal{S} \times \mathcal{S}_* \mapsto \mathcal{S}$ is continuous, $(\pi^{\mathcal{S}}_\sharp \gamma_n)_n$ converges to $\pi^{\mathcal{S}}_\sharp \gamma$ (cf. e.g. [2]). This verifies that $\mu = \pi^{\mathcal{S}}_\sharp \gamma$. Similarly, $\nu = \pi^{\mathcal{S}_*}_\sharp \gamma$, which proves that $\gamma \in \Gamma(\mu, \nu)$. QED.

We continue this section with a series of definitions followed by useful results.

Definition 3.5. Let $(\mathcal{S}, \Sigma, \mu)$ and $(\mathcal{S}_*, \Sigma_*, \mu_*)$ be measure spaces. We call $X : \mathcal{S} \mapsto \mathcal{S}_*$ a measurable function if $X^{-1}(B) \in \Sigma$ for each $B \in \Sigma_*$. Further assume that $\mathcal{S}$ and $\mathcal{S}_*$ are topological spaces and $\Sigma = \mathcal{B}_{\mathcal{S}}$ and $\Sigma_* = \mathcal{B}_{\mathcal{S}_*}$.

(i) any measurable function is called a Borel map.

(ii) If in addition $\mathcal{S}_* = \mathbb{R}^d$, we call any measurable function a d–dimensional Radon variable (or simply a Radon variable).

(iii) Any collection $\{X_t \mid t \ge 0\}$ of Random variables is called a stochastic process.

(iv) If $X : \mathcal{S} \to \mathbb{R}^d$ is a d–dimensional Radon variable, we call $X_\sharp \mu$ the law of X.

The following deep results from measure theory (cf. e.g.Theorem 16 page 409 [46]), provide a complete list of finite Borel probability measures on complete separable metric space.

Theorem 3.6. *Let μ be a probability Borel measure on a complete separable metric space $\mathcal{S}$. Then the following hold:*

(i) there exist $I \subset \mathbb{N}$, $E := \{e_i \mid i \in I\} \subset (1, 2)$, $\alpha_0 \ge 0$ and $\{\alpha_i \mid i \in I\} \subset [0, \infty]$ such that $(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \mu)$ is isomorphic to $(\mathcal{S}_*, \mathcal{B}_{\mathcal{S}_*}, \mu_*)$ where

$$\mathcal{S}_* = [0, 1] \cup E, \quad \mu_* = \alpha_0 \mathcal{L}^1_{[0,1]} + \sum_{i \in I} \alpha_i \delta_{e_i}.$$

(ii) If we further assume that $\mathcal{S}$ is uncountable and μ has no atoms then $\alpha_0 = 1$ and $I = \emptyset$. This means that $(\mathcal{S}, \mathcal{B}_{\mathcal{S}}, \mu)$ is isomorphic to the one dimensional Lebesgue measure space $([0,1], \mathcal{B}_{[0,1]}, \mathcal{L}^1_{[0,1]})$.

Corollary 3.7. *Let μ_* be a probability Borel measure on a complete separable metric space $\mathcal{S}_*$. Then there exists a Borel map $X : \mathbb{B}_d \mapsto \mathcal{S}_*$ such that $X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = \mu_*$.*

Proof: We may assume without loss $(\mathcal{S}_*, \mu_*)$ is as in Theorem 3.6. Note

$$\alpha_0 + \sum_{i \in I} \alpha_i = 1.$$

We skip the trivial case when $\alpha_0 = 0$ or $I = \emptyset$. Theorem 3.6 (i) gives an isomorphism Z between $([0, \alpha_0], \mathcal{L}^1_{[0,\alpha_0]})$ and $([0,1], \alpha_0 \mathcal{L}^1_{[0,1]})$ and such that

$$Z_\sharp \mathcal{L}^1_{[0,\alpha_0]} = \alpha_0 \mathcal{L}^1_{[0,1]}.$$

We readily construct a Borel map $Y : (\alpha_0, 1] \to E$ such that

$$Y_\sharp \mathcal{L}^1_{(\alpha_0,1]} = \sum_{i \in I} \alpha_i \delta_{e_i}.$$

We combine Y and Z to obtain a Borel map

$$X^* : [0,1] \mapsto [0,1] \cup E$$

such that

$$X^*_\sharp \mathcal{L}^1_{[0,1]} = \mu_*.$$

We use Theorem 3.6 (ii) to obtain an isomorphism W of $(\mathbb{B}_d, \mathcal{B}_{\mathbb{B}_d}, \mathcal{L}^d_{\mathbb{B}_d})$ onto $([0,1], \mathcal{B}_{[0,1]}, \mathcal{L}^1_{[0,1]})$. Setting $X := W \circ X^*$, we obtain the desired result. QED.

3.1.2 Wasserstein space viewed as a quotient space

Throughout this subsection, $(\mathcal{S}, \langle \cdot, \cdot \rangle)$ is a complete separable Hilbert space. Let $\| \cdot \|$ denote the associated norm and let dist denote the associated metric. Given $p \in [1, \infty)$ and using the notation at the beginning of the Section, when $\mu, \nu \in \mathcal{P}_p(\mathcal{S})$, to define

$$W_p(\mu, \nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathcal{S} \times \mathcal{S}} \|x - y\|^p \gamma(dx, dy) \right)^{1/p}. \tag{3.2}$$

The p–moment of μ is $M_p[\mu] \geq 0$ where

$$M_p^p[\mu] := \int_{\mathcal{S}} \|x\|^p \mu(dx).$$

Let

$$\mathbb{H}_p := L^p(\mathbb{B}_b; \mathcal{S}, \mathcal{L}^d_{\mathbb{B}_d}), \quad \mathbb{H} := L^2(\mathbb{B}_b; \mathcal{S}, \mathcal{L}^d_{\mathbb{B}_d})$$

be endowed with the standard norms so that (because $\mathcal{L}^d_{\mathbb{B}_d}$ is σ–finite) $\mathbb{H}_p$ is a normed space and $\mathbb{H}$ is a Hilbert space (cf. e.g. [21]).

Let $G_{\mathbb{B}_d}$ denote the set of Borel maps $S : \mathbb{B}_d \mapsto \mathbb{B}_d$ which have a Borel inverse $S^{-1} : \mathbb{B}_d \mapsto \mathbb{B}_d$ such that both S and S^{-1} preserving Lebesgue measures. This is a non communative group when endowed with $\circ$, the composition operator. When $d \geq 2$, the closure of $G_{\mathbb{B}_d}$ in $L^2(\mathbb{B}_d; \mathbb{B}_d)$ is $S_{\mathbb{B}_d}$ of all measure preserving map on $\mathbb{B}_d$ (cf. e.g. Theorem 1.4 [7]).

Corollary 3.8. *The push forward operator $\sharp$, which associates to a Borel map $X : \Omega \to \mathcal{S}$– the measure $X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$– maps $\mathbb{H}_p$ onto $\mathcal{P}_p(\mathcal{S})$. It induces an equivalence relation, where, the class of equivalence of $X \in \mathbb{H}_p$ is*

$$[X]_p = \{\bar{X} \in \mathbb{H}_p \mid X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = \bar{X}_\sharp \mathcal{L}^d_{\mathbb{B}_d}\}.$$

Proof: By Corollary 3.7, any $\mu \in \mathcal{P}_p(\mathcal{S})$ there is a Borel map $X : \Omega \mapsto \mathcal{S}$ such that $X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = \mu$. We have

$$M_p^p[\mu] = \int_{\mathbb{B}_d} \|X(\omega)\|^p d\omega < \infty$$

and so, $\mu \in \mathcal{P}_p(\mathcal{S})$. This concludes the proof of the corollary. QED.

Theorem 3.9. *The following hold:*

(i) the infimum in (3.2) *is finite and a minimizer γ_0 is achieved there.*

(ii) We have

$$W_p^p(\mu, \nu) = \inf_{X,Y \in \mathbb{H}_p} \Big\{ \big\|X - Y\big\|^p_{L^p(\mathbb{B}_b)} \ : \ \mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}, \ \nu = Y_\sharp \mathcal{L}^d_{\mathbb{B}_d} \Big\}. \tag{3.3}$$

and a minimizer is achieved in (3.3).

Proof: (i). Let $\gamma := \mu \otimes \nu$ be the product of μ by ν. Since

$$\int_{\mathcal{S}\times\mathcal{S}} \|x - y\|^p d\gamma(x, y) \leq 2^{p-1} \Big(\int_{\mathcal{S}\times\mathcal{S}} \|x\|^p \gamma(dx, dy) + \int_{\mathcal{S}\times\mathcal{S}} \|y\|^p \gamma(dx, dy) \Big)$$

and γ is arbitrary that has μ and ν as marginals, we conclude

$$W_p^p(\mu, \nu) \leq 2^{p-1} \Big(\int_{\mathcal{S}} \|x\|^p \mu(dx) + \int_{\mathcal{S}} \|y\|^p \nu(dy) \Big) < \infty$$

By Corollary 3.4, $\Gamma(\mu, \nu)$ is compact for the narrow convergence. To prove that (3.2) is a minimum, it suffices to show that $\gamma \to \int_{\mathcal{S}\times\mathcal{S}} \|x - y\|^p \gamma(dx, dy)$ is lower semicontinuous for the narrow convergence. To achieve this goal, let $(\gamma_n)_n$ be a sequence in $\Gamma(\mu, \nu)$ converging narrowly to some γ. Since for each natural number k, the function

$$(x, y) \to c_k(x, y) := \min\Big\{ \|x - y\|^p, k \Big\}$$

is continuous and bounded, we have

$$\liminf_{n\to\infty} \int_{\mathcal{S}\times\mathcal{S}} \|x - y\|^p \gamma_n(dx, dy) \geq \lim_{n\to\infty} \int_{\mathcal{S}\times\mathcal{S}} c_k(x, y) \gamma_n(dx, dy) = \int_{\mathcal{S}\times\mathcal{S}} c_k(x, y) \gamma(dx, dy)$$

and so, by Fatou's Lemma

$$\liminf_{n\to\infty}\int_{\mathcal{S}\times\mathcal{S}}\|x-y\|^p\gamma_n(dx,dy)\geq\sup_{k\in\mathbb{N}}\int_{\mathcal{S}\times\mathcal{S}}c_k(x,y)\gamma(dx,dy)=\int_{\mathcal{S}\times\mathcal{S}}\|x-y\|^p\gamma(dx,dy).$$

This verifies (i).

(ii) Let $X,Y\in\mathbb{H}$ are such that $\mu=X_\sharp\mathcal{L}^d_{\mathbb{B}_d}$ and $\nu=Y_\sharp\mathcal{L}^d_{\mathbb{B}_d}$ then

$$\gamma=:(X\times Y)_\sharp\mathcal{L}^d_{\mathbb{B}_d}\in\Gamma(\mu,\nu)$$

and so,

$$W_p^p(\mu,\nu)\leq\int_{\mathcal{S}\times\mathcal{S}}\|x-y\|^p\gamma(dx,dy)=\int_{\mathbb{B}_b}\|X(\omega)-Y(\omega)\|^pd\omega.$$

This proves the infimum at the right handside of (3.3) is bigger than or equal to $W_p^p(\mu,\nu)$. To show the reverse inequality, let γ_0 be a minimizer (3.2). By Corollary 3.7, there is a Borel map $Z:=(X,Y):\mathbb{B}_d\times\mathcal{S}^2$ such that $Z_\sharp\mathcal{L}^d_{\mathbb{B}_d}=\gamma_0$. As γ_0 has μ and ν as marginals, we have $X,Y\in\mathbb{H}$ and $\mu=X_\sharp\mathcal{L}^d_{\mathbb{B}_d}$ and $\nu=Y_\sharp\mathcal{L}^d_{\mathbb{B}_d}$. Since

$$W_p^p(\mu,\nu)=\int_{\mathcal{S}\times\mathcal{S}}\|x-y\|^p\gamma_0(dx,dy)=\int_{\mathbb{B}_b}\|X(\omega)-Y(\omega)\|^pd\omega,$$

we conclude that (X,Y) minimizes the right handside of (3.3) and so, (ii) holds. QED.

Definition 3.10. Given $\mu,\nu\in\mathcal{P}_p(\mathcal{S})$, we denote as $\Gamma_0^p(\mu,\nu)$, the set of γ in $\Gamma(\mu,\nu)$ such that

$$W_p^p(\mu,\nu)=\int_{\mathcal{S}\times\mathcal{S}}\|x-y\|^p\gamma(dx,dy).$$

By Theorem 3.9, $\Gamma_0^p(\mu,\nu)\neq\emptyset$. When $p=2$ we simply write $\Gamma_0(\mu,\nu)$ instead of $\Gamma_0^2(\mu,\nu)$.

Theorem 3.11. *For any $p\in[1,\infty)$, $\big(\mathcal{P}_p(\mathcal{S}),W_p\big)$ is a metric space.*

Proof: Let $\mu_0,\mu_1,\mu_2\in\mathcal{P}_p(\mathcal{S})$. By Theorem 3.9 there exist

$$X^0,X^1,\bar{X}^1,\bar{X}_2\in\mathbb{H}_p,\quad\text{and}\quad\gamma^{01}\in\Gamma_0^p(\mu_0,\mu_1),\ \gamma^{12}\in\Gamma_0^p(\mu_1,\mu_2)\tag{3.4}$$

such that

$$\mu_0=X^0_\sharp\mathcal{L}^d_{\mathbb{B}_d},\ \mu_1=X^1_\sharp\mathcal{L}^d_{\mathbb{B}_d},\ \mu_1=\bar{X}^1_\sharp\mathcal{L}^d_{\mathbb{B}_d},\ \mu_2=\bar{X}^2_\sharp\mathcal{L}^d_{\mathbb{B}_d}\tag{3.5}$$

and

$$W_p^p(\mu_0,\mu_1)=\big\|X^0-X^1\big\|^p_{L^p(\mathbb{B}_b)},\quad W_p^p(\mu_1,\mu_2)=\big\|\bar{X}^1-\bar{X}^2\big\|^p_{L^p(\mathbb{B}_b)}.\tag{3.6}$$

Claim 1: $W_p(\mu_0,\mu_1)=0$ iff $\mu_0=\mu_1$.

Proof of Claim 1: If $W_p^p(\mu_0,\mu_1)=0$ the first identity in (3.6) implies $X^0=X^1$ almost everywhere and so, by the first and second identities in (3.5), we have $\mu_0=\mu_1$. Conversely, when $\mu_0=\mu_1$, we use the fact that $\gamma^{00}:=(X^0\times X^0)_\sharp\mathcal{L}^d_{\mathbb{B}_d}\in\Gamma(\mu_0,\mu_1)$ to verify that $W_p(\mu_0,\mu_1)=0$.

Claim 2: W_p is symmetric.

Proof of Claim 2: Indeed, since $\bar\gamma^{10} := (X^1 \times X^0)_\sharp \mathcal{L}^d_{\mathbb{B}_d} \in \Gamma(\mu_1, \mu_0)$ we conclude that

$$W_p^p(\mu_1, \mu_0) \leq \int_{\mathcal{S}\times\mathcal{S}} \|x-y\|^p \bar\gamma^{10}(dx, dy) = \int_{\mathbb{B}_b} \|X^0(\omega) - X^1(\omega)\|^p d\omega \leq W_p^p(\mu_0, \mu_1).$$

We interchange the role of μ_0 and μ_1 to obtain the reverse inequality and conclude that $W_p^p(\mu_1, \mu_0) = W_p^p(\mu_0, \mu_1)$.

Claim 3: W_p satisfies the triangle inequality.

Proof of Claim 3: Thanks to the second and third identity in (3.5), we use Theorem 3.1 to obtain a sequence $(S_n)_n \subset G_{\mathbb{B}_d}$ such that

$$\lim_{n\to\infty} \big\|\bar X^1 - X^1 \circ S_n\big\|_{L^p(\mathbb{B}_b)} = 0. \tag{3.7}$$

Since S_n preserves Lebesgue measure, the first identity in (3.6) implies

$$\big\|X^0 \circ S_n - X^1 \circ S_n\big\|_{L^p(\mathbb{B}_b)} = \big\|X^0 - X^1\big\|_{L^p(\mathbb{B}_b)} = W_p(\mu_0, \mu_1).$$

This, together with (3.7) yields

$$W_p(\mu_0, \mu_1) = \lim_{n\to\infty} \big\|X^0 \circ S_n - \bar X^1\big\|_{L^p(\mathbb{B}_b)}.$$

Thus,

$$W_p(\mu_0, \mu_1) + W_p(\mu_1, \mu_2) = \lim_{n\to\infty} \big\|X^0 \circ S_n - \bar X^1\big\|_{L^p(\mathbb{B}_b)} + \big\|\bar X^1 - \bar X^2\big\|_{L^p(\mathbb{B}_b)} \geq \liminf_{n\to\infty} \big\|X^0 \circ S_n - \bar X^1\big\|_{L^p(\mathbb{B}_b)}.$$

We use the fact that

$$(X^0 \circ S_n \times \bar X^1)_\sharp \mathcal{L}^d_{\mathbb{B}_d} \in \Gamma(\mu_0, \mu_2)$$

to conclude that

$$W_p(\mu_0, \mu_1) + W_p(\mu_1, \mu_2) \geq W_p(\mu_0, \mu_2).$$

This completes the verification of the triangle inequality and concludes the proof of the theorem. QED.

Corollary 3.12. *The push forward operator is an isometry of $\mathbb{H}/\sharp$ onto $\mathcal{P}_p(\mathcal{S})$.*

Proof: The metric metric induced on $\mathbb{H}_p/\sharp$ is

$$\operatorname{dist}_p^p([X]_p, [Y]_p) := \inf_{\bar X, \bar Y \in \mathbb{H}} \Big\{ \int_{\mathbb{B}_b} \|X(\omega) - Y(\omega)\|^p d\omega \ : \ \bar X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = X_\sharp \mathcal{L}^d_{\mathbb{B}_d},\ \bar Y_\sharp \mathcal{L}^d_{\mathbb{B}_d} = Y_\sharp \mathcal{L}^d_{\mathbb{B}_d} \Big\}$$

We use Theorem 3.9 to conclude the proof of the remark. QED.

Remark 3.13. Remark 3.12 provides the first hint that when $\mathcal{S} = \mathbb{R}^d$, the intrinsic differential structures on $\mathcal{P}_2(\mathbb{R}^d)$ introduced in [2], may be inherited from the natural differential structure on the Hilbert space $\mathbb{H}$. When we endow $G_{\mathbb{B}_d}$ with the operation $\circ$ which associates to two maps their composition, we obtain a non commutative group with a right action on $\mathbb{H}$. For the latter action, the orbit of $X \in \mathbb{H}$ is the $X \cdot G_{\mathbb{B}_d} = \{X \circ S \mid S \in G_{\mathbb{B}_d}\}$ and these orbits form a partition of $\mathbb{H}$. We henceforth have another equivalence relation; two elements are equivalent if and only if their orbits are the same. Note that the set $X \cdot G_{\mathbb{B}_d}$ is strictly contained in $[X]_2$ but the closure of $X \cdot G_{\mathbb{B}_d}$ is $[X]_2$. Such a conclusion is based on Theorem 3.1.

Theorem 3.14. *Let $1 \leq p < \infty$, let $s_0 \in \mathbb{S}$ and let $\varphi_0 : \mathbb{S} \mapsto \mathbb{R}$ be defined as $\varphi_0(x) \equiv \text{dist}^p(x, s_0)$. A sequence $(\mu_n)_\subset \mathcal{P}_p(\mathbb{S})$ W_p–converges to $\mu \in \mathcal{P}_p(\mathbb{S})$ if and only if*

$$\lim_{n\to\infty} \int_{\mathbb{M}} \varphi(x)\mu_n(dx) = \int_{\mathbb{M}} \varphi(x)\mu(dx) \qquad \forall \varphi \in C_b(\mathbb{M}) \cup \{\varphi_0\}.$$

In other words, adding or removing φ_0 from the list of test function is what makes the difference between narrow convergence and W_p–convergence. Theorem 3.14 is obtained as a consequence of Subsection 5.1.1 and Proposition 7.1.5 [2].

3.1.3 Exercise

Exercise 3.15. Let $\mathcal{S}$ and $\mathcal{T}$ be two topological spaces, let μ be a Borel measure on $\mathcal{S}$ and let $X : \mathcal{S} \mapsto \mathcal{T}$ be Borel map. Set

$$\nu(B) = \mu\big(X^{-1}(B)\big)$$

for any B Borel subset of $\mathcal{T}$. Show that ν is a Borel map on $\mathcal{T}$.

3.2 Lectures 7, 8: Convex analysis for optimal transport (Sep 13, 20)

In this section, we assume $p = 2$ and $\mathcal{S} = \mathbb{M} \in \{\mathbb{T}^d, \mathbb{R}^d\}$ so that

$$\mathbb{H} = L^2(\mathbb{B}_d, \mathbb{M}, \mathcal{L}^d_{\mathbb{B}_d}).$$

Definition 3.16 (cf. Rockafellar [45])**.** Let $B \subset \mathbb{R}^d$ and let $\phi : B \mapsto (-\infty, \infty]$.

(i) We say that ϕ is lower semicontinuous at $x_0 \in B$ if for any sequence $(x_k)_{k=1}^\infty$ in B converging to x_0,

$$\liminf_{k\to\infty} \phi(x_k) \geq \phi(x_0)$$

(ii) The Legendre transform of ϕ is $\phi^* : \mathbb{R}^d \mapsto (-\infty, \infty]$ defined as

$$\phi^*(p) := \sup_{x\in B} \langle x, p\rangle - \phi(x)$$

(iii) When ϕ <u>is convex</u>, we say that $p_0 \in \mathbb{R}^d$ belongs to the subdifferential of ϕ at $x_0 \in B$ and we write $p_0 \in \partial_\cdot\phi(x_0)$ if

$$\phi(x) \geq \phi(x_0) + \langle p_0, x - x_0\rangle \qquad \forall x \in B$$

(iv) We define $\partial_\cdot\phi$ to be the set of $(x_0, p_0) \in B \times \mathbb{R}^d$ such that $p_0 \in \partial_\cdot\phi(x_0)$.

Theorem 3.17 (Important; cf. Rockafellar [45])**.** *Let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ be convex.*

(ii) We have that ϕ is locally Lipschitz in the interior of $\{\phi < \infty\}$.

(i) If ϕ is lower semicontinuous $\phi = \phi^{**}$.

Definition 3.18 (cf. Rockafellar [44]). A set $C \subset \mathbb{R}^d \times \mathbb{R}^d$ is cyclically monotone if for any natural number n, any permutation of n letters τ and any $\{(x_1, y_i)\}_{i=1}^n \subset C$ we have

$$\sum_{i=1}^n |x_i - y_i|^2 \le \sum_{i=1}^n |x_{\tau(i)} - y_i|^2$$

Exercise 3.19 (cf. Rockafellar [44] [45]). Assume $B \subset \mathbb{R}^d$ is a convex set, $\phi : B \mapsto (-\infty, \infty]$ is a convex function.

(i) Show that $(x_0, p_0) \in \partial.\phi$ if and only if $\langle x_0, p_0\rangle = \phi(x_0) + \phi^*(p_0)$.

(ii) Show that $\partial.\phi$ is cyclically monotone.

Proof: (i) Assume $(x_0, p_0) \in \partial.\phi$ so that

$$\phi(x) \ge \phi(x_0) + \langle p_0, x - x_0\rangle = \phi(x_0) + \langle p_0, x\rangle - \langle p_0, x_0\rangle \qquad \forall x \in B$$

Rearranging, we have

$$\langle p_0, x_0\rangle - \phi(x_0) \ge \langle p_0, x\rangle - \phi(x) \qquad \forall x \in B$$

which means that $\langle p_0, x\rangle - \phi(x)$ achieves its maximum at x_0 and so, $\phi^*(p_0) = \langle p_0, x_0\rangle - \phi(x_0)$.

Conversely, assume $\phi^*(p_0) = \langle p_0, x_0\rangle - \phi(x_0)$. Using the definition of $\phi^*(p_0)$ we observe

$$\langle p_0, x_0\rangle - \phi(x_0) \ge \langle x, p_0\rangle - \phi(x) \qquad \forall x \in B$$

Rearranging, we obtain

$$\phi(x) \ge \phi(x_0) + \langle x - x_0, p_0\rangle \qquad \forall x \in B.$$

This proves that $(x_0, p_0) \in \partial.\phi$.

(ii) Let n be a natural number, τ be a permutation of n letters and $(x_i, p_i)_{i=1}^n \subset \partial.\phi$. By (i) we have

$$\langle x_i, p_i\rangle = \phi(x_i) + \phi^*(p_i), \quad \langle x_{\tau(i)}, p_i\rangle \le \phi(x_{\tau(i)}) + \phi^*(p_i).$$

Summing up over the set of i's we obtain

$$\sum_{i=1}^n \langle x_i, p_i\rangle = \sum_{i=1}^n \phi(x_i) + \sum_{i=1}^n \phi^*(p_i), \quad \sum_{i=1}^n \langle x_{\tau(i)}, p_i\rangle \le \sum_{i=1}^n \phi(x_{\tau(i)}) + \sum_{i=1}^n \phi^*(p_i).$$

Since τ is a permutation then $\sum_{i=1}^n \phi(x_i) = \sum_{i=1}^n \phi(x_{\tau(i)})$ and so,

$$-\sum_{i=1}^n \langle x_i, p_i\rangle \le -\sum_{i=1}^n \langle x_{\tau(i)}, p_i\rangle.$$

This implies

$$\frac{1}{2}\sum_{i=1}^n |x_i|^2 + \frac{1}{2}\sum_{i=1}^n |p_i|^2 - \sum_{i=1}^n \langle x_i, p_i\rangle \le \frac{1}{2}\sum_{i=1}^n |x_{\tau(i)}|^2 + \frac{1}{2}\sum_{i=1}^n |p_i|^2 - \sum_{i=1}^n \langle x_{\tau(i)}, p_i\rangle.$$

In over words,

$$\frac{1}{2}\sum_{i=1}^{n}|x_i - p_i|^2 \leq \frac{1}{2}\sum_{i=1}^{n}|x_{\tau(i)} - p_i|^2.$$

This verifies (ii). QED.

Exercise 3.20. Show that $C \subset \mathbb{R}^d \times \mathbb{R}^d$ is cyclically monotone if and only if there exists a lower semicontinuous convex function $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ such that $C \subset \partial.\phi$.

Proof: (i) By Exercise 3.19, if $C \subset \partial.\phi$ and ϕ is convex then C is cyclically monotone.

(ii) The following proof is due to Rockafellar [44]. Consersely, assume $C \subset \mathbb{R}^d \times \mathbb{R}^d$ is cyclically monotone. Let $(\bar{x}, \bar{y}) \in C$. Define

$$\phi(x) := \sup_{n\in\mathbb{N}} \sup_{(x_i,y_i)_{i=1}^n \subset C} \Big\{ \langle x - x_1, y_1\rangle + \langle x_1 - x_2, y_2\rangle + \cdots + \langle x_{n-1} - x_n, y_n\rangle + \langle x_n - \bar{x}, \bar{y}\rangle \Big\}$$

In other words, using the convention $(\bar{x}, \bar{y}) = (x_{n+1}, y_{n+1})$ we have

$$\phi(x) := \sup_{n\in\mathbb{N}} \sup_{(x_i,y_i)_{i=1}^n \subset C} \Big\{ \langle x - x_1, y_1\rangle + \sum_{i=1}^{n}\langle x_i - x_{i+1}, y_{i+1}\rangle \mid (x_i, y_i)_{i=1}^n \subset C \Big\}.$$

Claim 1. $\phi \not\equiv \infty$.

Proof of Claim 1. Choosing $n = 1$ and $(\bar{x}, \bar{y}) = (x_1, y_1)$, we obtain $\phi(\bar{x}) \geq 0$. We use the fact that $(x_i, y_i)_{i=1}^{n+1} \subset C$ to conclude that if $x = \bar{x}$ then $\phi(\bar{x}) \leq 0$. In conclusion, we have $\phi(\bar{x}) = 0$.

Claim 2. ϕ is convex, lower semicontinuous and $C \subset \partial.\phi$.

Proof of Claim 2. The convexity and lower semicontinuity property of ϕ is due to the fact that ϕ is the supremum of linear functions. I

Let $(\tilde{x}, \tilde{y}) \in C$. Set $(\tilde{x}, \tilde{y}) = (x_1, y_1)$. For any arbitrary $(x_i, y_i)_{i=2}^n \subset C$ and $x \in \mathbb{R}^d$, by the definition of $\phi(x)$, we have

$$\phi(x) - \langle x - x_1, y_1\rangle \geq \langle \tilde{x} - x_2, y_2\rangle + \cdots + \langle x_{n-1} - x_n, y_n\rangle + \langle x_n - \bar{x}, \bar{y}\rangle$$

and so,

$$\phi(x) - \langle x - x_1, y_1\rangle \geq \sup_{n\geq 2} \sup_{(x_i,y_i)_{i=2}^n \subset C} \Big\{ \langle \tilde{x} - x_2, y_2\rangle + \cdots + \langle x_{n-1} - x_n, y_n\rangle + \langle x_n - \bar{x}, \bar{y}\rangle \Big\}.$$

Thus,

$$\phi(x) - \langle x - \tilde{x}, \tilde{y}\rangle \geq \phi(\tilde{x})$$

This proves $(\tilde{x}, \tilde{y}) \in \partial.\phi$. QED.

Theorem 3.21. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma \in \Gamma(\mu, \nu)$. The following are equivalent*

(i) $\gamma_0 \in \Gamma_0(\mu, \nu)$.

(ii) There exists a cyclically monotone Borel set $C \subset \mathbb{R}^d \times \mathbb{R}^d$ such that $\gamma_0[C] = 1$.

Proof: 1. Assume there exists $C \subset \mathbb{R}^d \times \mathbb{R}^d$ such that $\gamma[C] = 1$. We are to show that for any $\gamma \in \Gamma(\mu, \nu)$, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \gamma_0(dx, dy) \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \gamma(dx, dy).$$

By Exercise 3.20, there exists a lower semicontinuous convex function $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ such that $C \subset \partial.\phi$. By Exercise 3.19,

$$\langle x, p\rangle = \phi(x) + \phi^*(p) \qquad \forall (x, y) \in C$$

Thus,

$$\int_C \langle x, p\rangle \gamma_0(dx, dy) = \int_C \big(\phi(x) + \phi^*(p)\big)\gamma_0(dx, dy)$$

Since γ_0 is a set of null measure, we conclude

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle x, p\rangle \gamma_0(dx, dy) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(\phi(x) + \phi^*(p)\big)\gamma_0(dx, dy) \tag{3.8}$$

The identity

$$\langle x, p\rangle \leq \phi(x) + \phi^*(p) \qquad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$$

implies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle x, p\rangle \gamma(dx, dy) \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(\phi(x) + \phi^*(p)\big)\gamma(dx, dy) \tag{3.9}$$

Since

$$\int_{\mathbb{R}^d} \frac{|x|^2}{2}\mu(dx) + \int_{\mathbb{R}^d} \frac{|y|^2}{2}\nu(dy) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x|^2+|y|^2}{2}\gamma(dx, dy) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|x|^2+|y|^2}{2}\gamma_0(dx, dy)$$

we combine (3.8) and (3.9) to conclude that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \gamma_0(dx, dy) \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \gamma(dx, dy).$$

We have verify that (ii) implies (i).

2. Assume now that (i) holds. The main idea for proving (ii) (cf. [6] [24]) is contained in the following special case where we assume

$$\mu = \frac{1}{k}\sum_{i=1}^k \delta_{x_i}, \quad \nu = \frac{1}{k}\sum_{i=1}^k \delta_{y_i}, \quad X := \{x_i \mid i = 1, \cdots, k\}, \quad Y := \{y_i \mid i = 1, \cdots, k\}.$$

As $\gamma \in \Gamma(\mu, \nu)$, it is supported by $X \times Y$ and so, there exists $\gamma_{ij} \geq 0$ such that

$$\gamma_0 = \sum_{i,j=1}^k \gamma_{ij}\delta_{(x_i, y_i)}.$$

The cost for using the strategy γ_0 is

$$\mathrm{Cost}[\gamma_0] := \sum_{i,j=1}^{k} \gamma_{ij} |x_i - y_j|^2.$$

The smallest Borel set C such that $\gamma[C] = 1$ is

$$C = \{(x_i, y_j) \mid \gamma_{ij} > 0\}.$$

Note

$$\bar{g} := \frac{1}{2} \min\{\gamma_{ij} \mid (x_i, y_j) \in C\} > 0.$$

Let $n \leq k$ is a natural number and τ be a permutation of n letters. Let $(x_{i_l}, y_{j_l})_{l=1}^n$ be in C and assume on the contrary that

$$\sum_{i=1}^{n} |x_{i_l} - y_{j_l}|^2 > \sum_{i=1}^{n} |x_{i_l} - y_{\tau(j_l)}|^2 \tag{3.10}$$

We plan to build a new strategy with a smaller cost by only modifying

$$\bar{\mu} := \bar{g} \sum_{l=1}^{n} \delta_{x_{i_l}}, \quad \bar{\nu} := \bar{g} \sum_{l=1}^{n} \delta_{x_{j_l}}.$$

Recall that

$$\{y_{j_{\tau(l)}}\}_{j=1}^n = \{y_{j_l}\}_{j=1}^n$$

and so if we set

$$\gamma^1 := \bar{g} \sum_{l=1}^{n} \delta_{(x_{i_l}, y_{j_l})}, \quad \bar{\gamma}^1 := \bar{g} \sum_{l=1}^{n} \delta_{(x_{i_l}, y_{j_{\tau(l)}})}, \quad \bar{\gamma} := \gamma_0 - \gamma^1 + \bar{\gamma}^1.$$

then $\bar{\gamma} \in \Gamma(\mu, \nu)$ and so, by the minimality property of γ_0,

$$\mathrm{Cost}[\gamma^1] + \mathrm{Cost}[\gamma_0 - \gamma^1] = \mathrm{Cost}[\gamma_0] \leq \mathrm{Cost}[\bar{\gamma}] = \mathrm{Cost}[\bar{\gamma}^1] + \mathrm{Cost}[\gamma_0 - \gamma^1].$$

This reads off

$$\mathrm{Cost}[\gamma^1] \leq \mathrm{Cost}[\bar{\gamma}^1],$$

which is at variance with (3.10). QED.

We shall use the projections $\pi_0, \pi^1 : \mathbb{R}^{2d} \mapsto \mathbb{R}^d$ defined as

$$\pi^0(x, y) = x, \quad \pi^1(x, y) = y \qquad \forall x, y \in \mathbb{R}^d$$

and their interpolations

$$\pi^t(x, y) = (1 - t)x + ty = (1 - t)\pi^0(x, y) + t\pi^1(x, y)$$

3.2.1 Properties of convex functions.

We are just going to recall classical results from convex analysis without reproving all of them because this course is not about convex analysis but Mean Fields Games. Rather we provide references where the proofs of these results can be found.

Definition 3.22 (Geometric measure theory)**.** Recall that for any integer $s \geq 0$, a set $B \subset \mathbb{R}^d$ is s–rectifiable if it is a countable union of s–hypersurfaces and a set of $\mathcal{H}^s$–null measure. In particular, any set of null Lebesgue measure in $\mathbb{R}^d$ is $(d-1)$–rectifiable.

Facts 1. (Convex analysis [45]). Let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ be a convex function. Denote as dom (ϕ) the set where $\{\phi < \infty\}$ and as dom$(\nabla\phi)$ the set where ϕ is differentiable. Then the set dom $(\phi) \setminus$ dom$(\nabla\phi)$ is $(d-1)$–rectifiable. In particular dom $(\phi) \setminus$ dom$(\nabla\phi)$ is a set of null Lebesgue measure.

Facts 2. (cf. [22]) Let $B \subset \mathbb{R}^d$ be a convex open set and let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$.

(i) Note $\varphi \in C^1(a, b)$ is convex if and only if φ' is monotone nondecreasing. Hence if $\varphi \in C^2(a, b)$, it is convex if and only if $\varphi'' \geq 0$.

(ii) Note ϕ is convex if and only if it is convex along any (one–dimensional) line segment in B. By (i), we conclude that when $\phi \in C^2(B)$ then ϕ is convex if and only if $\nabla^2\phi$ is nonnegative definite. Indeed, the convexity of ϕ is equivalent to that of the map $t \to \phi(x_0 + tu)$ for any $x_0 \in B$ and any $u \in \mathbb{R}^d$.

(iii) Assume ϕ is convex. Then the entries of matrix of the distributional second derivatives are signed Radon measures which satisfies $\nabla^2\phi \geq 0$.

Exercise 3.23. Assume $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is a convex function and let $x \in \mathbb{R}^d$ and $y \in \partial.\phi(x)$.

(i) Show that if ϕ differentiable x at then $y = \nabla\phi(x)$ (in fact, one draws a stronger conclusion: $\partial.\phi(x)$ has a cardinality 1 if and only if ϕ is differentiable at x. In this case $\partial.\phi(x) = \{\nabla\phi(x)\}$). Similarly, show that if ϕ^* is differentiable at y then $x = \nabla\phi^*(y)$.

(ii) Further assume ϕ is lower semicontinuous. Show that $\partial.\phi$ is a closed subset of $\mathbb{R}^{2d}$.

Proof: (i) Recall that by Exercise 3.19 if $y \in \partial.\phi(x)$ then $\phi(x) + \phi^*(y) = \langle x, y\rangle$. Hence, Note $(z, w) \to H(z, w) := \phi(z) + \phi^*(w) - \langle z, w\rangle$ achieves its minimum at (x, y). If ϕ is differentiable at x then $H(\cdot, y)$ is differentiable at x and its partial derivatives with respect to z are null at x. Similarly, if ϕ^* is differentiable at y then $H(x, \cdot)$ is differentiable at y and its partial derivatives with respect to w at x are null. Since

$$\nabla_z H(z, y)|_{z=x} = \nabla\phi(x) - y, \quad \nabla_w H(x, w)|_{w=y} = \nabla\phi^*(x) - x$$

we complete the proof of (i).

(ii) Further assume ϕ is lower semicontinuous and let $(x_n, y_n)_n \subset \partial.\phi$ be a sequence converging to (x, y).By Exrecise 3.19 we have

$$\phi(x_n) + \phi^*(y_n) = \langle x_n, y_n\rangle.$$

Since as a supremum of linear functions, ϕ^* is lower semicontinuous and by assumption ϕ is lower semicontinuous we obtain

$$\phi(x)+\phi^*(y) \leq \liminf_{n\to\infty} \phi(x_n)+\liminf_{n\to\infty} \phi^*(y_n) \leq \liminf_{n\to\infty}\Big(\phi(x_n)+\phi^*(y_n)\Big) = \liminf_{n\to\infty}\langle x_n, y_n\rangle = \langle x, y\rangle.$$

The reverse inequality holding in general, we conclude $\phi(x) + \phi^*(y) = \langle x, y\rangle$ and so, using Exrecise 3.19 once more, we verify that $(x, y) \in \partial.\phi$. QED.

Exercise 3.24. Let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ be a convex lower semicontinuous function be such that $\phi \not\equiv \infty$. Set

$$A := \Big\{x \in \mathrm{dom}(\nabla\phi) \mid \nabla\phi(x) \in \mathrm{dom}(\nabla\phi^*)\Big\}, \quad B := \nabla\phi(A).$$

Show that $\nabla\phi : A \mapsto B$ is a bijection with $\nabla\phi^*$ as its inverse.

Proof: Note that $\nabla\phi$ is onto B. Let $x \in A$ and set $y = \nabla\phi(x) \in \mathrm{dom}(\nabla\phi^*)$. We have

$$\phi(x) + \phi^*(y) = \langle x, y\rangle$$

and so, by Exercise 3.23 (i), $x = \nabla\phi^*(y)$. This means

$$x = \nabla\phi^*\Big(\nabla\phi(x)\Big).$$

This verifies that $\nabla\phi$ is one-to-one on A and the inverse of $\nabla\phi$ is $\nabla\phi^*$. QED.

Exercise 3.25. Let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ be a convex lower semicontinuous function be such that $\phi \not\equiv \infty$. Define the interpolation

$$\phi_t(x) = (1-t)\frac{|x|^2}{2} + t\phi(x).$$

(i) Show that for any $t \in (0, 1]$, the Legendre transform ϕ_t^* is of class $C^1(\mathbb{R}^d)$ and $\nabla\phi_t^*$ is $(1-t)^{-1}$–Lipschitz.

(ii) Show that if $(x, y) \in \partial.\phi$ then $(x, (1-t)x + ty) \in \partial.\phi_t$.

Proof: (i) Since ϕ is convex, the second order distributional derivatives satisfy, for $t \in [0, 1)$,

$$\nabla^2\phi_t = (1-t)I + t\nabla^2\phi \geq (1-t)I.$$

Thus, the Legendre transforms ϕ_t^* satisfy

$$\nabla^2\phi_t^* \leq \frac{I}{1-t}.$$

This proves that $\nabla\phi_t^*$ is $(1-t)^{-1}$–Lipschitz.

(ii) Let $(x, y) \in \partial.\phi$. For any $w \in \mathbb{R}^d$ we have

$$\phi(w) \geq \phi(x) + \langle y, w - x\rangle$$

and so,

$$\phi_t(w) \geq (1-t)\frac{|w|^2}{2}+t\phi(x)+\langle ty, w-x\rangle = (1-t)\Big(\frac{|x|^2}{2}+\frac{|w-x|^2}{2}+\langle x, w-x\rangle\Big)+t\phi(x)+\langle ty, w-x\rangle.$$

This reads off

$$\phi_t(w) \geq \phi_t(x) + \big\langle (1-t)x + ty, w-x \big\rangle + (1-t)\frac{|w-x|^2}{2}.$$

This verifies the claim.

QED.

3.2.2 Borel measures and their support.

Definition 3.26. Let γ be a Borel measure on $\mathbb{R}^D$. We call the support of γ the set $\mathrm{spt}(\gamma)$ which consists of all $z \in \mathbb{R}^D$ such that $\gamma[B_r(z)] > 0$ for all $r > 0$.

Remark 3.27. Assume γ is a Borel measure on $\mathbb{R}^D$. If $\gamma[B_{3r}(z)] = 0$ then for every $w \in B_r(z)$ we have $\gamma[B_r(w)] = 0$. This proves the support of γ is a closed set.

Exercise 3.28. Assume γ is a probability Borel measure on $\mathbb{R}^D$.

(i) Show that the support of γ is the smallest closed set K such that $\gamma(K) = 1$.

(ii) Suppose $D = 2d$ and $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is a convex lower semicontinuous function so that $\partial.\phi$ is a closed set. Show there exists a Borel set $C \subset \partial.\phi$ such that $\gamma[C] = 1$ if and only if $\mathrm{spt}(\gamma) \subset \partial.\phi$.

Proof: (i) If $\gamma[\mathrm{spt}(\gamma)] < 1$ then the open set $O := \mathbb{R}^d \setminus \mathrm{spt}(\gamma)$ is of positive measure. For each $x \in O$ choose $r(x) > 0$ such that $B_{r(x)} \subset O$ and $\gamma\big[B_{r(x)}(x)\big] = 0$. Let $L \subset O$ be an arbitrary compact set. Then there exists $x_1, \cdots, x_n \in L$ such that

$$L \subset \cup_{i=1}^n B_{r(x_i)}(x_i).$$

Hence, $\gamma[L] = 0$. Since γ is Borel measure, $\gamma[O] = 0$ as it is the supremum of $\gamma[L]$ over the set of compact set $L \subset O$. The latter conclusion is at a variance with the fact that O is of positive measure.

Let $K \subset \mathbb{R}^D$ be a closed set such that $\gamma[K] = 1$. Assume on the contrary we can find $x \in \mathrm{spt}(\gamma) \setminus K$. Since K^c, the complement of K is open, there exists $r > 0$ such that $B_r(x) \subset K^c$. We have $\gamma[B_r(x)] \leq \gamma[K^c] = 0$, which contradicts the fact that $x \in \mathrm{spt}(\gamma)$.

(ii) Suppose $D = 2d$ and $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is a convex lower semicontinuous function. By Exercise 3.23, $\partial.\phi$ is a closed set. If $C \subset \partial.\phi$ is a Borel set such that $\gamma[C] = 1$ then $\bar{C} \subset \partial.\phi$ and $\gamma[\bar{C}] = 1$. By (i) $\mathrm{spt}(\gamma) \subset \bar{C} \subset \partial.\phi$. Conversely, assume $\mathrm{spt}(\gamma) \subset \partial.\phi$. Then $C := \mathrm{spt}(\gamma)$ is a Borel set contains in $\partial.\phi$ and by (i), it satisfies $\gamma[C] = 1$. QED.

3.2.3 Monge problem: optimal maps.

Throughout this section $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$.

Definition 3.29. Let $p \in [1, \infty)$ be a real number.

(i) We define $\mathcal{T}(\mu, \nu)$ to be the set of Borel maps $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $g_\sharp \mu = \nu$. In other words

$$\int_{\mathbb{R}^d} \varphi(g(x))\mu(dx) = \int_{\mathbb{R}^d} \varphi(y)\nu(dy) \qquad \forall \varphi \in C_b(\mathbb{R}^d).$$

(ii) We define p–Monge's minimal cost for transporting μ onto ν to be the nonnegative number $\bar{W}_p(\mu, \nu)$ defined as

$$\bar{W}_p^p(\mu, \nu) := \inf_{g \in \mathcal{T}(\mu,\nu)} \int_{\mathbb{R}^d} |x - g(x)|^p \mu(dx). \tag{3.11}$$

Note that if for instance $d = 1$, $\mu = \delta_0$ and $\nu = 1/2(\delta_0 + \delta_1)$ then $\mathcal{T}(\mu, \nu) = \emptyset$ and so, $\bar{W}_p(\mu, \nu) = \infty$.

For any $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ Borel map, we set

$$\gamma_g := (Id \times g)_\sharp \mu.$$

By definition

$$\int_{\mathbb{R}^{2d}} \varphi(x, y)\gamma_g(dx, dy) := \int_{\mathbb{R}^d} \varphi(x, g(x))\mu(dx), \qquad \forall \varphi \in C_b(\mathbb{R}^{2d}). \tag{3.12}$$

Remark 3.30. Let $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a Borel map.

(i) If $\gamma \in \Gamma(\mu, \nu)$ is supported by the graph of g then $\gamma = \gamma_g$. Using special functions $\varphi(x, y) \equiv \varphi(y)$, we verify that $g \in \mathcal{T}(\mu, \nu)$.

(ii) If $g \in \mathcal{T}(\mu, \nu)$, choosing $\varphi \equiv \varphi(x)$ and then $\varphi \equiv \varphi(y)$ in (3.12), we obtain $\gamma_g \in \Gamma(\mu, \nu)$:

$$\Big\{\gamma_g \mid g \in \mathcal{T}(\mu, \nu)\Big\} \subset \Gamma(\mu, \nu). \tag{3.13}$$

An approximation argument shows (3.12) continue to hold for the nonnegative continuous functions $\varphi(x, y) := |x - y|^p$. This means

$$\int_{\mathbb{R}^d} |x - g(x)|^p \mu(dx) = \int_{\mathbb{R}^d} |x - y|^p \gamma_g(dx, dy) \tag{3.14}$$

This, together with (3.13) implies $\bar{W}_p(\mu, \nu) \geq W_p(\mu, \nu)$.

Theorem 3.31. *Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ be such that μ vanishes on $(d-1)$–rectifiable sets and assume $p = 2$ (in fact a variant of the same conclusions hold for $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ and $p \in (1, \infty)$).*

(i) There exists a unique $\bar{g}$ minimizer in (3.11). The measure $\gamma_{\bar{g}}$ (cf. (3.12)) is the unique minimizer of (3.2) and $\bar{W}_2(\mu, \nu) = W_2(\mu, \nu)$. Furthermore, there is $\bar{\phi} : \mathbb{R}^d \to (-\infty, \infty]$ convex, lower semicontinuous such that $\bar{g} = \nabla\bar{\phi}$ μ–almost everywhere.

(ii) Assume $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is a convex lower semicontinuous function. If $\nabla\phi \in \mathcal{T}(\mu, \nu)$ then $\bar g = \nabla \phi$ μ–almost everywhere. Furthermore

Proof: (i) By Theorem 3.9, there exists γ minimizer in $W_2(\mu, \nu)$, which means $\gamma \in \Gamma_0(\mu, \nu)$. By Theorem 3.21 there is a Borel set $C \subset \mathbb{R}^{2d}$ such that $\gamma[C] = 1$ and C is cyclically monotone. By Exercise 3.28, we may assume without loss of generality that $C = \text{spt}(\gamma)$ and so, it is a closed set. By Exercise 3.20 there exists a lower semicontinuous convex function $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ such that $C \subset \partial.\phi$ and so, by Exercise 3.19

$$\phi(x) + \phi^*(y) = \langle x, y\rangle \quad \forall (x, y) \in C. \tag{3.15}$$

Let $\Omega_0 := \pi^0(C)$. As C is a closed set, Ω_0 is a Borel set.

Claim 1. $\mu\Big[\Omega_0 \cap \text{dom}(\nabla\phi)\Big] = 1.$

Proof of Claim 1.

$$\mu[\Omega_0] = \gamma[\Omega_0 \times \mathbb{R}^d] \geq \pi^0(C) = 1. \tag{3.16}$$

and

$$\Omega_0 \subset \text{dom}(\phi). \tag{3.17}$$

Hence,

$$\Omega_0 \setminus \text{dom}(\nabla\phi) \subset \text{dom}(\phi) \setminus \text{dom}(\nabla\phi). \tag{3.18}$$

Since $\text{dom}(\phi) \setminus \text{dom}(\nabla\phi)$ is $(d-1)$–rectifiable, its μ–measure is null and so by (3.18)

$$\mu\Big[\Omega_0 \setminus \text{dom}(\nabla\phi)\Big] = 0. \tag{3.19}$$

This, together with (3.16) verifies Claim 1.

Claim 2. Given $x \in \Omega_0 \cap \text{dom}(\nabla\phi)$ and $y \in \mathbb{R}^d$ such that (3.15) holds, by Exercise 3.23 we have $\nabla\phi(x) = y$.

Claim 3. $\gamma = \gamma_{\bar g}$.

Proof of Claim 3. Let $f \in C_b(\mathbb{R}^{2d})$. We first use Claim 1 and then Claim 2 to obtain

$$\int_{\mathbb{R}^{2d}} f(x, y)\gamma(dx, dy) = \int_{\Big[\Omega_0 \cap \text{dom}(\nabla\phi)\Big] \times \mathbb{R}^d} f(x, y)\gamma(dx, dy) = \int_{\Big[\Omega_0 \cap \text{dom}(\nabla\phi)\Big] \times \mathbb{R}^d} f\big(x, \nabla\phi(x)\big)\gamma(dx, dy).$$

Thus, as μ is the first marginal of γ we have

$$\int_{\mathbb{R}^{2d}} f(x, y)\gamma(dx, dy) = \int_{\Omega_0 \cap \text{dom}(\nabla\phi)} f\big(x, \nabla\phi(x)\big)\mu(dx). \tag{3.20}$$

Let $\bar c$ be any arbitrary constant and set

$$\bar g = \begin{cases} \nabla\phi(x) & \text{if} \quad x \in \text{dom}(\nabla\phi) \\ \bar c & \text{if} \quad x \notin \text{dom}(\nabla\phi). \end{cases} \tag{3.21}$$

We use Claim 1 and (3.20) to conclude

$$\int_{\mathbb{R}^{2d}} f(x, y)\gamma(dx, dy) = \int_{\mathbb{R}^d} f\big(x, \bar g(x)\big)\mu(dx).$$

Thus, $\gamma = \gamma_{\bar{g}}$ and so, it is uniquely determined.

We have

$$\int_{\mathbb{R}^d} |x - \bar{g}(x)|^2 \mu(dx) = \int_{\mathbb{R}^{2d}} |x-y|^2 \gamma(dx,dy) = W_2^2(\mu,\nu) \leq \bar{W}_2^2(\mu,\nu).$$

The last inequality has been obtained thanks to Remark 3.30 (ii). This proves $\bar{g}$ is a minimizer in $\bar{W}_2^2(\mu,\nu)$ and is uniquely determined up to a set of null μ–measure. This completes the verification of (i).

(ii) Assume $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is a convex lower semicontinuous function and $\nabla\phi \in \mathcal{T}(\mu,\nu)$. By Remark 3.30 (ii), $\gamma_{\nabla\phi} \in \Gamma(\mu,\nu)$. Note

$$\gamma_{\nabla\phi}\Big[\mathbb{R}^d \setminus \big\{(x, \nabla\phi(x)) \mid x \in \mathrm{dom}(\nabla\phi)\big\}\Big] = 0$$

and so, $C := \{(x, \nabla\phi(x)) \mid x \in \mathrm{dom}(\nabla\phi)\}$ is Borel set of full measure, contained in $\partial.\phi$. By Theorem 3.21, $\gamma_{\nabla\phi}$ is a minimizer in $W_2(\mu,\nu)$ and so, by (i) $\gamma_{\nabla\phi} = \gamma_{\bar{g}}$. The identity

$$\int_{\mathbb{R}^{2d}} \varphi(x,y)\gamma_{\nabla\phi}(dx,dy) = \int_{\mathbb{R}^{2d}} \varphi(x,y)\gamma_{\bar{g}}(dx,dy)$$

holding also for special functions of the form $\varphi(x,y) = \langle a(x), y\rangle$, we have $\bar{g} = \nabla\phi$ μ–almost everywhere. This concludes the proof of the theorem. QED.

Example 3.32. Assume $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$ are such that μ vanishes on $(d-1)$–rectifiable sets. By Theorem 3.31 the Monge problem admits a minimizer g such that $g = \nabla\phi$ μ–almost everywhere, for a convex function $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$. Furthermore, $\gamma_{\nabla\phi}$ is the unique minimizer in Monge problem. Let ϕ_t be as in Exercise 3.25. The exercise asserts that $\nabla\phi_t^*$ is Lipschitz. We have

$$v_t(z) = (\nabla\phi - Id) \circ \nabla\phi_t^* = \left(\frac{\nabla\phi_t - Id}{t}\right) \circ \nabla\phi_t^* = \frac{Id - \nabla\phi_t^*}{t} = \nabla F_t(z)$$

where

$$F_t(z) = \frac{|z|^2}{2t} - \frac{\phi_t^*(z)}{t}$$

3.3 Lecture 9: Absolutely continuous curves (Sep 24).

Definition 3.33. Let $p \in [1,\infty)$ and $t \in [0,T] \to \sigma_t \in \mathcal{P}_2(\mathbb{M})$.

(i) We say that σ is p–absolutely continuous and we write $\sigma \in AC_p\big(0,T; \mathcal{P}_2(\mathbb{M})\big)$ if there exists $\beta \in L^p(0,T)$ such that

$$W_p(\sigma_t, \sigma_s) \leq \int_s^t \beta(\tau)d\tau \qquad \forall 0 \leq s < t \leq T.$$

(ii) We say that σ is a geodesic of constant speed if

$$W_p(\sigma_t,\sigma_s) = \frac{|t-s|}{T} W_p(\sigma_1,\sigma_0) \qquad \forall\ 0 \leq s < t \leq T.$$

We would like to study special paths which interpolate between two prescribed measures. For instance, if $\mu_0 = \delta_{x_0}$ and μ_{x_1} are dirac masses, the interpolating measure we use is the path

$$\sigma_t := \delta_{x_t}, \qquad x_t := (1-t)x_0 + tx_1.$$

Note we have used a path which connect two points in $\mathbb{R}^d$ to construct a path which connect two measures on $\mathbb{R}^d$ while remaining itself a dirac mass at all times. Furthermore, since $W_2(\sigma_t, \sigma_s) = |x_t - x_s| = |t-s|\,|x_0 - x_1|$ we conclude that $t \to \sigma_t$ is a geodesic of constant speed. We shall extend this consideration to measures, more general than dirac masses (cf. Theorem 3.34). The theorem also provides us with a first hint that when considering paths of probability measures $t \to \sigma_t \in \mathcal{P}_2(\mathbb{M})$, the velocities which are gradient of functions play a role.

Theorem 3.34. *Let $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ be a convex lower semicontinuous function be such that $\phi \not\equiv \infty$. Define the interpolation*

$$\phi_t(x) = (1-t)\frac{|x|^2}{2} + t\phi(x).$$

Let $\gamma \in \Gamma(\mu, \nu)$ be such that $\operatorname{spt}(\gamma) \subset \partial.\phi$ (so that $\phi \not\equiv \infty$) and we set

$$\gamma_t := (\pi^0 \times \pi^t)_\sharp \gamma, \quad \gamma_{st} := (\pi^s \times \pi^t)_\sharp \gamma \quad \sigma_t := \pi^1_\sharp \gamma_t, .$$

(i) For any $t \in (0,1]$, $\operatorname{spt}(\gamma_t) \subset \partial.\phi_t$ and σ_t is supported by $\pi^1\big(\operatorname{spt}(\gamma_t)\big)$.

(ii) For any $t \in (0,1)$ and $(x,y) \in \partial.\phi$, $(x, (1-t)x + ty)$ is uniquely determined and we can define σ_t–almost everywhere $v_t \circ \pi^t = \pi^1 - \pi^0$:

$$v_t\big((1-t)x + ty\big) = y - x. \tag{3.22}$$

(iii) We have σ as a geodesic of constant speed connecting μ to ν and

$$W_2(\sigma_t, \sigma_s) = \int_s^t \|v_t\|_{\sigma_t} \qquad \forall 0 \le s < t \le 1.$$

(iii) We have that v is a velocity for σ in the sense that (4.1) holds. We write

$$\partial_t \sigma + \nabla \cdot (\sigma v) = 0 \qquad \mathcal{D}'\big((0,1) \times \mathbb{R}^d\big)$$

Proof: (i) By definition, γ_t is supported by $\{(x, (1-t)x + ty) \mid (x,y) \in \operatorname{spt}(\gamma)\}$. Similarly, σ_t is supported by $\{(1-t)x + ty \mid (x,y) \in \operatorname{spt}(\gamma)\}$. We use Exercise 3.25 to conclude σ_t is supported by $\pi^1(\gamma_t)$.

(ii) *Claim 1.* We claim optimal routes don't cross except maybe at the endpoints. This means if $(x,y), (\bar{x}, \bar{y}) \in \operatorname{spt}(\gamma)$ and $t \in (0,1)$ then $\pi^t(x,y) \neq \pi^t(\bar{x}, \bar{y})$ unless $(x,y) = (\bar{x}, \bar{y})$.

Proof of Claim 1. Assume $t \in (0,1)$, $(x,y), (\bar{x}, \bar{y}) \in \partial.\phi$ and assume $\pi^t(x,y) = \pi^t(\bar{x}, \bar{y})$. Then $(1-t)(x - \bar{x}) = t(\bar{y} - y)$. We multiply both sides of the previous identity by $\bar{y} - y$ and then by $\bar{x} - x$ and use the fact that $\partial.\phi$ is cyclically monotone to obtain

$$0 \ge (1-t)\langle x - \bar{x}, \bar{y} - y\rangle = t|\bar{y} - y|^2, \quad (1-t)|x - \bar{x}|^2 = t\langle x - \bar{x}, \bar{y} - y\rangle \le 0.$$

Thus, $\bar{x} = x$ and $\bar{y} = y$.

(iii) Note that if $s, t \in [0,1]$ then $\gamma_{st} \in \Gamma(\mu_s, \mu_t)$ and so,

$$W_2^2(\mu_s, \mu_t) \leq \int_{\mathbb{R}^{2d}} |w - z|^2 \gamma_{st}(dw, dz) = \int_{\mathbb{R}^{2d}} |((1-s)x + sy) - ((1-t)x + ty)|^2 \gamma(dx, dy).$$

Thus,

$$W_2^2(\mu_s, \mu_t) \leq (s-t)^2 \int_{\mathbb{R}^{2d}} |x - y|^2 \gamma(dx, dy) = (s-t)^2 W_2^2(\mu_0, \mu_1). \tag{3.23}$$

We use the fact that W_2 satisfies the triangle inequality and (3.23) to obtain for $0 \leq s < t \leq 1$

$$\begin{aligned} W_2(\mu_0, \mu_1) &\leq W_2(\mu_0, \mu_s) + W_2(\mu_s, \mu_t) + W_2(\mu_t, \mu_1) && (3.24) \\ &\leq s W_2(\mu_0, \mu_1) + (t-s) W_2(\mu_0, \mu_1) + (1-t) W_2(\mu_0, \mu_1) && (3.25) \\ &= W_2(\mu_0, \mu_1). \end{aligned}$$

Thus, the inequalities in (3.24–3.25) are in fact equalities, in particular

$$W_2(\mu_s, \mu_t) = (t-s) W_2(\mu_0, \mu_1). \tag{3.26}$$

By the definition of v_t in (3.22) we have

$$\|v_t\|_{\sigma_t}^2 = \int_{\mathbb{R}^d} |v_t(z)|^2 \sigma_t(dz) = \int_{\mathbb{R}^{2d}} |v_t((1-t)x + ty)|^2 \gamma(dx, dy) = \int_{\mathbb{R}^{2d}} |y - x|^2 \gamma(dx, dy) = W_2^2(\mu_0, \mu_1)$$

This, together with (3.26) verifies (iii).

(iv) Let $\varphi \in C_c^\infty((0,1) \times \mathbb{R}^d)$. For any $t \in (0,1)$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} \langle \nabla \varphi(t, z), v_t(z) \rangle \sigma_t(dz) &= \int_{\mathbb{R}^{2d}} \langle \nabla \varphi\big(t, (1-t)x + ty\big), v_t\big((1-t)x + ty\big) \rangle \gamma(dx, dy) \\ &= \int_{\mathbb{R}^{2d}} \langle \nabla \varphi\big(t, (1-t)x + ty\big), y - x \rangle \gamma(dx, dy) \\ &= \frac{d}{dt} \int_{\mathbb{R}^{2d}} \varphi\big(t, (1-t)x + ty\big) \gamma(dx, dy) - \int_{\mathbb{R}^{2d}} \partial_t \varphi\big(t, (1-t)x + ty\big) \gamma(dx, dy). \\ &= \frac{d}{dt} \int_{\mathbb{R}^d} \varphi\big(t, z\big) \sigma_t(dz) - \int_{\mathbb{R}^d} \partial_t \varphi\big(t, z\big) \sigma_t(dz). \end{aligned}$$

Consequently, since $\varphi(0, \cdot) = \varphi(1, \cdot) \equiv 0$, integrating with respect to t, we conclude

$$\int_0^1 dt \int_{\mathbb{R}^d} \Big(\partial_t \varphi\big(t, z\big) + \langle \nabla \varphi(t, z), v_t(z) \rangle \Big) \sigma_t(dz) = 0$$

QED.

Chapter 4

Geometry on $\mathcal{P}_2(\mathbb{M})$ and concepts of differentials

4.1 Lecture 10; Velocities for geodesic paths (Sep 27)

Definition 4.1. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

(i) We define $L^2(\mu)$ to be the set of Borel maps $v : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$\|v\|_\mu^2 := \int_{\mathbb{R}^d} |v(x)|^2 \mu(dx) < \infty.$$

(ii) Note $L^2(\mu)$ is a Hilbert space which contains $\nabla C_c^\infty(\mathbb{R}^d)$. We define $T_\mu\mathcal{P}_2(\mathbb{R}^d)$ to be the closure of $\nabla C_c^\infty(\mathbb{R}^d)$ in $L^2(\mu)$:

$$T_\mu\mathcal{P}_2(\mathbb{R}^d) = \overline{\nabla C_c^\infty(\mathbb{R}^d)}^{L^2(\mu)}$$

(iii) We call $T_\mu\mathcal{P}_2(\mathbb{R}^d)$ the tangent space to $\mathcal{P}_2(\mathbb{R}^d)$ at μ and refer to any element of $T_\mu\mathcal{P}_2(\mathbb{R}^d)$ as a tangent vector to $\mathcal{P}_2(\mathbb{R}^d)$ at μ.

Remark 4.2. Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let γ and v be as in Theorem 3.34.

(i) By Theorem 3.34, any $\gamma \in \Gamma(\mu, \nu)$ can be used to construct a geodesic of (minimal length) constant speed connecting μ to ν. Furthermore, (although we did not prove it here) one can show that every geodesic of constant speed connecting μ to ν is necessarily of the form that appear in Theorem 3.34 (cf. Theorem 7.2.2 [2]).

(ii) Let ϕ and ϕ_t be as in the theorem and set

$$\psi_t(y) := (1-t)\phi_t^*(y) + t\frac{|y|^2}{2}.$$

Let $(x, y) \in \mathrm{spt}(\gamma)$ and for $t \in (0, 1)$ set $z = (1-t)x + ty$. By Exercise 3.25 (ii), $z \in \partial_\cdot\phi_t(x)$ and so, by (i) of the same exercise, $x \in \partial_\cdot\phi_t^*(z) = \{\nabla\phi_t^*(z)\}$. We use Theorem 3.34 to obtain

$$v_t(z) = y - x = \frac{z-x}{t} = \frac{z - \nabla\phi_t^*(z)}{t} = \nabla\left(\frac{\frac{|z|^2}{2} - \phi_t^*(z)}{t}\right) =: \nabla F_t(z).$$

Hence, v_t is the gradient of a function, which we have denoted as F_t.

(iii) Not only Theorem 3.34 ensures that $\|v_t\|_{\sigma_t} = W_2(\mu, \nu)$, by Exercise 3.25 (i)

$$\frac{-I}{1-t} \leq \nabla^2 F_t(z) = \frac{I - \nabla^2 \phi_t^*(z)}{t} \leq \frac{I}{t}.$$

Consequently, ∇F_t is a Lipschitz map for $t \in (0,1)$. One can check that in fact $v_t \in T_{\sigma_t}\mathcal{P}_2(\mathbb{R}^d)$.

Conclusions 4.3. *Every geodesic $t \in [0,1] \to \sigma_t \in \mathcal{P}_2(\mathbb{R}^d)$ of constant speed is driven by a velocity vector field $v : (0,1) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that v_t is the gradient of a function and $v_t \in L^2(\sigma_t)$ and we further have*

$$\int_0^1 \|v_t\|_{\sigma_t}^2 dt < \infty.$$

This fact is only a partial justification of for terming $T_\mu\mathcal{P}_2(\mathbb{R}^d)$ the tangent space. A full justification is given by the following remarkable theorem due to Ambrosio, Gigli and Savaré [2].

Theorem 4.4 (Chapter 8 [2])**.** *Let $t \in [0,1] \to \sigma_t \in \mathcal{P}_2(\mathbb{R}^d)$ be a curve continuous for the narrow convergence. The following are equivalent:*

(i) $\sigma \in AC_2\big(0,1;\mathcal{P}_2(\mathbb{R}^d)\big)$.

(ii) There exists a Borel velocity field $v : (0,1) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$v(t,\cdot) \in T_{\sigma_t}\mathcal{P}_2(\mathbb{R}^d) \quad a.e. \quad t \in (0,1), \qquad \int_0^1 \|v(t,\cdot)\|_{\sigma_t}^2 dt < \infty$$

and

$$\partial_t \sigma + \nabla \cdot \big(\sigma v\big) = 0 \qquad in \quad (0,1) \times \mathbb{R}^d$$

in the sense of distributions, meaning

$$\int_0^1 dt \int_{\mathbb{R}^d} \Big(\partial_t \varphi(t,x) + \langle \nabla \varphi(t,x), v(t,x) \rangle\Big) \sigma_t(dx) = 0 \qquad \forall \varphi \in C_c^\infty\big((0,1) \times \mathbb{R}^d\big) \tag{4.1}$$

In either case the metric derivative

$$|\sigma'|(t) := \lim_{h \to 0} \frac{W_2(\sigma_{t+h}, \sigma_t)}{|h|}$$

exists almost everywhere. For any velocity v, we have $\|v(t,\cdot)\|_{L^2(\sigma_t)} \geq |\sigma'|(t)$ almost everywhere but, we can choose v to be the unique velocity of minimal norm in the sense that $\|v(t,\cdot)\|_{L^2(\sigma_t)} = |\sigma'|(t)$ almost everywhere.

4.2 Lecture 11: Wasserstein gradient and chain rule (Oct 04)

Throughout this lecture we fix $U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \cup \{\pm\infty\}$ and let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. We denote as π_μ : the orthogonal projection operator of $L^2(\mu)$ onto $T_\mu\mathcal{P}_2(\mathbb{R}^d)$.

Remark 4.5. Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma \in \Gamma_0(\mu, \nu)$.

(i) If μ vanishes on $(d-1)$–rectifiable sets and $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ is a convex lower semicontinuous function such that $Id - \nabla\phi \in L^2(\mu)$ then $Id - \nabla\phi \in T_\mu\mathcal{P}_2(\mathbb{R}^d)$.

(ii) For any $w \in \Big[T_\mu\mathcal{P}_2(\mathbb{R}^d)\Big]^\perp$ (cf. [2]) we have

$$\int_{\mathbb{R}^{2d}} \langle w(x), y - x\rangle \gamma(dx, dy) = 0.$$

When μ vanishes on $(d-1)$–rectifiable sets so that there exists a convex lower semicontinuous function $\phi : \mathbb{R}^d \mapsto (-\infty, \infty]$ such that $\gamma = (Id \times \nabla\phi)_\sharp\mu$ then (i) is easy to prove if we further assume $\nabla\phi \equiv Id$ outside a ball of large radius. Indeed, in this case (i) ensures

$$\int_{\mathbb{R}^{2d}} \langle w(x), y - x\rangle \gamma(dx, dy) = \int_{\mathbb{R}^d} \Big\langle w(x), \nabla\Big(\phi(x) - \frac{|x|^2}{2}\Big)\Big\rangle \mu(dx) = 0.$$

We would like to define the differential of U at μ. Since $\mathcal{P}_2(\mathbb{R}^d)$ is not a linear space the following expansion

$$U(\nu) = U(\mu) + \text{Linear}(\nu - \mu) + o\big(W_2(\mu, \nu)\big)$$

does not make sense. We propose to replace $\text{Linear}(\nu - \mu)$ by

$$\int_{\mathbb{R}^{2d}} \langle y - x, \xi(x)\rangle \gamma(dx, dy)$$

which is a convex function of γ on $\Gamma_0(\mu, \nu)$. The first natural question is how to choose $\gamma \in \Gamma_0(\mu, \nu)$. The next theorem supports the fact that we can choose $\gamma \in \Gamma_0(\mu, \nu)$ any way we want.

Theorem 4.6 (cf. [30])**.** *Assume $\mu \in \mathrm{dom}(U)$ and let $\xi \in L^2(\mu)$. The following are equivalent:*

(i)

$$U(\nu) \geq U(\mu) + \inf_{\gamma \in \Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle y - x, \xi(x)\rangle \gamma(dx, dy) + o\big(W_2(\mu, \nu)\big)$$

(ii)

$$U(\nu) \geq U(\mu) + \inf_{\gamma \in \Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle y - x, \pi_\mu(\xi)(x)\rangle \gamma(dx, dy) + o\big(W_2(\mu, \nu)\big)$$

(iii)

$$U(\nu) \geq U(\mu) + \sup_{\gamma \in \Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle y - x, \xi(x)\rangle \gamma(dx, dy) + o\big(W_2(\mu, \nu)\big)$$

(iv)

$$U(\nu) \geq U(\mu) + \sup_{\gamma \in \Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle y - x, \pi_\mu(\xi)(x)\rangle \gamma(dx, dy) + o\big(W_2(\mu, \nu)\big)$$

Proof: Recall $\pi_\mu : L^2(\mu) \mapsto T_\mu \mathcal{P}_2(\mathbb{R}^d)$ is orthogonal projection operator. Since $\xi - \pi_\mu(\xi) \in \Big[T_\mu \mathcal{P}_2(\mathbb{R}^d)\Big]^\perp$, thanks to Remark 4.5 we conclude that (i) is equivalent to (ii) and (iii) is equivalent to (iv). It remains to show that (ii) is equivalent to (iv). In the sequel, we assume $\xi \in T_\mu \mathcal{P}_2(\mathbb{R}^d)$, which means $\xi = \pi_\mu(\xi)$.

(a) At this step, further make the stronger assumption that $\xi = \nabla\varphi$ where $\varphi \in C_c^\infty(\mathbb{R}^d)$. Note if we set $C := \|\nabla^2\varphi\|_{L^\infty}$ then there is a bounded function $r : \mathbb{R}^{2d} \mapsto [-C, C]$ such that

$$\varphi(y) = \varphi(x) + \langle \nabla\varphi(x), y - x\rangle + \frac{r(x,y)}{2}|x-y|^2 \qquad \forall x, y \in \mathbb{R}^d.$$

Let $\gamma, \bar\gamma \in \Gamma_0(\mu,\nu)$. We have

$$i[\gamma] =: \int_{\mathbb{R}^{2d}} (\varphi(y) - \varphi(x))\gamma(dx,dy) = \int_{\mathbb{R}^{2d}} \langle \nabla\varphi(x), y-x\rangle \gamma(dx,dy) + \int_{\mathbb{R}^{2d}} \frac{r(x,y)}{2}|x-y|^2\gamma(dx,dy)$$

and

$$i[\bar\gamma] =: \int_{\mathbb{R}^{2d}} (\varphi(y) - \varphi(x))\bar\gamma(dx,dy) = \int_{\mathbb{R}^{2d}} \langle \nabla\varphi(x), y-x\rangle \bar\gamma(dx,dy) + \int_{\mathbb{R}^{2d}} \frac{r(x,y)}{2}|x-y|^2\bar\gamma(dx,dy).$$

Since

$$i[\gamma] = \int_{\mathbb{R}^d} \varphi(y)\nu(dy) - \int_{\mathbb{R}^d} \varphi(x)\mu(dx) = i[\bar\gamma]$$

are independent of γ and $\bar\gamma$ but depends only on μ and ν, we conclude

$$\int_{\mathbb{R}^{2d}} \langle \nabla\varphi(x), y-x\rangle (\gamma - \bar\gamma)(dx,dy) = \int_{\mathbb{R}^{2d}} \frac{r(x,y)}{2}|x-y|^2(\gamma-\bar\gamma)(dx,dy)$$

Thus,

$$\left|\int_{\mathbb{R}^{2d}} \langle \nabla\varphi(x), y-x\rangle (\gamma - \bar\gamma)(dx,dy)\right| \le \int_{\mathbb{R}^{2d}} \frac{C}{2}|x-y|^2(\gamma+\bar\gamma)(dx,dy) \le C W_2^2(\mu,\nu).$$

This concludes the proof of the Lemma when $\xi = \nabla\varphi$.

(b) We use an approximation argument to extend the proof to $\xi \in T_\mu \mathcal{P}_2(\mathbb{R}^d)$. QED.

Definition 4.7. Assume $\mu \in \mathrm{dom}(U)$.

(i) The subgradient of U at $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is the set of all $\xi \in L^2(\mu)$ such that

$$U(\nu) \ge U(\mu) + \inf_{\gamma\in\Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle \xi(x), y-x\rangle \gamma(dx,dx) + o\big(W_2(\mu,\nu)\big),$$

for $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. We denote this set by $\partial_\cdot U(\mu)$.

(ii) We say that ξ belongs to the supergradient of U at μ if

$$U(\nu) \le U(\mu) + \sup_{\gamma\in\Gamma_0(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle \xi(x), y-x\rangle \gamma(dx,dx) + o\big(W_2(\mu,\nu)\big),$$

for $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. We denote this set by $\partial^\cdot U(\mu)$.

Theorem 4.8. *Assume* $\mu \in \mathrm{dom}(U)$.

(i) If both $\partial_{\cdot} U(\mu)$ *and* $\partial^{\cdot} U(\mu)$ *are not empty then they are equal and* $\pi_\mu(\xi_1) = \pi_\mu(\xi_2)$ *for any* $\xi_1 \in \partial_{\cdot} U(\mu)$ *and* $\xi_2 \in \partial^{\cdot} U(\mu)$.

(ii) If $\partial_{\cdot} U(\mu)$ *is not empty, then it is a convex set, closed for the weak convergence topology.*

(iii) If $\partial^{\cdot} U(\mu)$ *is not empty, then it is a convex set, closed for the weak convergence topology.*

Proof: (i) Assume both $\partial_{\cdot} U(\mu)$ and $\partial^{\cdot} U(\mu)$ and let $\xi_1 \in \partial_{\cdot} U(\mu)$ and $\xi_2 \in \partial^{\cdot} U(\mu)$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. For any $\epsilon \in (-1,1)$ such that $|\epsilon|$ is small enough the function $x \to \phi_\epsilon(x) := |x|^2/2 + \epsilon\varphi(x)$ is convex and so, if we define $\mu_\epsilon := (\nabla\phi_\epsilon)_\sharp\mu$ then $\gamma_\epsilon := (Id \times \nabla\phi_\epsilon)_\sharp\mu \in \Gamma_0(\mu, \mu_\epsilon)$. Thus

$$W_2(\mu, \mu_\epsilon) = \int_{\mathbb{R}^{2d}} |x-y|^2 \gamma_\epsilon(dx, dy) = \int_{\mathbb{R}^d} |x - \nabla\phi_\epsilon(x)|^2 \mu(dx) = \epsilon^2 \|\nabla\varphi\|_\mu^2$$

and so,

$$U(\mu_\epsilon) \geq U(\mu) + \int_{\mathbb{R}^{2d}} \langle \xi_1(x); y - x\rangle \gamma_\epsilon(dx, dy) + o\big(W_2(\mu, \mu_\epsilon)\big).$$

This means

$$U(\mu_\epsilon) \geq U(\mu) + \epsilon\langle \xi_1; \nabla\varphi\rangle_\mu + o(\epsilon).$$

Similarly,

$$U(\mu_\epsilon) \leq \epsilon\langle \xi_2; \nabla\varphi\rangle_\mu + o(\epsilon).$$

We combine the two previous identities to conclude

$$0 \geq \epsilon\langle \xi_1 - \xi_2; \nabla\varphi\rangle_\mu + o(\epsilon).$$

We divide both sides of the previous inequality by $\epsilon > 0$ and then by $\epsilon < 0$ and let ϵ tend to 0 in the subsequent inequalities to observe

$$0 = \langle \xi_1 - \xi_2; \nabla\varphi\rangle_\mu \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^d).$$

Thus, $\pi_\mu(\xi_1) = \pi_\mu(\xi_2)$. By Theorem 4.6, since $\pi_\mu(\xi_1) = \pi_\mu(\xi_2) \in \partial_{\cdot} U(\mu)$, we have $\xi_2 \in \partial_{\cdot} U(\mu)$. Similarly, $\xi_1 \in \partial^{\cdot} U(\mu)$.

(ii) Assume $\partial_{\cdot} U(\mu)$ is not empty. It is easy to check it is a convex set. We skip the argument showing the set is closed for the weak convergence topology.

(iii) The proof of (iii) follows the same lines of arguments as that of (ii). QED.

Definition 4.9. Assume $\mu \in \mathrm{dom}(U)$.

(i) Since $\|\cdot\|_\mu$ is uniformly convex, then unless $\partial_{\cdot} U(\mu)$ is empty, as a convex set which is closed for the weak topology, it admits a unique element of minimal norm which is referred to as the Wasserstein gradient of U at μ and denoted as $\nabla_{\omega_2} U(\mu)$.

(ii) Similarly, if $\partial^{\cdot} U(\mu)$ is empty, it admits a unique element of minimal norm which is referred to as the Wasserstein gradient of U at μ and denoted as $\nabla_{\omega_2} U(\mu)$. Note by Theorem 4.8, $\nabla_{\omega_2} U(\mu)$ is well defined if both convex sets are not empty.

(iii) If both $\partial.U(\mu)$ and $\partial^{\cdot}U(\mu)$ are not empty, we say that U is differentiable at μ.

Lemma 4.10. *Assume U is differentiable at $\mu \in \text{dom}(U)$. Then*

$$\partial.U(\mu) = \nabla_{\omega_2} U(\mu) + \Big[T_\mu \mathcal{P}_2(\mathbb{R}^d)\Big]^\perp$$

.

Proof: By Remark 4.5 the inclusion

$$\xi + \Big[T_\mu \mathcal{P}_2(\mathbb{R}^d)\Big]^\perp \subset \partial.U(\mu)$$

holds for any $\xi \in \partial.U(\mu)$. Thus, it only remains to prove the converse inclusion when U is differentiable at $\mu \in \text{dom}(U)$. Let $\xi_0 := \nabla_{\omega_2} U(\mu)$ and let $\xi \in \partial.U(\mu)$. By Theorem 4.8, $\pi_\mu(\xi) = \xi_0$ and so,

$$\xi = \xi_0 + \xi - \pi_\mu(\xi) \in \xi_0 + \Big[T_\mu \mathcal{P}_2(\mathbb{R}^d)\Big]^\perp.$$

This concludes the proof. QED.

Exercise 4.11. Let $W : \mathcal{P}_2(\mathbb{M}) \mapsto [-\infty, \infty]$ and let $\xi \in L^2(\mu)$. Define

$$w(q) = W(\delta_q) \quad q \in \mathbb{M}.$$

Show that if $q_0 \in \mathbb{M}$ and $\xi \in \partial.W(\delta_{q_0})$ then $\xi(q_0) \in \partial.w(q_0)$. Similarly, if $\bar{\xi} \in \partial^{\cdot}W(\delta_{q_0})$ then $\bar{\xi}(q_0) \in \partial^{\cdot}w(q_0)$. In conclusion, if W is differentiable at δ_{q_0} then w is differentiable at q_0 and $\nabla w(q_0) = \nabla_{\omega_2} W(\delta_{q_0})$.

Proof: Assume $\xi \in \partial.W(\delta_{q_0})$ Then for $\mu := \delta_{q_0}$ and $\nu := \delta_q$ we have $\Gamma_0(\mu, \nu) = \{\delta_{(q_0,q)}\}$ and so, the identity

$$W(\nu) \geq W(\mu) + \int_{\mathbb{M}^2} \langle \xi(x), y - x\rangle \gamma(dx, dy) + o\big(W_2(\mu, \nu)\big)$$

reads off

$$w(q) \geq w(q_0) + \langle \xi(q_0), q - q_0\rangle + o(|q_0 - q|)$$

Thus, $\xi(q_0) \in \partial.w(q_0)$.

$$\nabla w(q_0) = \nabla_{\omega_2} W(\delta_q)(q).$$

The proof for the sub-differential is identical and so, if W is differentiable at δ_{q_0} then w is differentiable at q_0 and $\nabla w(q_0) = \nabla_{\omega_2} W(\delta_{q_0})$. QED.

Remark 4.12 (Proposition 8.4.6 [2]). Let $\sigma \in AC_2(a, b; \mathcal{P}_2(\mathbb{R}^d))$ and let v be the velocity of minimal norm of σ as given in Theorem 4.4. There exists a set $I \subset (a, b)$ of full measure such that for any $t \in I$ and any $\gamma_h \in \Gamma_0\big(\sigma_t, \sigma_{t+h}\big)$

(i)

$$\lim_{h \to 0} \frac{W_2(\sigma_{t+h}, \sigma_t)}{|h|} = |\sigma'|(t).$$

(ii)

$$\lim_{h\to 0}\Big(\pi^0 \times \frac{\pi^1-\pi^0}{h}\Big)_\sharp \gamma_h = (Id\times v_t)_\sharp \sigma_t \quad \text{in} \quad \mathcal{P}_2(\mathbb{R}^{2d}).$$

The latter statement means

$$\lim_{h\to 0}\int_{\mathbb{R}^{2d}} F\Big(x, \frac{y-x}{h}\Big)\gamma_h(dx,dy) = \int_{\mathbb{R}^d} F\Big(x, v_t(x)\Big)\sigma_t(dx) \tag{4.2}$$

for any $F\in C(\mathbb{R}^{2d})$ which grows at most quadratically at infinity: there exists $C>0$ such that

$$|F(x,z)|\le C(1+|x|^2+|z|^2) \qquad \forall x,z\in\mathbb{R}^d.$$

Lemma 4.13. *Let σ, v and $I\subset(0,1)$ be as in Remark 4.12. If U is differentiable at σ_t and $t\in I$ then*

$$\lim_{h\to 0}\frac{U(\sigma_{t+h})-U(\sigma_t)}{h} = \int_{\mathbb{R}^{2d}}\langle v_t(x), \nabla_{\omega_2}U(\sigma_t)\rangle\sigma_t(dx).$$

Proof: Set $\xi := \nabla_{\omega_2}U(\sigma_t)$. We have

$$\frac{U(\sigma_{t+h})-U(\sigma_t)}{h} - \int_{\mathbb{R}^d}\langle\xi(x), v_t(x)\rangle\sigma_t(dx) = a^1(h)+a^2(h)$$

where

$$a^1(h) := \frac{U(\sigma_{t+h})-U(\sigma_t)-\int_{\mathbb{R}^{2d}}\langle\xi(x), y-x\rangle\gamma_h(dx,dy)}{h}$$

and

$$a^2(h) := \int_{\mathbb{R}^{2d}}\Big\langle\xi(x), \frac{y-x}{h}\Big\rangle\gamma_h(dx,dy) - \int_{\mathbb{R}^d}\langle\xi(x), v_t(x)\rangle\sigma_t(dx)$$

Let $\epsilon>0$. Since $\xi\in T_\mu\mathcal{P}_2(\mathbb{R}^d)$, we can choose $\varphi\in C_c^\infty(\mathbb{R}^d)$ such that

$$\|\xi-\nabla\varphi\|_{\sigma_t}\le\epsilon.$$

Write $a^2(h)=a^2_\epsilon(h)+\bar a^2_\epsilon(h)$ where

$$a^2_\epsilon(h) := \int_{\mathbb{R}^{2d}}\Big\langle\nabla\varphi(x), \frac{y-x}{h}\Big\rangle\gamma_h(dx,dy) - \int_{\mathbb{R}^d}\langle\nabla\varphi(x), v_t(x)\rangle\sigma_t(dx)$$

and

$$\bar a^2_\epsilon(h) := \int_{\mathbb{R}^{2d}}\Big\langle\xi(x)-\nabla\varphi(x), \frac{y-x}{h}\Big\rangle\gamma_h(dx,dy) - \int_{\mathbb{R}^d}\langle\xi(x)-\nabla\varphi(x), v_t(x)\rangle\sigma_t(dx).$$

We have

$$|\bar a^2_\epsilon(h)|\le\|\xi-\nabla\varphi\|_{\sigma_t}\frac{W_2(\sigma_{t_h},\sigma_t)}{|h|}+\|\xi-\nabla\varphi\|_{\sigma_t}\|v_t\|_{\sigma_t}\le\epsilon\Big(\frac{W_2(\sigma_{t_h},\sigma_t)}{|h|}+\|v_t\|_{\sigma_t}\Big).$$

Thus,

$$\limsup_{h\to 0}|\bar a^2_\epsilon(h)|\le\epsilon\Big(|\sigma'|(t)+\|v_t\|_{\sigma_t}\Big). \tag{4.3}$$

Using $F(x,z) \equiv \langle \nabla\varphi(x), z\rangle$ in (4.2) we obtain

$$\limsup_{h\to 0} a^2_\epsilon(h) = 0. \tag{4.4}$$

By the fact that $\xi = \nabla_{\omega_2} U(\sigma_t)$ we obtain

$$\limsup_{h\to 0} a^1(h) = 0. \tag{4.5}$$

We combine (4.3)-(4.5) to infer

$$\limsup_{h\to 0}\left|\frac{U(\sigma_{t+h}) - U(\sigma_t)}{h} - \int_{\mathbb{R}^d} \langle \xi(x), v_t(x)\rangle \sigma_t(dx)\right| \leq \epsilon\Big(|\sigma'|(t) + \|v_t\|_{\sigma_t}\Big) \qquad \forall \epsilon > 0. \tag{4.6}$$

Thus, the lim sup at the left handside of (4.6) is null. This completes the verification of the lemma. QED.

4.3 Lectures 12, 13: Gradient on $\mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{H}$ (Oct 09, 11)

Continuing with the notation of Section 4.2, we fix $U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}\cup\{\pm\infty\}$ and let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Following the notation of Section 3.2 we set

$$\mathbb{H} = L^2(\mathbb{B}_d, \mathbb{R}^d, \mathcal{L}^d_{\mathbb{B}_d}),$$

where $\mathbb{B}_d$ is the d–dimensional ball in $\mathbb{R}^d$ of volume 1, centered at the origin. This is a Hilbert space endowed with the inner product which we denote as $\langle\cdot,\cdot\rangle_{L^2}$, and it induces a norm $\|\cdot\|_{L^2}$.

We lift the function U to obtain a function $\hat{U} : \mathbb{H} \mapsto \mathbb{R}\cup\{\pm\infty\}$ defined as

$$\hat{U}(Y) := U\big(Y_\sharp \mathcal{L}^d_{\mathbb{B}_d}\big)$$

Definition 4.14. Let $X \in \mathrm{dom}(\hat{U})$. We recall the following standard definitions.

(i) The subgradient of $\hat{U}$ at X is the set of all $\zeta \in \mathbb{H}$ such that

$$\hat{U}(Y) \geq \hat{U}(X) + \langle \zeta, Y - X\rangle_{L^2} + o\big(\|Y - X\|_{L^2}\big),$$

for $Y \in \mathbb{H}$. We denote this set by $\partial_\cdot \hat{U}(X)$.

(ii) We say that ζ belongs $\partial^\cdot \hat{U}(X)$, the supergradient of $\hat{U}$ at X if $-\zeta \in \partial_\cdot(-\hat{U})(X)$.

(iii) We say that $\hat{U}$ is differentiable at X if both $\partial_\cdot \hat{U}(X)$ and $\partial^\cdot \hat{U}(X)$ are not empty.

Remark 4.15. Let $X \in \mathbb{H}$. As done in Section 4.2 for U, have that if $\partial_\cdot \hat{U}(X)$ and $\partial^\cdot \hat{U}(X)$ are not empty then they must be equal. Furthermore, when these sets are not empty, they are closed convex set and so, they have a unique element of minimal norm which is referred to as the gradient of $\hat{U}$ at X. In this case, we denote the gradient of $\hat{U}$ at X as $\nabla_{L^2}\hat{U}(X)$.

Theorem 4.16. *Assume $X \in \mathbb{H}$ is such that $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$ and let $\xi \in L^2(\mu)$.*

(i) If $\xi \circ X \in \partial_{\cdot}\hat{U}(X)$ *then* $\xi \in \partial_{\cdot}U(\mu)$.

(ii) If $\xi \in \partial_{\cdot}U(\mu) \cap T_\mu \mathcal{P}_2(\mathbb{R}^d)$ *then* $\xi \circ X \in \partial_{\cdot}\hat{U}(X)$

Proof: The proof of the theorem is not difficutl and is due to [30]. QED.

The previous theorem leads us to introduce the following special subset of $\mathbb{H}$.

Definition 4.17. Let $X \in \mathbb{H}$ be such that $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$. We define

(i)
$$F(X) := \{\xi \circ X \mid \xi \in L^2(\mu)\}, \qquad .$$

(ii)
$$\nabla F(X) := \{\xi \circ X \mid \xi \in T_\mu \mathcal{P}_2(\mathbb{R}^d)\}$$

Exercise 4.18. Let $X \in \mathbb{H}$ be such that $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$. Show the following:

$$F(X) = \overline{\{\phi \circ X \mid \phi \in C_c(\mathbb{R}^d, \mathbb{R}^d)\}}^{L^2(\mu)} =: A \quad \text{and} \quad \nabla F(X) = \overline{\{\nabla \varphi \circ X \mid \varphi \in C^1_c(\mathbb{R}^d)\}}^{L^2(\mu)} =: B.$$

Proof: (i) Recall that if $\xi_1, \xi_2 \in L^2(\mu)$ then

$$\|\xi_1 - \xi_2\|_\mu = \|\xi_1 \circ X - \xi_2 \circ X\|_{L^2}.$$

Thus,

$$(\xi_n)_n \subset L^2(\mu) \text{ is a Cauchy sequence if and only if } (\xi_n \circ X)_n \subset \mathbb{H} \text{ is a Cauchy sequence.} \tag{4.7}$$

Claim 1. $F(X)$ is a closed set

Proof of Claim 1. Assume $(\xi_n \circ X)_n \subset F(X)$ converges to $\zeta \in \mathbb{H}$ and so, it is a Cauchy sequence. By (4.7) $(\xi_n)_n \subset L^2(\mu)$ is a Cauchy sequence and so, converges to some $\xi \in L^2(\mu)$. We use again (4.7) conclude that $(\xi_n \circ X)_n \subset \mathbb{H}$ converges to $\xi \circ X$ and so, $\zeta = \xi \circ X$. This verifies the claim.

Since $\{\phi \circ X \mid \phi \in C_c(\mathbb{R}^d, \mathbb{R}^d)\} \subset F(X)$ and by Claim 1, the latter set is closed, we conclude $A \subset F(X)$. It remains to show the reverse inequality.

Claim 2. We have $F(X) \subset A$.

Proof of Claim 2. Let $\zeta = \xi \circ X \in F(X)$, where $\xi \in L^2(\mu)$. Since $C_c(\mathbb{R}^d, \mathbb{R}^d)$ is dense in $L^2(\mu)$, there exists $(\xi_n)_n \subset L^2(\mu)$ converging to ξ. By (4.7), $(\xi_n \circ X)_n \subset A$ converges to $\xi \circ X = \zeta$ and so, $\zeta \in A$. This verifies the claim and yields to the conclusion that $A = F(X)$.

(ii) The proof for $\nabla F(X)$ is done in a similar manner. QED.

Theorem 4.19 (Hard to prove)**.** *Assume* $\mu \in \text{dom}(U)$ *and there exists a convex function* $u : \mathbb{B}_d \mapsto \mathbb{R}$ *such that* $(\nabla u)_\sharp \mathcal{L}^d_{\mathbb{B}_d} = \mu$. *If we set* $X := \nabla u$ *and* $\zeta \in \mathbb{H}$ *is a Borel map such that* $\zeta \in \partial^{\cdot} \hat{U}(X)$ *then*

$$\text{proj}_{F[X]}\zeta \in \partial^{\cdot} \hat{U}(X) \text{ and } \text{proj}_{\nabla F[X]}\zeta \in \partial^{\cdot} \hat{U}(X).$$

Proof: We cannot present the proof of the theorem here as in it a non trivial result which relies on a deep decomposition result due to Caravenna and Daneri [10] which asserts the following: given any convex function $u : \mathbb{B}_d \to \mathbb{R}$, one can disintegrate $\mathcal{L}^d_{\mathbb{B}_d}$ into probability measures $\{\nu_y \ : \ y \in \nabla u(\mathbb{B}_d)\}$ such that each ν_y is supported by the level set $\{\nabla u = y\}$ and is comparable to the Hausdorff measure $\mathcal{H}^{k(y)}$, where $k(y) \in \{0, 1, ..., d\}$. The proof of Theorem 4.19 is due to [30]. QED.

Exercise 4.20. Let $X, \bar X \in \mathbb{H}$ be such that $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d} = \bar X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$. Show that $\partial.\hat U(X) \neq \emptyset$ if and only if $\partial.\hat U(\bar X) \neq \emptyset$. In this case, we have

$$\min_{\zeta \in}\{\|\zeta\| \mid \zeta \in \partial.\hat U(X)\} = \min_{\bar\zeta \in}\{\|\bar\zeta\| \mid \bar\zeta \in \partial.\hat U(\bar X)\} \tag{4.8}$$

Proof: We prove for instance that any $\zeta \in \partial.\hat U(X)$ can be used construct an element of $\partial.\hat U(\bar X)$. Let $G_{\mathbb{B}_d}$ be the set of almost invertible Lebesgue measure preserving maps introduced in Section 3.1.1. The same section ensures existence of a sequence $(S_n)_n \subset G_{\mathbb{B}_d}$ such that

$$\|X \circ S_n - \bar X\| = \|X - \bar X \circ S_n^{-1}\| \le \frac{1}{n}. \tag{4.9}$$

Claim 1. Any $\bar\zeta$ point of accumulation of $(\zeta \circ S_n)_n \subset \mathbb{H}$ for the weak topology, belongs to $\partial.\hat U(\bar X)$.

Proof of the Claim 1. We have

$$\hat U(\bar X + H) - \hat U(\bar X) = \hat U(\bar X \circ S_n^{-1} + H \circ S_n^{-1}) - \hat U(X) = \hat U\Big(X + H_n\Big) - \hat U(X) \ge \langle \zeta; H_n\rangle + o\big(\|H_n\|\big), \tag{4.10}$$

where we have set

$$H_n := \bar X \circ S_n^{-1} - X + H \circ S_n^{-1}.$$

Note

$$\|H_n\| \le \|\bar X \circ S_n^{-1} - X\| + \|H\| \le \frac{1}{n} + \|H\|. \tag{4.11}$$

Letting n tend to ∞ in (4.10) and thanks to (4.11), we conclude

$$\hat U(\bar X + H) - \hat U(\bar X) \ge \langle \bar\zeta; H\rangle + o\big(\|H\|\big).$$

This concludes the proof of the claim.

Claim 2. The identity in (4.8) holds.

Proof of the Claim 2. Let $r = \|\nabla_{L^2} \hat U(X)\|$ and let $(S_n)_n$ be as above a let $\bar\zeta$ be a point of accumulation of $(\zeta \circ S_n)_n \subset \mathbb{H}$ for the weak topology. We have

$$\min_{\zeta^* \in}\{\|\zeta^*\| \mid \zeta^* \in \partial.\hat U(\bar X)\} \le \|\bar\zeta\| \le \lim_n \|\zeta \circ S_n\| = \|\nabla_{L^2} \hat U(X)\| = \min_{\zeta^* \in}\{\|\zeta^*\| \mid \zeta^* \in \partial.\hat U(X)\}.$$

By symmetry, we obtain the reverse inequality. QED.

Exercise 4.21 (Generalization of Theorem 4.19)**.** Let $X \in \mathbb{H}$ be such that $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$. Show that

$$\mathrm{proj}_{F[X]}\zeta, \ \mathrm{proj}_{\nabla F[X]}\zeta \in \partial.\hat U(X).$$

Proof: We refer the reader to Theorem 3.19 [30].

Corollary 4.22. *Let $X \in \mathbb{H}$ and assume $\mu = X_\sharp \mathcal{L}^d_{\mathbb{B}_d}$. The following are equivalent*

(i) $\partial.U(\mu) \neq \emptyset$.

(ii) $\partial.\hat{U}(X) \neq \emptyset$.

In either case, $\nabla_{\omega_2} U(\mu) \circ X = \nabla_{L^2} \hat{U}(X)$.

Proof: *Part 1.* If $\partial.U(\mu) \neq \emptyset$ then $\nabla_{\omega_2} U(\mu) \in \partial.U(\mu) \cap T_\mu \mathcal{P}_2(\mathbb{R}^d)$ and so, by Theorem 4.16 $\nabla_{\omega_2} U(\mu) \circ X \in \partial.\hat{U}(X)$. Hence,

$$\|\nabla_{L^2} \hat{U}(X)\| \leq \|\nabla_{\omega_2} U(\mu) \circ X\| = \|\nabla_{\omega_2} U(\mu)\|_\mu \tag{4.12}$$

Conversely, assume $\partial.\hat{U}(X) \neq \emptyset$ and so, $\zeta =: \nabla_{L^2} \hat{U}(X)$. exists. By Exercise 4.21 $\mathrm{proj}_{\nabla F[X]} \zeta \in \partial.\hat{U}(X)$. Since ζ is of minimal norm, we have $\zeta = \mathrm{proj}_{\nabla F[X]} \zeta$ and so, there exists $\xi \in T_\mu \mathcal{P}_2(\mathbb{R}^d)$ such that $\zeta = \xi \circ X$. By Theorem 4.16, $\xi \in \partial.U(\mu)$. Thus,

$$\|\nabla_{\omega_2} U(\mu)\|_\mu \leq \|\xi\|_\mu = \|\zeta\| \tag{4.13}$$

Part 2. Assume (i) and (ii) hold. We combine (4.12) and (4.13) to conclude that

$$\|\nabla_{L^2} \hat{U}(X)\| = \|\nabla_{\omega_2} U(\mu) \circ X\|.$$

Thus, $\nabla_{L^2} \hat{U}(X) = \nabla_{\omega_2} U(\mu) \circ X$. QED.

Lemma 4.23. *Assume $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is intrinsically continuously differentiable in $\mathcal{O}$ (cf. Definition 2.2 in 2) and there for each compact set $\mathcal{K} \subset \mathcal{P}_2(\mathbb{M})$ there is a constant $c(\mathcal{K}) > 0$ such that* (2.5) *holds. If $\nabla_{\omega_2} U(\mu)$ the Wasserstein gradient of U at μ exists then $\nabla_{\omega_2} U(\mu) = D_\mu U(\cdot, \mu)$.*

Proof: 1. Claim. Since both $\nabla_{\omega_2} U(\mu)$ and $D_\mu U(\cdot, \mu)$ belong to $T_\mu \mathcal{P}_2(\mathbb{M})$, it suffices to show that

$$\big\langle \nabla_{\omega_2} U(\mu), \nabla\varphi \big\rangle_\mu = \big\langle D_\mu U(\cdot, \mu), \nabla\varphi \big\rangle_\mu \qquad \forall \varphi \in C_c^\infty(\mathbb{M}). \tag{4.14}$$

Fix such a φ and define $\sigma_t := (Id + t\nabla\varphi)_\sharp \mu$. There exists $T > 0$ such that $|Id|^2/2 + t\varphi$ is a convex function for any $t \in [0, T]$ and so,

$$\gamma_t =: \big(Id \times (Id + t\nabla\varphi)\big)_\sharp \mu \in \Gamma_0(\mu, \sigma_t) \quad \text{and so} \quad W_2^2(\mu, \sigma_t) = \int_{\mathbb{M}^2} |y - x|^2 \gamma_t(dx, dy) = t^2 \|\nabla\varphi\|_\mu^2$$

Thus,

$$U(\sigma_t) = U(\sigma_0) + \int_{\mathbb{M}^2} \langle \nabla_{\omega_2} U(\mu)(x), y - x \rangle \gamma_t(dx, dy) + o\big(W_2(\mu, \sigma_t)\big) = U(\sigma_0) + t \langle \nabla_{\omega_2} U(\mu), \nabla\varphi \rangle_\mu + o(t)$$

Hence,

$$\lim_{t \to 0} \frac{U(\sigma_t) - U(\sigma_0)}{t} = \langle \nabla_{\omega_2} U(\mu), \nabla\varphi \rangle_\mu. \tag{4.15}$$

2. We use Lemma 2.4 with $S(t, q) := q + t\nabla\varphi(q)$ to conclude that

$$\lim_{t \to 0} \frac{U(\sigma_t) - U(\sigma_0)}{t} = \big\langle D_\mu U(\cdot, \mu), \nabla\varphi \big\rangle_\mu$$

This, together with (4.15) proves (4.14). QED.

4.4 Lectures 14, 15: Hessian and Partial Laplacian on $\mathcal{P}_2(\mathbb{M})$ (Oct 16, 18)

Throughout this lecture

$$U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}, \quad \text{and} \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).$$

Definition 4.24. Suppose U is differentiable in a neighborhood of $\mu \in \mathrm{dom}(U)$ and for any $\zeta \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\nu \mapsto \langle \zeta, \nabla_{\omega_2} U(\nu)\rangle_\nu =: \zeta \cdot (U(\nu))$ is differentiable at μ.

(i) We define $\bar{\mathrm{H}}\mathrm{ess}\, U[\mu] : \nabla C_c^\infty(\mathbb{R}^d) \times \nabla C_c^\infty(\mathbb{R}^d) \mapsto \mathbb{R}$ if the following exists:

$$\bar{\mathrm{H}}\mathrm{ess}\, U[\mu](\zeta_1, \zeta_2) = \zeta_1 \cdot \Big(\zeta_2 \cdot \big(U(\mu)\big)\Big) - \Big(\bar{\nabla}_{\zeta_1}\zeta_2\Big) \cdot (U(\mu))$$

for any $\zeta_1, \zeta_2 \in \nabla C_c^\infty(\mathbb{R}^d)$. Here, $\bar{\nabla}_{\zeta_1}\zeta_2 = \nabla\zeta_2\zeta_1$.

(ii) If there is a constant C such that

$$|\mathrm{Hess}\, U[\mu](\zeta_1, \zeta_2)| \leq C\, \|\zeta_1\|_\mu \, \|\zeta_2\|_\mu$$

for all $\zeta_1, \zeta_2 \in \nabla C_c^\infty(\mathbb{R}^d)$ then $\bar{\mathrm{H}}\mathrm{ess}[\mu]$ has a unique continuous linear extension onto $T_\mu\mathcal{P}_2(\mathbb{R}^d) \times T_\mu\mathcal{P}_2(\mathbb{R}^d)$ which we denote as $\mathrm{Hess}\, U[\mu]$. In that case, we say that U has a Hessian at μ.

Remark 4.25. Assume $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ satisfies the assumptions in Lemma 4.23 and $\frac{\delta U}{\delta \mu}(\cdot, \mu)$ is of class C^2. Then (cf. Subsection 2.2.1 [12])

$$\nabla_q\big(D_\mu U(q, \mu)\big) = \nabla_q^2\Big(\frac{\delta U}{\delta \mu}(q, \mu)\Big)$$

and so, this is a symmetric matrix. Further assume U is twice continuously Fréchet differentiable continuously differentiable, $\frac{\delta^2 U}{\delta \mu^2}$ twice continuously differentiable on $\mathbb{M}^2$ with uniformly bounded second derivative and $(q, x, \mu) \to \nabla_q\Big(\frac{\delta^2 U}{\delta \mu^2}(q, x, \mu)\Big), \nabla_q^2\Big(\frac{\delta^2 U}{\delta \mu^2}(q, x, \mu)\Big)$ are continuous. If U is twice continuously differentiable in the sense of Wasserstein then

$$\nabla^2_{\omega_2} U(q, x, \mu) = \nabla^2_{qx}\Big(\frac{\delta^2 U}{\delta \mu^2}(q, x, \mu)\Big).$$

We would like to state a sufficient condition for $\bar{\mathrm{H}}\mathrm{ess}[\mu]$ to exists and would like to compute this object. Let us assume U is differentiable in a neighborhood of $\mu \in \mathrm{dom}(U)$. The map $x \mapsto \nabla_{\omega_2} U(\nu)(x)$ is normally defined only μ almost everywhere. However, let assume it admits an extension we still denote as $\nabla_{\omega_2} U(\nu)$ and the extension is continuous for ν in a neighborhood of μ. A sufficient condition to have that $\nabla_{\omega_2} U(\nu) \in L^2(\nu)$ is to assume there exists a constant C_μ such that

$$|\nabla_{\omega_2} U(\nu)(x)| \leq C_\mu(1 + |x|) \tag{4.16}$$

for any $x \in \mathbb{R}^d$ and any ν in the neighborhood of μ.

Exercise 4.26. Let $\mu \in \mathcal{P}_2(\mathbb{M})$, $\zeta \in \nabla C_c^\infty(\mathbb{M})$. For $t \in [0,1]$ we set

$$r_t(x,y) = (1-t)x + ty, \quad E(x,y) := \int_0^1 \left|\nabla\zeta(r_t(x,y)) - \nabla\zeta(x)\right|^2 dt \qquad \forall x, y \in \mathbb{M}.$$

Define

$$\epsilon_\zeta(r) := \sup_{W_2(\mu,\nu)\le r} \sup_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{M}^2} (1+|x|)^2 E(x,y)\gamma(dx,dy), \qquad r > 0.$$

Show that $\lim_{r\to 0^+} \epsilon_\zeta(r) = 0$.

Proof: Let $(r_n)_n \subset (0,\infty)$ be a sequence decreasing to 0. It suffices to show that $\lim_{n\to\infty} \epsilon_\zeta(r_n) = 0$ and use the arbitrariness of $(r_n)_n$ to verify the statement in the Exercise. Let $(\nu_n)_n \subset \mathcal{P}_2(\mathbb{M})$ and $\gamma_n \in \Gamma_0(\mu,\mu_n)$ be such that $W_2(\mu,\mu_n) \le r_n$

$$\epsilon_\zeta(r_n) \le \frac{1}{n} + \int_{\mathbb{M}^2} (1+|x|)^2 E(x,y)\gamma_n(dx,dy)$$

Recall (cf. [2] Subsection 5.1.1) a Borel function $g : \mathbb{M} \mapsto [\mathbb{R}]0,\infty)$ is uniformly integrable with respect to $\mathcal{K} \subset \mathcal{P}(\mathbb{M})$ if

$$\lim_{k\to\infty} \int_{\{g\ge k\}} g(x)m(dx) = 0 \qquad \text{uniformly in } m \in \mathcal{K}.$$

Since $(\mu_n)_n$ converges to μ in $\mathcal{P}_2(\mathbb{R}^d)$ we have (cf. [2] Proposition 7.1.5) that $|x|^2$ is uniformly integrable with respect to $\{\mu_n \mid n \in \mathbb{N}\} \cup \{\mu\}$. Since E is bounded, we conclude that $(1+|x|)^2 E(x,y)$ is uniformly integrable with respect to $\{\gamma_n \mid n \in \mathbb{N}\}$ and up to a subsequence which we don't relabel, $(\gamma_n)_n$ converges narrowly (cf. [2]) to some $\gamma \in \Gamma_0(\mu,\mu)$. In fact, the uniform integrability of $(1+|x|)^2 E(x,y)$ implies (cf. Lemma 5.1.7 [2])

$$\lim_{n\to\infty} \int_{\mathbb{M}^2} (1+|x|)^2 E(x,y)\gamma_n(dx,dy) = \int_{\mathbb{M}^2} (1+|x|)^2 E(x,y)\gamma(dx,dy)$$

Since $\Gamma_0(\mu,\mu) = \{(Id \times Id)_\sharp \mu\}$ this reads off

$$\lim_{n\to\infty} \int_{\mathbb{M}^2} (1+|x|)^2 E(x,y)\gamma_n(dx,dy) = \int_{\mathbb{M}} (1+|x|)^2 E(x,x)\mu(dx) = 0.$$

Thus, $\lim_{n\to\infty} \epsilon_\zeta(r_n) = 0$ which concludes the proof. QED.

Definition 4.27. Denoting $[0,\infty)$ as $\mathbb{R}_+$, we say that $\rho \in C(\mathbb{R}_+,\mathbb{R}_+)$ is a modulus if ρ is monotone nondecreasing, subadditive, and $\rho(0) = 0$.

Exercise 4.28. Let $\rho \in C(\mathbb{R}_+,\mathbb{R}_+)$ be a concave modulus.

(i) Show $\rho(t)/t$ is monotone nonincreasing and so, for any $t \ge 0$ and $\epsilon > 0$, we have $\rho(t) \le \rho(\epsilon) + t/\epsilon\rho(\epsilon)$.

(ii) If $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \Gamma_0(\mu,\nu)$ then

$$\int_{\mathbb{R}^{2d}} |x-y|\rho\big(|x-y|\big)\gamma(dx,dy) \le \rho\big(W_2^{\frac{1}{2}}(\mu,\nu)\big)W_2(\mu,\nu)\Big(W_2^{\frac{1}{2}}(\mu,\nu)+1\Big)$$

Proof: (i) Mollifying ρ if necessary, it is not a loss of generality to assume that ρ is of class C^1. We have $t^2(\rho(t)/t)' = \rho'(t)t - \rho(t)$. But $R(t) := -\rho(t)$ is convex and so, for $t > 0$ we have $R(0) - R(t) \geq R'(t)(0 - t)$. This is equivalent to $t^2(\rho(t)/t)' \leq 0$ which proves the first part of the remark. If $s \in [0, \epsilon]$ and $t \in [\epsilon, \infty)$ then

$$\rho(s) \leq \rho(\epsilon) \leq \rho(\epsilon) + \frac{s}{\epsilon}\rho(\epsilon) \quad \text{and so,} \quad \frac{\rho(t)}{t} \leq \frac{\rho(\epsilon)}{\epsilon} \leq \frac{\rho(\epsilon)}{t} + \frac{\rho(\epsilon)}{\epsilon}.$$

This proves (i).

(ii) Let $\gamma \in \Gamma_0(\mu, \nu)$. By (i) for any $\epsilon > 0$,

$$\int_{\mathbb{R}^{2d}} |x - y|\rho\big(|x - y|\big)\gamma(dx, dy) \leq \rho(\epsilon)\Big(\frac{W_2^2(\mu, \nu)}{\epsilon} + \int_{\mathbb{R}^{2d}} |x - y|\gamma(dx, dy)\Big).$$

We apply Cauchy–Schwarz inequality to obtain

$$\int_{\mathbb{R}^{2d}} |x - y|\rho\big(|x - y|\big)\gamma(dx, dy) \leq \rho(\epsilon)W_2(\mu, \nu)\Big(\frac{W_2(\mu, \nu)}{\epsilon} + 1\Big).$$

We conclude the proof by setting $\epsilon := W_2^{1/2}(\mu, \nu)$. QED.

We would like to compute $\nabla_x\nabla_{\omega_2}U(\mu)$ which we will denote as $\tilde{A}(\nu)$ and compute $\nabla_{\omega_2}\nabla_{\omega_2}U(\mu)$ which we will denote as $A_{\mu\mu}$. This will be done through a first order Taylor expansion type on $\mathcal{P}_2(\mathbb{R}^d)$. For that we suppose we are given $\rho, \epsilon : [0, \infty) \to \mathbb{R}$ are nonnegative function (depending on μ) such that $\lim_{t\to 0^+} \epsilon(t) = 0$ and ρ is a concave modulus.

Set

$$\Upsilon(s, t) = (t + s)\Big(\rho(t) + \epsilon(s)\Big).$$

Theorem 4.29 (Sufficient condition for a Hessian to exist; cf. [16])**.** *Suppose U satisfies the assumptions imposed at the beginning of this section. In particular* (4.16) *holds. Suppose there are Borel bounded matrix valued functions $\tilde{A}(\mu) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $A_{\mu\mu} : \mathbb{R}^{2d} \to \mathbb{R}^{d\times d}$ satisfying the following properties: for any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$*

$$\sup_{\gamma\in\Gamma_0(\mu,\nu)} \Big|\nabla_{\omega_2}U(\nu)(y) - \nabla_{\omega_2}U(\mu)(x) - P_\gamma[\mu](x, y)\Big| \leq \Upsilon\Big(W_2(\mu, \nu), |x - y|\Big). \tag{4.17}$$

Here, for $\gamma \in \mathcal{P}(\mathbb{R}^{2d})$ and $x, y \in \mathbb{R}^d$, we have set

$$P_\gamma[\mu](x, y) := \tilde{A}(\mu)(x)(y - x) + \int_{\mathbb{R}^{2d}} A_{\mu\mu}(x, a)(b - a)\gamma(da, db)\,. \tag{4.18}$$

Then, U has a Hessian at μ and

$$\operatorname{Hess} U(\mu)(\zeta_1, \zeta_2) = \int_{\mathbb{R}^d} \langle\tilde{A}(\mu)(x)\zeta_1(x), \zeta_2(x)\rangle\mu(dx) + \int_{\mathbb{R}^{2d}} \langle A_{\mu\mu}(x, a)\zeta_1(a), \zeta_2(x)\rangle\mu(dx)\mu(da)$$

for $\zeta_1, \zeta_2 \in T_\mu\mathcal{P}_2(\mathbb{R}^d)$.

Proof: Fix $\zeta_1, \zeta_2 \in \nabla C_c^\infty(\mathbb{R}^d)$. We are to show that the map $\nu \mapsto \Lambda(\nu) := dU(\nu)(\zeta_2)$ is differentiable at μ, that $\zeta_1 \cdot \big(dU(\nu)(\zeta_2)\big)$ exists and then we need to explicitly compute $\zeta_1 \cdot \big(dU(\nu)(\zeta_2)\big)$.

Let $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\gamma \in \Gamma_0(\mu,\nu)$ and set $A_\nu := \nabla_{\omega_2} U(\nu)$. Since ζ_2 is of compact support, its first and second derivatives are bounded, and so, we may choose a bounded vector field $w \in C(\mathbb{R}^{2d}, \mathbb{R}^d)$ and a bounded matrix field $v \in C(\mathbb{R}^{2d}, \mathbb{R}^{d\times d})$ such that $\|v\|_\infty, \|w\|_\infty \le C$

$$\zeta_2(y) = \zeta_2(x) + \nabla\zeta_2(x)(y-x) + |y-x|^2 v(x,y), \quad \zeta_2(y) - \zeta_2(x) = |y-x| w(x,y) \qquad \forall x,y \in \mathbb{R}^d. \tag{4.19}$$

Note we in fact have the identity

$$|y-x|^2 v(x,y) := \int_0^1 \Big(\nabla\zeta_2\big((1-t)x+ty\big) - \nabla\zeta_2(x)\Big)(y-x)dt \tag{4.20}$$

By assumption, for each $\gamma \in \mathcal{P}(\mathbb{R}^{2d})$, there exists $l(\gamma,\nu,y) \in \mathbb{M}$ such that $|l(\gamma,\nu,y)| \le 1$ and

$$A_\nu(y) - A_\mu(x) - P_\gamma[\mu](x,y) = l(\gamma,\nu,y)\Upsilon\Big(W_2(\mu,\nu), |x-y|\Big). \tag{4.21}$$

We have then by definition of the map Λ that

$$\begin{aligned}\Lambda(\nu) - \Lambda(\mu) &= \int_{\mathbb{R}^{2d}} \big(\langle A_\nu(y), \zeta_2(y)\rangle - \langle A_\mu(x), \zeta_2(x)\rangle\big)\gamma(dx,dy) \\ &= \int_{\mathbb{R}^{2d}} \big(\langle A_\nu(y) - A_\mu(x), \zeta_2(y)\rangle + \langle A_\mu(x), \zeta_2(y) - \zeta_2(x)\rangle\big)\gamma(dx,dy).\end{aligned}$$

This, combined with (4.19) yields

$$\begin{aligned}\Lambda(\nu) - \Lambda(\mu) &= \int_{\mathbb{R}^{2d}} \langle A_\nu(y) - A_\mu(x), \zeta_2(x) + |y-x| w(x,y)\rangle\gamma(dx,dy) \\ &\quad + \int_{\mathbb{R}^{2d}} \langle A_\mu(x), \nabla\zeta_2(x)(y-x) + |y-x|^2 v(x,y)\rangle\big)\gamma(dx,dy) \\ &= I + II.\end{aligned} \tag{4.22}$$

Set

$$R := \Big| II - \int_{\mathbb{R}^{2d}} \langle \nabla\zeta_2^T(x) A_\mu(x), (y-x)\rangle\gamma(dx,dy)\Big|.$$

Note

$$R = \Big|\int_{\mathbb{R}^{2d}} \langle A_\mu(x); |y-x|^2 v(x,y)\rangle\gamma(dx,dy)\Big|.$$

We use (4.20) to conclude that

$$\begin{aligned}R &\le \int_{\mathbb{R}^{2d}} \big|A_\mu(x)\big| \int_0^1 \big|\nabla\zeta_2\big((1-t)x+ty\big) - \nabla\zeta_2(x)\big| |y-x| dt\gamma(dx,dy) \\ &\le W_2(\mu,\nu)\sqrt{\int_{\mathbb{R}^{2d}} \big|A_\mu(x)\big|^2 \int_0^1 \big|\nabla\zeta_2\big(r_t(x,y)\big) - \nabla\zeta_2(x)\big|^2 dt\gamma(dx,dy)}\end{aligned}$$

We use (4.16) to conclude that

$$R \leq C_\mu W_2(\mu,\nu)\sqrt{\int_{\mathbb{R}^{2d}}(1+|x|)^2\int_0^1 \big|\nabla\zeta_2\big(r_t(x,y)\big)-\nabla\zeta_2(x)\big|^2 dt\gamma(dx,dy)}.$$

Using the notation of Exercise 4.26, the previous inequality reads off

$$R \leq C_\mu W_2(\mu,\nu)\sqrt{\epsilon_{\zeta_2}\big(W_2(\mu,\nu)\big)}, \qquad \lim_{s\to 0^+}\sqrt{\epsilon_{\zeta_2}(s)}=0. \tag{4.23}$$

By (4.21)

$$\begin{aligned} I &= \int_{\mathbb{R}^{2d}}\Big\langle \tilde{A}(\mu)(x)(y-x)+\int_{\mathbb{R}^{2d}}A_{\mu\mu}(x,a)(b-a)\gamma(da,db),\zeta_2(x)\Big\rangle\gamma(dx,dy)\\ &+\int_{\mathbb{R}^{2d}}\Big\langle \tilde{A}(\mu)(x)(y-x)+\int_{\mathbb{R}^{2d}}A_{\mu\mu}(x,a)(b-a)\gamma(da,db),|y-x|w(x,y)\Big\rangle\gamma(dx,dy)\\ &+\int_{\mathbb{R}^{2d}}\Big\langle l(\gamma,\nu,y)\Upsilon\Big(W_2(\mu,\nu),|x-y|\Big),\zeta_2(y)\Big\rangle\gamma(dx,dy)\\ &= III+IV+V. \end{aligned}\tag{4.24}$$

We have by Cauchy–Schwarz inequality that

$$|V| \leq \|\zeta_2\|_{L^\infty}\int_{\mathbb{R}^{2d}}\Upsilon\big(W_2(\mu,\nu),|x-y|\big)\gamma(dx,dy). \tag{4.25}$$

By Jensen's inequality

$$\int_{\mathbb{R}^{2d}}|x-y|\epsilon\big(W_2(\mu,\nu)\big)\gamma(dx,dy)\leq W_2(\mu,\nu)\epsilon\big(W_2(\mu,\nu)\big). \tag{4.26}$$

Since ρ is concave, we may apply first Jensen's inequality and second use the fact that it is monotone nondecreasing to obtain

$$\int_{\mathbb{R}^{2d}}\rho(|x-y|)W_2(\mu,\nu)\gamma(dx,dy)\leq W_2(\mu,\nu)\rho\bigg(\int_{\mathbb{R}^{2d}}|x-y|\gamma(dx,dy)\bigg)\leq W_2(\mu,\nu)\rho\big(W_2(\mu,\nu)\big). \tag{4.27}$$

We combine (4.25)-(4.27) and use Exercise 4.28 to infer

$$|V|\leq \|\zeta_2\|_{L^\infty}W_2(\mu,\nu)\bigg(2\epsilon(W_2(\mu,\nu))+\rho\Big(W_2(\mu,\nu)\Big)+\rho\Big(W_2^{\frac{1}{2}}(\mu,\nu)\Big)\Big(W_2^{\frac{1}{2}}(\mu,\nu)+1\Big)\bigg) \tag{4.28}$$

Checking also that

$$\bigg\|\tilde{A}(\mu)(x)(y-x)+\int_{\mathbb{R}^{2d}}A_{\mu\mu}(x,a)(b-a)\gamma(da,db)\bigg\|\leq\Big(\|\tilde{A}(\mu)\|_{L^\infty}+\|A_{\mu\mu}\|_{L^\infty}\Big)W_2(\mu,\nu),$$

we conclude

$$|IV|\leq\Big(\|\tilde{A}(\mu)\|_{L^\infty}+\|A_{\mu\mu}\|_{L^\infty}\Big)W_2^2(\mu,\nu). \tag{4.29}$$

We combine (4.22) (4.23) (4.24) (4.28) and (4.29) and make the substitution $(a,b) \leftrightarrow (x,y)$ to obtain

$$\begin{aligned}\Lambda(\nu) - \Lambda(\mu) = &\int_{\mathbb{R}^{2d}} \langle \tilde{A}(\mu)^T(x)\zeta_2(x) + \nabla\zeta_2^T(x)A_\mu(x), (y-x)\rangle \gamma(dx,dy)\\ &+ \int_{\mathbb{R}^{2d}} \Big\langle \int_{\mathbb{R}^{2d}} (y-x), A_{\mu\mu}^T(a,x)\zeta_2(a)\Big\rangle \gamma(da,db)\gamma(dx,dy)\\ &+ o\big(W_2(\mu,\nu)\big)\end{aligned}$$

Thus, $\xi \in \partial.\Lambda(\mu) \cap \partial^{\cdot}\Lambda(\mu)$ if we set

$$\xi := \tilde{A}(\mu)^T(x)\zeta_2(x) + \nabla\zeta_2^T(x)A_\mu(x) + \int_{\mathbb{R}^d} A_{\mu\mu}^T(a,x)\zeta_2(a)\mu(da)$$

Since $\zeta_1 \in \nabla C_c^\infty(\mathbb{R}^d) \subset T_\mu\mathcal{P}_2(\mathbb{R}^d)$ and by Lemma 4.10 $\xi - \nabla_{\omega_2}\Lambda(\mu) \in \Big[T_\mu\mathcal{P}_2(\mathbb{R}^d)\Big]^\perp$ we obtain

$$\zeta_1 \cdot (\Lambda(\mu)) = \xi \cdot (\Lambda(\mu))$$

and so,

$$\begin{aligned}\zeta_1 \cdot (\Lambda(\mu)) = &\int_{\mathbb{R}^d} \langle \tilde{A}(\mu)^T(x)\zeta_2(x), \zeta_1(x)\rangle \mu(dx) + \int_{\mathbb{R}^{2d}} \langle A_{\mu\mu}(a,x)\zeta_1(x), \zeta_2(a)\rangle \mu(da)\mu(dx)\\ &+ \int_{\mathbb{R}^d} \langle A_\mu(x), \nabla\zeta_2(x)\zeta_1(x)\rangle \mu(dx). \qquad (4.30)\end{aligned}$$

Making the substitution $a \leftrightarrow x$ in (4.30) and using definition 4.24, we read from the subsequent identity that

$$\bar{\text{H}}\text{ess}\, U(\mu)(\zeta_1,\zeta_2) = \int_{\mathbb{R}^d} \langle \tilde{A}(\mu)(x)\zeta_1(x), \zeta_2(x)\rangle \mu(dx) + \int_{\mathbb{R}^{2d}} \langle A_{\mu\mu}(x,a)\zeta_1(a), \zeta_2(x)\rangle \mu(dx)\mu(da). \qquad (4.31)$$

Obviously, there is a constant C such that $|\bar{\text{H}}\text{ess}\, U(\mu)(\zeta_1,\zeta_2)| \leq C\,\|\zeta_1\|_\mu\,\|\zeta_2\|_\mu$. Thus $\text{Hess}\, U(\mu)$ is well–defined and (4.31) remains valid for $\zeta_1,\zeta_2 \in T_\mu\mathcal{P}_2(\mathbb{R}^d)$. QED.

Definition 4.30. Under the assumptions of Theorem 4.29, we say that U is twice differentiable at $\mu \in \text{dom}(U)$. If $A_{\mu\mu}$ satisfies (4.18), so does $\pi_\mu\big(A_{\mu\mu}\big)$ which is the matrix whose rows are the orthogonal projections onto $T_\mu\mathcal{P}_2(\mathbb{R}^d)$ of the rows of $A_{\mu\mu}$. Note that although $A_{\mu\mu}$ may not be unique, $\pi_\mu\big(A_{\mu\mu}\big)$ is uniquely determined. In the sequel, we tacitly assume that $A_{\mu\mu} \equiv \pi_\mu\big(A_{\mu\mu}\big)$.

(i) We define the Wasserstein partial Laplacian operator $\triangle_{\omega_2}$ such that $\triangle_{\omega_2} U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, by

$$(\triangle_{\omega_2} U)[\mu] = \sum_{i=1}^d \text{Hess}\, U[\mu](\zeta_i,\zeta_i),$$

where ζ_i is the Wasserstein gradient of the i–th moment $\mu \mapsto \int_{\mathbb{R}^d} x_i\mu(dx)$.

(ii) Given $\epsilon \geq 0$, the ϵ–perturbation of $\triangle_{\omega_2}$ is $\triangle_{\omega_2,\epsilon}$ such that $\triangle_{\omega_2,\epsilon}U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$, and

$$(\triangle_{\omega_2,\epsilon}U)[\mu] = \sum_{i=1}^{d}\Big(\mathrm{Hess}\, U[\mu](\zeta_i,\zeta_i) + \epsilon \int_{\mathbb{R}^d} \langle \tilde{A}[\mu](x)\zeta_i(x), \zeta_i(x)\rangle \mu(dx)\Big)$$

(iii) We define the second order Wasserstein gradient of U at m to be $\pi_\mu\big(A_{\mu\mu}\big)$ and we denote it as $\nabla^2_{\omega_2}U[\mu]$.

(iv) Suppose further that U is twice differentiable in a neighborhood of μ. If $(x,\nu) \mapsto \tilde{A}[\nu](x)$ and $(x,y,\nu) \mapsto \pi_\nu\big(A_{\nu\nu}\big)(x,y)$ are continuous, we say that U is twice continuously differentiable on that neighborhood.

Remark 4.31. Suppose the assumptions in Theorem 4.29 holds, i.e. U is twice differentiable at $\mu \in \mathrm{dom}(U)$. Then

(i) $A_\mu := \nabla_{\omega_2}U(\mu]$ is differentiable on $\mathbb{R}^d$ and its gradient (w.r.t. the x variable) is $\tilde{A}[\mu]$, whose rows belong to $T_\mu\mathcal{P}_2(\mathbb{R}^d)$.

(ii) We have

$$\triangle_{\omega_2}U[\mu] = \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_{\omega_2}U[\mu](x)\big)\mu(dx) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla^2_{\omega_2}U[\mu](x,a)\Big)\mu(dx)\mu(da)$$

and

$$(\triangle_{\omega_2,\epsilon}U)[\mu] = (1+\epsilon)\int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_{\omega_2}U[\mu](x)\big)\mu(dx) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla^2_{\omega_2}U[\mu](x,a)\Big)\mu(dx)\mu(da)$$

(iii) Note that the expressions in (ii) continue to make sense if we merely assume that

$$\mathrm{div}_x\big(A_\mu(x)\big) \in L^1(\mathbb{R}^d,\mu) \quad \text{and} \quad \mathrm{Tr}\Big(\nabla^2_{\omega_2}U[\mu](\cdot,\cdot)\Big) \in L^1(\mathbb{R}^{2d},\mu\otimes\mu).$$

(iv) Let $u : \mathbb{R}^{kd} \to \mathbb{R}$ be defined for $x := (x_1,\cdots,x_k) \in \mathbb{R}^{kd}$ by

$$u(x) = U\big(\mu_x\big), \quad \mu_x = \frac{1}{k}\sum_{j=1}^{k}\delta_{x_j}.$$

If $\mu = \mu_x$, then u is differentiable in a neighborhood of x, ∇u is differentiable at x and

$$\triangle_{\omega_2}U\big[\mu_x\big] = \sum_{j,l=1}^{k}\mathrm{div}_{x_j}\big(\nabla_{x_l}u\big)(x).$$

Proposition 4.32 (cf. [16])**.** *Let U be as in Theorem 4.29, which in particular means that we have fixed $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ such that U is differentiable in a neighborhood of $\mu \in \mathrm{dom}(U)$ and U is twice differentiable at m. Let $T > 0$ and suppose $\sigma \in AC_2(0,T,\mathcal{P}_2(\mathbb{R}^d))$ is a path which has a velocity of minimal norm $\mathbf{v} \in C^1((0,T)\times\mathbb{M})$ which is bounded and has bounded first order time and space derivatives. If $s \in (0,T)$ and $m = \sigma_s$ then*

$$\frac{d^2}{dt^2}U(\sigma_t))\Big|_{t=s} = \mathrm{Hess}\, U[\sigma_s](\mathbf{v}_s,\mathbf{v}_s) + \langle \partial_t\mathbf{v}_s + \nabla\mathbf{v}_s\mathbf{v}_s; \nabla_{\omega_2}U(\sigma_s)\rangle_{\sigma_s}$$

4.5 Lectures 16, 17: Polynomials on $\mathcal{P}_2(\mathbb{R}^d)$ (Oct 23, 25)

Throughout this section to alleviate the notation, we set $\mathbb{M} := \mathbb{R}^d$. We fix a natural number $k \geq 1$ and denote as P_k the set of permutations of k letters. For $x = (x_1, \cdots, x_k) \in \mathbb{M}^k$ and $\sigma \in P_k$ we set $x^\sigma := (x_{\sigma(1)}, \cdots, x_{\sigma(k)})$. We denote as $\mathcal{S}ym[k](\mathbb{R})$ the set of $\Phi \in C(\mathbb{M}^k)$ such that there exists $C > 0$ such that

$$\Phi(x) = \Phi(x^\sigma), \qquad |\Phi(x)| \leq C(1 + |x|^2) \qquad \forall (x, \sigma) \in \mathbb{M}^k \times P_k.$$

We set

$$F_\Phi[\mu] = \frac{1}{k} \int_{\mathbb{M}^k} \Phi(x) \mu^{\otimes k}(dx).$$

We denote as $\mathcal{S}ym_2[k](\mathbb{R})$ the set of $\Phi \in \mathcal{S}ym[k](\mathbb{R}) \cap C(\mathbb{M}^k)$ such that $\nabla^2 \Phi$ is bounded, uniformly continuous and has a concave modulus of continuity.

Fix $\Phi \in \mathcal{S}ym_2[k](\mathbb{R})$. For $x_1 \in \mathbb{M}$ and $\mu \in \mathcal{P}_2(\mathbb{M})$ we set

$$A_\mu(x_1) := \begin{cases} \nabla \Phi(x_1) & \text{if} \quad k = 1, \\ \int_{\mathbb{M}^{k-1}} \nabla_{x_1} \Phi(x_1, \cdots, x_k) \mu(dx_2) \cdots \mu(dx_k) & \text{if} \quad k \geq 2, \end{cases} \tag{4.32}$$

If $x_1, x_2 \in \mathbb{M}$ and $\mu \in \mathcal{P}_2(\mathbb{M})$ we set

$$A_{\mu\mu}(x_1, x_2) := \begin{cases} 0 & \text{if} \quad k = 1, \\ \nabla^2_{x_2 x_1} \Phi(x_1, x_2) & \text{if} \quad k = 2, \\ & \\ (k-1) \int_{\mathbb{M}^{k-2}} \nabla_{x_2 x_1} \Phi(x) \mu(dx_3) \cdots \mu(dx_k) & \text{if} \quad k \geq 3 \end{cases} \tag{4.33}$$

For $\gamma \in \mathcal{P}_2(\mathbb{M}^2)$, if μ is the first marginal of γ we set

$$P_\gamma(x_1, y_1) := \nabla_{x_1} A_\mu(x_1)(y_1 - x_1) + \int_{\mathbb{M}^2} A_{\mu\mu}(x_1, x_2)(y_2 - x_2) \gamma(dx_2, dy_2),$$

Then we have the following lemma.

Lemma 4.33. *For any $\mu \in \mathcal{P}_2(\mathbb{M})$, the following hold.*

(i) The function F_Φ is differentiable in the sense of Wasserstein at any μ and $\nabla_{\omega_2} F_\Phi[\mu] = A_\mu \in T_\mu \mathcal{P}_2(\mathbb{M})$.

(ii) Further assume that $\nabla^2 \Phi$ has a modulus of continuity $\rho : [0, \infty) \to [0, \infty)$ which is concave. If $\nu \in \mathcal{P}_2(\mathbb{M})$, then there exists $\tilde{\rho}, \epsilon_2 : [0, \infty) \to \mathbb{R}$ are nonnegative function (depending on μ and Φ) such that $\lim_{t \to 0^+} \tilde{\rho}(t) = \lim_{t \to 0^+} \epsilon_2(t) = 0$ and $\tilde{\rho}$ is a concave modulus and

$$\sup_{\gamma \in \Gamma_0(\mu, \nu)} \Big| A_\nu(y_1) - A_\mu(x_1) - P_\gamma(x_1, y_1) \Big| \leq \Big(|x - y| + W_2(\mu, \nu) \Big) \Big(\tilde{\rho}\big(|x_1 - y_1|\big) + \epsilon_2\big(W_2(\mu, \nu)\big) \Big).$$

Proof: Note that since $\nabla^2 \Phi$ is bounded there exists a constant $C > 0$ such that

$$|\Phi(x)| \leq C(1 + |x|^2), \quad |\nabla \Phi(x)| \leq C(1 + |x|), \quad |\nabla^2 \Phi(x)| \leq C \qquad \forall \quad x \in \mathbb{M}. \tag{4.34}$$

The second inequality in (4.34) ensures that for any $\mu \in \mathcal{P}_2(\mathbb{M})$, $\nabla_{x_1}\Phi \in L^1(\mu^{\otimes(k-1)})$ and so, A_μ is well–defined. Furthermore, Φ is bounded. Similarly, the second and third inequalities in (4.34) ensure that $\nabla_{x_1}A_\mu$ and $A_{\mu\mu}$ are well–defined and bounded.

(i) The proof of (i) is easier when $k=1$. We assume in the sequel that $k\geq 2$. Using the fact that Φ is symmetric, for any $i\in\{2,\cdots,k\}$ if σ is the permutation such that $\sigma(1)=i$, $\sigma(i)=1$ and $\sigma(j)=j$ for any $j\notin\{1,i\}$ we have

$$\nabla_{x_1}\Phi(x)=\nabla_{x_i}\Phi(x^\sigma). \tag{4.35}$$

Applying Taylor expansion, thanks to the third inequality in (4.34) there exists a uniformly continuous function $f:\mathbb{M}^k\times\mathbb{M}^k\to\mathbb{R}$ bounded by C such that

$$\Phi(y)=\Phi(x)+\sum_{i=1}^k\nabla_{x_i}\Phi(x)\cdot(y_i-x_i)+\frac{1}{2}f(x,y)|x-y|^2. \tag{4.36}$$

Let $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma\in\Gamma_0(\mu,\nu)$. By changing variables

$$\int_{\mathbb{M}^2\times\mathbb{M}^2}\nabla_{x_i}\Phi(x)\cdot(y_i-x_i)\gamma(dx_1,dy_1)\gamma(dx_i,dy_i)=\int_{\mathbb{M}^2}\nabla_{x_i}\Phi(x^\sigma)\cdot(y_1-x_1)\gamma(dx_i,dy_i)\gamma(dx_1,dy_1).$$

Using (4.35) we conclude that

$$\int_{\mathbb{M}^2\times\mathbb{M}^2}\Big(\nabla_{x_i}\Phi(x)\cdot(y_i-x_i)-\nabla_{x_1}\Phi(x)\cdot(y_1-x_1)\Big)\gamma(dx_1,dy_1)\gamma(dx_i,dy_i)=0. \tag{4.37}$$

We have

$$F_\Phi[\nu]-F_\Phi[\mu]=\frac{1}{k}\int_{\mathbb{M}^k\times\mathbb{M}^k}(\Phi(y)-\Phi(x))\gamma^{\otimes k}(dx,dy).$$

This, together with (4.36) and (4.37) implies

$$F_\Phi[\nu]-F_\Phi[\mu]=\int_{\mathbb{M}^2}A_\mu(x_1)\cdot(y_1-x_1)\gamma(dx_1,dy_1)+\frac{1}{2k}\int_{\mathbb{M}^k\times\mathbb{M}^k}f(x,y)|x-y|^2\gamma^{\otimes k}(dx,dy) \tag{4.38}$$

and so,

$$\Big|F_\Phi[\nu]-F_\Phi[\mu]-\int_{\mathbb{M}^2}A_\mu(x_1)\cdot(y_1-x_1)\gamma(dx_1,dx_2)\Big|\leq\frac{C}{2}W_2^2(\mu,\nu).$$

This proves that $A_\mu\in\partial F_\Phi[\mu]$. Note that A_μ is the gradient of

$$x_1\mapsto\Phi_1(x_1):=\int_{\mathbb{M}^{k-1}}\Phi(x_1,\cdots,x_k)\mu(dx_2)\cdots\mu(dx_k)$$

which is a bounded function with bounded first derivatives. Thus, $A_\mu\in T_\mu\mathcal{P}_2(\mathbb{R}^d)$. For any $\zeta\in\partial F_\Phi[\mu]$, we use Lemma 4.10 to infer $\pi_\mu(\zeta-A_\mu)=0$, which means that $\pi_\mu(\zeta)=\pi_\mu(A_\mu)=A_\mu$. In particular $\|\zeta\|_\mu\geq\|\pi_\mu(\zeta)\|_\mu=\|A_\mu\|_\mu$, which proves that A_μ is the element of minimal norm in $\partial F_\Phi[\mu]$.

(ii) Since the proof of (ii) is easier in the case $k=2$ compared to the case when $k\geq 3$, we assume in the sequel that $k\geq 3$.

Let $i \in \{3, \cdots, k\}$ and let σ be the permutation such that $\sigma(2) = i$, $\sigma(i) = 2$ and $\sigma(j) = j$ for any $j \notin \{2, i\}$. Given $x = (x_1, \cdots, x_k) \in \mathbb{M}^k$, using the fact that Φ is symmetric, we have

$$(\nabla^2_{x_2 x_1}\Phi)(x) = (\nabla^2_{x_1 x_2}\Phi)(x) = (\nabla^2_{x_1 x_i}\Phi)(x^\sigma) = (\nabla^2_{x_i x_1}\Phi)(x^\sigma). \tag{4.39}$$

For any $y = (y_1, \cdots, y_k) \in \mathbb{M}^k$,

$$(\nabla_{x_1}\Phi)(y) - (\nabla_{x_1}\Phi)(x) = \int_0^1 \sum_{i=1}^k (\nabla^2_{x_i x_1}\Phi)(x + t(y - x))(y_i - x_i)dt. \tag{4.40}$$

Let $\nu \in \mathcal{P}_2(\mathbb{M})$ and let $\gamma \in \Gamma_0(\mu, \nu)$. The change of variables which exchanges x_2 with x_i is used to obtain

$$\int_{\mathbb{M}^2 \times \mathbb{M}^2} \Big((\nabla^2_{x_i x_1}\Phi)(x)(y_i - x_i) - (\nabla^2_{x_i x_1}\Phi)(x^\sigma)(y_2 - x_2) \Big) \gamma(dx_2, dy_2)\gamma(dx_i, dy_i) = 0.$$

Combining the latter with (4.39) we infer

$$\int_{\mathbb{M}^2 \times \mathbb{M}^2} \Big((\nabla^2_{x_i x_1}\Phi)(x)(y_i - x_i) - (\nabla^2_{x_2 x_1}\Phi)(x)(y_2 - x_2) \Big) \gamma(dx_2, dy_2)\gamma(dx_i, dy_i) = 0. \tag{4.41}$$

Set

$$e_t(x, y) := \int_0^1 \sum_{i=1}^k \Big((\nabla^2_{x_i x_1}\Phi)(x + t(y - x)) - (\nabla^2_{x_i x_1}\Phi)(x) \Big)(y_i - x_i)dt$$

By (4.40)

$$A_\nu(y_1) - A_\mu(x_1) = \int_{\mathbb{M}^{k-1} \times \mathbb{M}^{k-1}} \int_0^1 \sum_{i=1}^k (\nabla^2_{x_i x_1}\Phi)(x + t(y - x))(y_i - x_i)dt\gamma(dx_2, dy_2) \cdots \gamma(dx_k, dy_k)$$

This, together with (4.41) yields

$$\begin{aligned} A_\nu(y_1) - A_\mu(x_1) = \nabla_{x_1} A_\mu(x_1)(y_1 - x_1) + \int_{\mathbb{M}^2} A_{\mu\mu}(x_1, x_2)(y_2 - x_2)\gamma(dx_2, dy_2) \\ + \int_{\mathbb{M}^{k-1} \times \mathbb{M}^{k-1}} \int_0^1 e_t(x, y)dt\gamma(dx_2, dy_2) \cdots \gamma(dx_k, dy_k). \end{aligned} \tag{4.42}$$

Let $\rho : [0, \infty) \to [0, \infty)$ be a concave function, modulus of continuity of $\nabla^2\Phi$. Then

$$|e_t(x, y)| \leq \sum_{i=1}^k \rho(|x - y|)|x_i - y_i| \leq \sum_{i=1}^k \rho\Big(\sum_{j=1}^k |x_j - y_j|\Big)|x_i - y_i| \leq \sum_{i,j=1}^k \rho\Big(|x_j - y_j|\Big)|x_i - y_i|$$

We divide the expression $\sum_{i,j=1}^k \rho\Big(|x_j - y_j|\Big)|x_i - y_i| =: G_1 + G_2 + G_3$ into 3 groups where

$$G_1 = |x_1 - y_1| \sum_{j=1}^k \rho\Big(|x_j - y_j|\Big), \quad G_2 := \rho\Big(|x_1 - y_1|\Big) \sum_{i=2}^k |x_i - y_i|$$

and

$$G_3 := \sum_{i,j=2}^{k} \rho\Big(|x_j - y_j|\Big)|x_i - y_i| = \sum_{i=2}^{k} \rho\Big(|x_i - y_i|\Big)|x_i - y_i| + \sum_{i\neq j, i,j\geq 2} \rho\Big(|x_j - y_j|\Big)|x_i - y_i|$$

We have

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_1 \gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) = |x_1-y_1|\rho\Big(|x_1-y_1|\Big)+|x_1-y_1|\sum_{j=2}^{k}\int_{\mathbb{M}^2}\rho\Big(|x_j-y_j|\Big)\gamma(dx_j, dy_j)$$

Since ρ is concave, we use Jensen's inequality to conclude

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_1 \gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) = |x_1-y_1|\rho\Big(|x_1-y_1|\Big)+(k-1)|x_1-y_1|\rho\Big(\int_{\mathbb{M}^2}|a-b|\gamma(da, db)\Big)$$

We use again Jensen's inequality and the fact that ρ is monotone nondecreasing to obtain

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_1 \gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) \leq |x_1-y_1|\Big(\rho\big(|x_1-y_1|\big)+(k-1)\rho\big(W_2(\mu,\nu)\big)\Big) \tag{4.43}$$

Similarly, we have

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_3 d\gamma^{\otimes(k-2)} = \sum_{i\neq j, i,j\geq 2}\int_{\mathbb{M}^2}|a-b|d\gamma\int_{\mathbb{M}^2}\rho\Big(|a-b|\Big)d\gamma + (k-1)\int_{\mathbb{M}^2}|a-b|\rho(|a-b|)d\gamma.$$

We use Jensen's inequality and Exercise 4.28 we conclude that

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_3 d\gamma^{\otimes(k-2)} \leq (k-1)W_2(\mu,\nu)\Big((k-2)\rho\big(W_2(\mu,\nu)\big)+\rho\big(W_2^{\frac{1}{2}}(\mu,\nu)\big)\Big(1+\rho\big(W_2^{\frac{1}{2}}(\mu,\nu)\big)\Big)\Big). \tag{4.44}$$

We have

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_2 \gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) = \rho\Big(|x_1-y_1|\Big)\sum_{i=2}^{k}\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}}|x_i-y_i|\gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k).$$

Using again Jensen's inequality we conclude that

$$\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}} G_2 \gamma(dx_2, dy_2)\cdots\gamma(dx_k, dy_k) \leq (k-1)\rho\Big(|x_1-y_1|\Big)W_2(\mu,\nu) \tag{4.45}$$

We combine (4.43), (4.44) and (4.45) to obtain a constant $C_k > 1$ depending only on k such that

$$\begin{aligned}\int_{\mathbb{M}^{k-1}\times\mathbb{M}^{k-1}}\int_0^1 |e_t(x,y)|dtd\gamma^{\otimes(k-1)} &\leq C_k|x_1-y_1|\Big(\rho\big(|x_1-y_1|\big)+\rho\big(W_2(\mu,\nu)\big)\Big)\\ &+ C_kW_2(\mu,\nu)\Big(\rho\big(W_2(\mu,\nu)\big)+\rho\big(W_2^{\frac{1}{2}}(\mu,\nu)\big)\Big(\rho\big(W_2^{\frac{1}{2}}(\mu,\nu)\big)+1\Big)\Big)\\ &+ C_k\rho\Big(|x_1-y_1|\Big)W_2(\mu,\nu).\end{aligned} \tag{4.46}$$

This, together with (4.42) proves (ii) if we set

$$\tilde{\rho}(t) = C_k \rho(t), \quad \epsilon_2(t) = C_k \rho(t) + C_k \rho(\sqrt{t})\Big(\rho(\sqrt{t}) + 1\Big).$$

QED.

We define on the set $\mathcal{S}ym_2[k](\mathbb{R})$ the operators $\mathcal{L}_{k,\epsilon}$ as

$$\mathcal{L}_{k,\epsilon}\big(\Phi\big) := \epsilon \triangle_x \Phi + \sum_{j,l=1}^{k} \mathrm{Tr}\big(\nabla_{x_j x_k} \Phi\big)$$

Corollary 4.34. *If Φ satisfies the conditions in Lemma 4.33 then*

$$(\triangle_{\omega_2,\epsilon} F_\Phi) = F_{\mathcal{L}_{k,\epsilon}(\Phi)}$$

Proof: We combine Theorem 4.29, Remark 4.31 and Lemma 4.33 to obtain

$$(\triangle_{\omega_2,\epsilon} F_\Phi) = \int_{\mathbb{M}^k} \Big(\triangle_{x_1} \Phi(x) + (k-1)\mathrm{Tr}\big(\nabla_{x_1 x_2} \Phi(x)\big)\Big)\mu(dx_1)\cdots\mu(dx_k). \tag{4.47}$$

We use the fact that k is symmetric to conclude for any $j \in \{1, \cdots, k\}$

$$\int_{\mathbb{M}^k} \triangle_{x_1} \Phi(x)\mu(dx_1)\cdots\mu(dx_k) = \int_{\mathbb{M}^k} \triangle_{x_j} \Phi(x)\mu(dx_1)\cdots\mu(dx_k). \tag{4.48}$$

Similarly, for any $1 \le j < k \le n$ we have

$$\int_{\mathbb{M}^k} \mathrm{Tr}\big(\nabla_{x_1 x_2} \Phi(x)\big)\mu(dx_1)\cdots\mu(dx_k) = \int_{\mathbb{M}^k} \mathrm{Tr}\big(\nabla_{x_j x_l} \Phi(x)\big)\mu(dx_1)\cdots\mu(dx_k) \tag{4.49}$$

We combine (4.47)-(4.49) to verify the Corollary. QED.

$$\mathrm{Sym}[\mathbb{C}^{\mathrm{k}}] := \mathcal{S}ym[k](\mathbb{R}) + i\mathcal{S}ym[k](\mathbb{R})$$

and let

$$\mathrm{Sym}_2[\mathbb{C}^{\mathrm{k}}] := \mathcal{S}ym_2[k](\mathbb{R}) + i\mathcal{S}ym_2[k](\mathbb{R}).$$

For $(\xi_1, \cdots, \xi_k) \in \mathbb{M}^k$, we define

$$\Psi_\xi^k(x) := \exp\Big(-2\pi i \sum_{j=1}^{k} \langle \xi_j, x_j \rangle\Big). \tag{4.50}$$

The symmetrization of Ψ_ξ^k is

$$\Phi_\xi^k(x) := \frac{1}{k!} \sum_{\sigma \in P_k} \exp\Big(-2\pi i \sum_{j=1}^{k} \langle \xi_{\sigma(j)}, x_j \rangle\Big), \quad \forall\, x = (x_1, \cdots, x_k) \in \mathbb{M}^k.$$

Set

$$\lambda_k^2(\xi) := 4\pi^2 \Big| \sum_{j=1}^k \xi_j \Big|^2 \quad \text{and} \quad \lambda_{k,\epsilon}^2(\xi) := \lambda_k^2(\xi) + 4\epsilon\pi^2 \sum_{j=1}^k |\xi_j|^2 \tag{4.51}$$

We have

$$\nabla_{x_l x_j} \Psi_\xi^k(x) = -4\pi^2 \xi_j \otimes \xi_l \Psi_\xi^k(x),$$

and so,

$$\mathcal{L}_{k,\epsilon}(\Psi_\xi^k) = -4\pi^2 \bigg(\epsilon \sum_{j=1}^k |\xi_j|^2 + \Big| \sum_{j=1}^k \xi_j \Big|^2 \bigg) \Psi_\xi^k = -\lambda_{k,\epsilon}^2(\xi) \Psi_\xi^k.$$

By symmetrization we conclude

$$\mathcal{L}_{k,\epsilon}(\Phi_\xi^k) = -\lambda_{k,\epsilon}^2(\xi) \Phi_\xi^k. \tag{4.52}$$

We sometimes use the notation F_ξ^k defined as

$$F_\xi^k[m] := F_{\Phi_\xi^k}[m] = \frac{1}{k} \int_{\mathbb{M}^k} \Phi_\xi^k(x) m(dx_1) \cdots m(dx_k) \qquad \forall m \in \mathcal{P}_2(\mathbb{R}^d).$$

Remark 4.35. Let $\mu \in \mathcal{P}_2(\mathbb{M})$ be of bounded support.

(i) Instead of assuming that Φ satisfies the conditions in Lemma 4.33, let us rather assume $\Phi \in \mathcal{S}ym[k](\mathbb{R}) \cap C^3(\mathbb{M})$ and there is a constant $C > 0$ such that

$$|\nabla \Phi(x)| \leq C(1 + |x|) \qquad \forall x \in \mathbb{M}^k.$$

Because $\nabla^2 \Phi$ is bounded and Lipschitz on the support of μ, one checks the conclusions in Theorem 4.29 and Lemma 4.33 are still verified. Similarly, Corollary 4.34 yields,

$$(\triangle_{\omega_2,\epsilon} F_\Phi) = F_{\mathcal{L}_{k,\epsilon}(\Phi)}.$$

(ii) By (i), (4.52) implies

$$\big(\triangle_{\omega_2,\epsilon} F_{\Phi_\xi^k}\big) = F_{-\lambda_{k,\epsilon}^2(\xi)\Phi_\xi^k} = -\lambda_{k,\epsilon}^2(\xi) F_{\Phi_\xi^k}.$$

This means $F_{\Phi_\xi^k}$ is an eigenfunction for $\triangle_{\omega_2,\epsilon}$ with $-\lambda_{k,\epsilon}^2(\xi)$ as the associated eigenvalue.

4.6 Lecture 18: Sobolev functions on $\mathcal{P}_2(\mathbb{R}^d)$ (Oct 30)

In this section, we continue using the notation of Section 4.5 and briefly recall some of the statements in [16]. We use the expressions of $\lambda_{k,\epsilon}(\xi)$ as defined in (4.51) and write $\lambda_k(\xi)$ in place of $\lambda_{k,0}(\xi)$.

Definition 4.36. We have the following definition.

(i) We call $\mathcal{A}$ the set of sequences of functions $(a_k)_{k=1}^\infty$ such that $a_k : \mathbb{M}^k \to \mathbb{C}$ is a Borel function that are symmetric in the sense that $a_k(\xi) = a_k(\xi^\sigma)$ for any $\xi \in \mathbb{M}^k$ and any $\sigma \in P_k$. In other words, a_k is defined on $\mathbb{M}^k/P_k$, the k–symmetric product of $\mathbb{M}$.

(ii) We call $\mathcal{A}^s$ the set of sequences $A := \big(a_k\big)_{k=1}^\infty \subset \mathcal{A}$ such that

$$||A||_{H^s}^2 := \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{M}^k} |a_k(\xi)|^2 \big(1 + \lambda_k^2(\xi)\big)^s d\xi < \infty. \tag{4.53}$$

(iii) If $B = \big(b_k\big)_{k=1}^\infty \in \mathcal{A}^s$, then the following is a well–defined sesquilinear form (cf. Exercise 4.37 below):

$$\langle A; B\rangle_{H^s} := \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{M}^k} a_k(\xi) b_k^*(\xi) \big(1 + \lambda_k^2(\xi)\big)^s d\xi$$

Exercise 4.37. Show that the sesquilinear form $\langle \cdot; \cdot\rangle_{H^s} : \mathcal{A}^s \times \mathcal{A}^s \to \mathbb{C}$ is well defined. Further more show the following hold.

(i) (Hölder's inequality)

$$|\langle A; B\rangle_{H^s}| \leq ||A||_{H^s} \cdot ||B||_{H^s}$$

(ii) (triangle inequality)

$$||A + B||_{H^s} \leq ||A||_{H^s} + ||B||_{H^s}$$

Proof: Let $A, B \in \mathcal{A}^s$ be as in Definition 4.36. Then for any $\lambda > 0$ we have

$$\frac{1}{k!} \Big| \int_{\mathbb{M}^k} a_k(\xi) b_k^*(\xi) \big(1 + \lambda_k^2(\xi)\big)^s d\xi \Big| \leq \frac{1}{2k!} \int_{\mathbb{M}^k} \Big(\frac{|a_k(\xi)|^2}{\lambda^2} + \lambda^2 |b_k(\xi)|^2 \Big) \big(1 + \lambda_k^2(\xi)\big)^s d\xi. \tag{4.54}$$

Therefore, the series produced by the left hand side of (4.54) converges absolutely, which concludes the proof.

Assume without loss of generality that $||B||_{H^0} \neq 0$.

(i) By (4.54)

$$2|\langle A; B\rangle_{H^s}| \leq \frac{||A||_{H^s}^2}{\lambda^2} + \lambda^2 ||B||_{H^s}^2.$$

We use $\lambda := ||A||_{H^s}^{\frac{1}{2}} ||B||_{H^s}^{-\frac{1}{2}}$ to conclude the proof of (i).

(ii) We use (i) and the identity

$$||A + B||_{H^s}^2 = ||A||_{H^s}^2 + ||B||_{H^s}^2 + \langle A; B\rangle_{H^s} + \langle G; F\rangle_{H^s}$$

to conclude the proof of (ii). QED.

For $\big(a_k\big)_{k=1}^\infty, \big(b_k\big)_{k=1}^\infty \in \mathcal{A}^s$ we sometimes impose the condition that there exist $\delta, C > 0$ such that

$$\int_{\mathbb{M}^k} \big(|a_k(\xi)| + |b_k(\xi)|\big) d\xi \leq \frac{Ck!}{k^\delta} \tag{4.55}$$

holds for all $k \in \mathbb{N}$.

Exercise 4.38. Let $(a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty \in \mathcal{A}^s$ be such that (4.55) holds. Show the series converge uniformly on $\mathcal{P}_2(\mathbb{R}^d)$.

Proof: By the fact that $|F_\xi^k[m]| \le k^{-1}$, we have

$$\Big|\sum_{k=1}^N \frac{1}{k!}\int_{\mathbb{M}^k} a_k(\xi)F_\xi^k[m]d\xi\Big| \le \sum_{k=1}^N \frac{1}{k!\,k}\int_{\mathbb{M}^k}|a_k(\xi)|d\xi \le \sum_{k=1}^N \frac{C}{k^{1+\delta}}.$$

This concludes the proof. QED.

Definition 4.39. We have the following definition.

(i) We call $H^s(\mathcal{P}_2(\mathbb{R}^d))$ the set of $F : \mathcal{P}_2(\mathbb{R}^d) \to [-\infty, \infty]$ for which there exists $(a_k)_{k=1}^\infty \subset \mathcal{A}^s$ such that

$$F[m] := \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} a_k(\xi)F_\xi^k[m]d\xi \tag{4.56}$$

converges for any $m \in \mathcal{P}_2(\mathbb{R}^d)$.

(ii) If we further assume (4.55) holds, we write $F \in \mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d))$.

Exercise 4.40. Let $(a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty \in \mathcal{A}^s$ be such that (4.55) holds. Show that if

$$\sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} a_k(\xi)F_\xi^k[m]d\xi = \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} b_k(\xi)F_\xi^k[m]d\xi$$

then $a_k \equiv b_k$ for any $k \in \mathbb{N}$.

Proof: Use Remark 5.2 [16] QED.

Definition 4.41. Let $(a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty \in \mathcal{A}^s$ be such that (4.55) holds. Setting

$$F[m] := \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} a_k(\xi)F_\xi^k[m]d\xi, \quad G[m] := \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} b_k(\xi)F_\xi^k[m]d\xi \qquad \forall m \in \mathcal{P}_2(\mathbb{M})$$

we define

$$\langle F, G\rangle_{H^s} := \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{M}^k} a_k(\xi)b_k^*(\xi)\big(1+\lambda_k^2(\xi)\big)^s d\xi$$

We write $L^2(\mathcal{P}_2(\mathbb{M})) = \mathcal{H}^0(\mathcal{P}_2(\mathbb{R}^d))$ and use the notation $\langle F, G\rangle_{L^2} = \langle F, G\rangle_{H^0}$.

From definition, we have

$$\mathcal{H}^s(\mathcal{P}_2(\mathbb{R}^d)) \subset H^s(\mathcal{P}_2(\mathbb{R}^d)) \cap C(\mathcal{P}_2(\mathbb{R}^d))$$

and the second inclusion results from the fact that the convergence of the series converges uniformly on $m \in \mathcal{P}_2(\mathbb{R}^d)$ (cf. Exercise 4.38 (iv)).

Chapter 5

An alternative derivation of the master equation

5.1 Lecture 19: Notation an assumptions (Nov 01)

Let $H \in C^2(\mathbb{M} \times \mathbb{R}^d)$ be such that the gradient DH is Lipschitz, the matrix of the second derivatives D^2H is uniformly continuous and there exists $C > 0$ such that

$$0 < D_{pp}H(q,p) \leq CI_d \qquad \forall (q,p) \in \mathbb{M} \times \mathbb{R}^d. \tag{5.1}$$

Let

$$U_*, \mathcal{F} : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

be sufficiently smooth and set

$$F(q,\mu) := \frac{\delta \mathcal{F}}{\delta \mu}(\mu)(q), \quad u_*(q,\mu) := \frac{\delta U_*}{\delta \mu}(\mu)(q), \qquad \forall (q,\mu) \in \mathbb{M} \times \mathcal{P}_2(\mathbb{M}).$$

We set

$$\mathcal{H}(\mu,\xi) := \int_{\mathbb{M}} H(q,\xi(q))\mu(dq) \qquad \forall \mu \in \mathcal{P}_2(\mathbb{M}),\ \forall \xi \in L^2(\mu).$$

We use H and $\mu \in \mathcal{P}_2(\mathbb{M})$ to define the operator N_μ, by setting for Borel vector fields $\xi, \zeta : \mathbb{M} \mapsto \mathbb{R}^d$,

$$N_\mu[\xi,\zeta] := \int_{\mathbb{M}} \big\langle \xi(x), D_pH(x,\zeta(x)) \big\rangle \mu(dx), \tag{5.2}$$

provided the integral exists.

If $U : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is smooth enough, we consider the following operators

$$O(U)(\mu) = \int_{\mathbb{M}} \operatorname{div}_q\big(\nabla_{\omega_2} U(\mu)(q)\big)\mu(dq), \qquad B(U)(\mu) = \int_{\mathbb{M}^2} \operatorname{Tr}\Big(\nabla^2_{\omega_2} U(q,q',\mu)\Big)\mu(dq)\mu(dq').$$

We recall the definition of the partial Wasserstein Laplacian and its perturbations introduced in [16] denoted as

$$\triangle_{\omega_2} U(\mu) = O(U)(\mu) + B(U)(\mu), \qquad \triangle_{\omega_2,r} = \triangle_\omega U + rO.$$

For $w : \mathbb{M} \times \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth, recall (cf. (1.1))

$$\mathcal{L}_{\epsilon_1}^{\epsilon_2}(w) := \mathcal{L}_{\epsilon_1}(w) + \mathcal{L}^{\epsilon_2}(w).$$

where

$$\mathcal{L}_{\epsilon_1}(w)(q,\mu) := \epsilon_1 \Big(\triangle_q w(q,\mu) + \int_{\mathbb{M}} \mathrm{div}_x \Big(\nabla_{\omega_2} w(q,\mu)(x) \Big) \mu(dx) \Big)$$

corresponds to the individual noises operator and

$$\mathcal{L}^{\epsilon_2}(w)(q,\mu) := \epsilon_2 \Big(2 \int_{\mathbb{M}} \mathrm{div}_q \Big(\nabla_{\omega_2} u(q,x,\mu) \Big) \mu(dx) + \int_{\mathbb{M}^2} \mathrm{Tr} \Big(\nabla^2_{\omega_2} u(q,x,y,\mu) \Big) \mu(dx)\mu(dy) \Big)$$

corresponds to the common noise operator.

The so called *Master Equation* in mean field games consists in finding $T > 0$ and

$$u : [0,T] \times \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

such that

$$\partial_t u(t,q,\mu) + H(q, \nabla_q u(t,q,\mu)) + N_\mu[\nabla_{\omega_2} u(t,q,\mu), \nabla_q u(t,\cdot,\mu)] + F(q,\mu) = \mathcal{L}_{\epsilon_1}^{\epsilon_2}(u)(t,q,\mu), \tag{5.3}$$

subject to the initial value condition

$$u(0,\cdot,\cdot) = u_*(\cdot,\cdot). \tag{5.4}$$

Remark 5.1. Assume $w : \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ and $\beta : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ are sufficiently smooth and set $u := w + \beta$. Then

$$\mathcal{L}_1(u)(q,\mu)(q,\mu) = \mathcal{L}_1(w)(q,\mu) + \int_{\mathbb{M}} \mathrm{div}_x \Big(\nabla_{\omega_2} \beta(\mu)(x) \Big) \mu(dx)$$

and

$$\mathcal{L}^1(u)(q,\mu)(q,\mu) = \mathcal{L}^1(w)(q,\mu) + \int_{\mathbb{M}^2} \mathrm{Tr} \Big(\nabla^2_{\omega_2} \beta(x,y,\mu) \Big) \mu(dx)\mu(dy)$$

Corollary 5.2. *Assume $u : \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ and $\beta : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ are sufficiently smooth and set $u := w + \beta$. Then*

$$N_\mu \big[\nabla_{\omega_2} u(q,\mu), \nabla_q u(t,\cdot,\mu) \big] = N_\mu \big[\nabla_{\omega_2} w(q,\mu), \nabla_q w(t,\cdot,\mu) \big] + N_\mu \big[\nabla_{\omega_2} \beta(\mu), \nabla_q w(t,\cdot,\mu) \big]$$

and

$$\mathcal{L}_{\epsilon_1}^{\epsilon_2}(u)(q,\mu) = \mathcal{L}_{\epsilon_1}^{\epsilon_2}(w)(q,\mu) + \epsilon_1 \int_{\mathbb{M}} \mathrm{div}_x \Big(\nabla_{\omega_2} \beta(\mu)(x) \Big) \mu(dx)$$
$$+ \epsilon_2 \int_{\mathbb{M}^2} \mathrm{Tr} \Big(\nabla^2_{\omega_2} \beta(x,y,\mu) \Big) \mu(dx)\mu(dy).$$

5.2 Conservation laws systems on $\mathcal{P}_2(\mathbb{M})$; formal computations

We define

$$f(q) := \mathcal{F}(\delta_q) \quad \forall q \in \mathbb{M}.$$

Part I. Hamilton Jacobi Equations. We will related a partial differential equation (PDE) on $\mathbb{M}$ to another one on $\mathcal{P}_2(\mathbb{M})$. We start with

$$U : [0,T] \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

such that

$$\partial_t U + \mathcal{H}(\mu, \nabla_{\omega_2} U) + \mathcal{F}(\mu) = 0. \tag{5.5}$$

Set

$$\varphi(t,q) := U(t, \delta_q) \qquad q \in \mathbb{M}$$

and assume U is smooth. By Exercise 4.11 , $\varphi(t,\cdot)$ is differentiable and

$$\nabla_q \varphi(t,q) = \nabla_{\omega_2} U(t, \delta_q)(q).$$

Thus,

$$\mathcal{H}(\mu, \nabla_{\omega_2} U(t,\delta_q)) = H\big(q, \nabla_{\omega_2} U(t,\delta_q)(q)\big) = H\big(q, \nabla_q \varphi(t,q)\big).$$

This implies

$$0 = \partial_t U + \mathcal{H}(\mu, \nabla_{\omega_2} U + \mathcal{F}(\mu) = \partial_t \varphi(t,q) + H(q, \nabla_q \varphi(t,q)) + f(q) = 0. \tag{5.6}$$

In other words, (5.6) is nothing but (5.5) restricted to one particle.

Set

$$v(t,q) = \nabla_q \varphi(t,q), \qquad\qquad V(t,x,\mu) = \nabla_{\omega_2} U(t,x,\mu)$$

The equation on $\mathbb{M}$ yields a system of equations on $\mathbb{M}$ while the equation on $\mathcal{P}_2(\mathbb{M})$ yields a system of equations on $\mathbb{M} \times \mathcal{P}_2(\mathbb{M})$.

System of Conservation Laws: *Vectorial Master Equation on* $\mathbb{M} \times \mathcal{P}_2(\mathbb{M})$

$$\partial_t v + \nabla_q \Big(H(q, v(t,q)) \Big) + \nabla_q f(q) = 0; \qquad \partial_t V + \nabla_q \Big(H(q,V) \Big) + \bar{N}[\nabla_{\omega_2} V, V] + \nabla_q F(q,\mu) = 0 \tag{5.7}$$

Here,

$$\bar{N}\big[\nabla_{\omega_2} V, V\big](t,q,\mu) := \int_{\mathbb{M}} \nabla_{\omega_2} V(t,q,x,\mu) \cdot D_p H(x, V(t,x,\mu)) d\mu$$

The second system in (5.7) is the so-called vectorial master equation. As the first system in (5.7) is a finite dimensional system of conservation laws derived from the last identity in (5.6), the vectorial master equation can viewed as an infinite dimensional system of conservation laws.

If we choose any $\beta(t,\mu)$ and set

$$u(t,q,\mu) := \frac{\delta U}{\delta \mu}(t,\mu)(q) + \beta(t,\mu)$$

then we have

$$\nabla_q u(t,q,\mu) = \nabla_{\omega_2} U(t,q,\mu).$$

The task to fulfill is to write a scalar version of the system of equations at the right handside of (5.7) by choosing β so that

$$\partial_t u + H(q, \nabla_q u) + N_\mu[\nabla_{\omega_2} u, \nabla_q u] + F(q,\mu) = 0 \tag{5.8}$$

where N_μ is the non local operator defined in (5.2). The equation is (5.8) is the so–called scalar master equation (in the absence of viscosity terms).

<u>*Perturbation of HJE on* $\mathbb{M}$</u> <u>*Perturbation of HJE on* $\mathcal{P}_2(\mathbb{M})$</u>

$$\partial_t \varphi(t,q) + H(q, \nabla \varphi(t,q)) + f(q) = \epsilon \triangle_q \varphi; \qquad \partial_t U + \mathcal{H}(\mu, \nabla_{\omega_2} U + \mathcal{F}(\mu) = \epsilon O(U) \tag{5.9}$$

Set

$$v(t,q) = \nabla_q \varphi(t,q), \qquad V(t,x,\mu) = \nabla_{\omega_2} U(t,x,\mu = \nabla_q u(t,q,\mu)$$

The equation on $\mathbb{M}$ yields a system of equations on $\mathbb{M}$ while the equation on $\mathcal{P}_2(\mathbb{M})$ yields a system of equations on $\mathbb{M} \times \mathcal{P}_2(\mathbb{M})$.

<u>*Navier-Stokes type:*</u> <u>*Scalar Master Equation on* $\mathbb{M} \times \mathcal{P}_2(\mathbb{R}^d)$</u>

$$\partial_t v + \nabla_q \Big(H(q, v(t,q)) \Big) + \nabla_q f(q) = \epsilon \triangle_q v; \quad \partial_t u + H(q, \nabla_q u) + N[\nabla_{\omega_2} u, \nabla_q u] + F(q,\mu) = \mathcal{L}_\epsilon(u) \tag{5.10}$$

Here, $\mathcal{L}_\epsilon$ is the operator defined below (1.1).

5.3 Lectures 20, 21: From HJEs to master equations (Nov 06, 08)

Let O, B, $\triangle_{\omega_2}$ be the operators defined at the beginning of this chapter. Fix $\mu \in \mathcal{P}_2(\mathbb{M})$ and let O be an open neighborhood of μ. We denote as I_d the d–dimensional identity matrix.

We suppose $\mathcal{F}, U : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ are sufficiently smooth functions and we set

$$u(q,\mu) := \frac{\delta U}{\delta \mu}(\mu)(q), \qquad F(q,\mu) := \frac{\delta \mathcal{F}}{\delta \mu}(\mu)(q) \qquad \forall \mu \in \mathcal{P}_2(\mathbb{M}),\ \forall q \in \mathbb{M}.$$

By Lemma 4.23 of Chapter 4

$$\nabla_q u(q,\mu) = D_\mu U(\mu)(q) = \nabla_{\omega_2} U(\mu)(q). \tag{5.11}$$

Thus

$$\mathcal{H}\big(\mu, \nabla_\mu U(\mu)\big) = \int_{\mathbb{M}} H\big(q, \nabla_q u(q,\mu)\big) \mu(dq). \tag{5.12}$$

Remark 5.3. While (5.11) means

$$\nabla_x\Big(\frac{\delta U}{\delta\mu}(x,\mu)\Big) = \nabla_{\omega_2}U(x,\mu),$$

one uses arguments similar to those in the proof of Lemma 4.23 of Chapter 4 to show that

$$\nabla_{xy}\bigg(\frac{\delta^2 U}{\delta^2\mu}(x,y,\mu)\bigg) = \nabla^2_{\omega_2}U(x,y,\mu)$$

The following expressions which depend on u will be useful:

$$\lambda_0(u)(\mu) = -\int_{\mathbb{M}} L\Big(q, D_pH\big(q,\nabla_q u(q,\mu)\big)\Big)\mu(dq),$$

$$\bar\lambda_0(u)(\mu) = \int_{\mathbb{M}}\Big(H\big(q,\nabla_q u(q,\mu)\big) + N_\mu\big[\nabla_{\omega_2}u(q,\mu)(\cdot),\nabla_x u(\cdot,\mu)\big](q)\Big)\mu(dq),$$

$$\lambda_1(u)(\mu) = \int_{\mathbb{M}}\Big(\triangle_q u(q,\mu) + \int_{\mathbb{M}} \operatorname{div}_x\big(\nabla_{\omega_2}u(x,\mu)(q)\big)\mu(dx)\Big)\mu(dq).$$

and

$$\lambda_2(u)(\mu) := \lambda_2^1(u)(\mu) + \lambda_2^2(u)(\mu)$$

where

$$\lambda_2^1(u)(\mu) := 2\int_{\mathbb{M}^2} \operatorname{div}_q\Big(\nabla_{\omega_2}u(q,x,\mu)\Big)\mu(dx)\mu(dq),$$

$$\lambda_2^2(u)(\mu) := \int_{\mathbb{M}^3} \operatorname{Tr}\Big(\nabla^2_{\omega_2}u(q,x,y,\mu)\Big)\mu(dx)\mu(dy)\mu(dq)$$

5.3.1 Useful identities

Recall (cf. [11])

$$\frac{\delta^2 U}{\delta\mu^2}(q,x,\mu) + \frac{\delta U}{\delta\mu}(x,\mu) = \frac{\delta^2 U}{\delta\mu^2}(x,q,\mu) + \frac{\delta U}{\delta\mu}(q,\mu). \tag{5.13}$$

This means,

$$\frac{\delta u}{\delta\mu}(q,\mu)(x) + u(x,\mu) = \frac{\delta u}{\delta\mu}(x,\mu)(q) + u(q,\mu).$$

Hence, by Lemma 4.23 of Chapter 4

$$\nabla_q\bigg(\frac{\delta u}{\delta\mu}(q,\mu)(x)\bigg) = \nabla_{\omega_2}u(x,\mu)(q) + \nabla_q u(q,\mu). \tag{5.14}$$

and so,

$$\nabla_x\bigg(\nabla_q\bigg(\frac{\delta u}{\delta\mu_x}(q,\mu)(x)\bigg)\bigg) = \nabla_x\bigg(\nabla_{\omega_2}u(x,\mu)(q)\bigg). \tag{5.15}$$

If $g : \mathbb{M}\times\mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ then

$$\nabla_q\bigg(\frac{\delta g(q,\mu)}{\delta\mu}\bigg)(x) = \frac{\delta}{\delta\mu}\bigg(\nabla_q g(q,\mu)\bigg)(x). \tag{5.16}$$

To emphasize the fact that q is a parameter and x is the action variable, we sometimes rather write

$$\nabla_q \Big(\frac{\delta g_q(\mu)}{\delta \mu_x} \Big)(x) = \Big(\frac{\delta (\nabla_q g_q(\mu))}{\delta \mu_x} \Big)(x).$$

We use (5.16) in the expression at the left handside of (5.15) to obtain

$$\nabla_{\omega_2} \Big(\nabla_q u(q,x,\mu) \Big) = \nabla_x \Big(\frac{\delta}{\delta \mu_x} \Big(\nabla_q u(q,x,\mu) \Big) \Big) = \nabla_x \Big(\nabla_{\omega_2} u(x,\mu)(q) \Big) \tag{5.17}$$

5.3.2 First variations; Example of weak Fréchet differentials on $\mathcal{P}_2(\mathbb{M})$

Thanks to Lemma 4.23 of Chapter 4, one checks that

$$O(U)(\mu) = \int_{\mathbb{M}} \triangle_q U(q,\mu)\mu(dq).$$

We use the last identity in Remark 4.25 of Chapter 4 to obtain

$$B(U)(\mu) = \int_{\mathbb{M}^2} \mathrm{Tr} \Big(\nabla_{qx} \Big(\frac{\delta^2 U}{\delta \mu^2}(q,x,\mu) \Big) \Big) \mu(dq)\mu(dx).$$

Set

$$V^{qx}(\mu) := \mathrm{Tr} \Big(\nabla_{qx} \frac{\delta^2 U}{\delta \mu^2}(q,x,\mu) \Big) = \mathrm{Tr} \Big(\nabla_{qx} \frac{\delta u(q,\mu)}{\delta \mu_x}(x) \Big) = \mathrm{Tr} \Big(\nabla_q \nabla_{\omega_2} u(q,x,\mu) \Big)$$

By (5.17)

$$V^{qx}(\mu) := \mathrm{Tr} \Big(\nabla_{\omega_2} \Big(\nabla_x u(x,q,\mu) \Big).$$

Proposition 5.4. *The following identities hold:*

(i)

$$\frac{\delta}{\delta \mu} \Big(\mathcal{H}\big(\mu, \nabla_{\omega_2} U(\mu)\big) \Big)(q) = H\big(q, \nabla_q u(\cdot,\mu)\big) + N_\mu \big[\nabla_{\omega_2} u(q,\mu)(\cdot), \nabla_x u(\cdot,\mu) \big] - \bar{\lambda}_0(u)(\mu).$$

(ii)

$$\frac{\delta}{\delta \mu} \Big(O(U(\mu)) \Big)(q) = \triangle_q u(q,\mu) + \int_{\mathbb{M}} \mathrm{div}_x \big(\nabla_{\omega_2} u(q,\mu)(x) \big) \mu(dx) - \lambda_1(u)(\mu),$$

Proof: (i) We use the expression in (5.12) and obtain

$$\begin{aligned} \mathcal{H}\Big((1-s)\mu + s\nu, \nabla_\mu U\big((1-s)\mu + s\nu\big) \Big) &= \int_{\mathbb{M}} H\big(q, \nabla_q u\big(q, (1-s)\mu + s\nu\big)\big)\mu(dq) \\ &+ s \int_{\mathbb{M}} H\big(q, \nabla_q u\big(q, (1-s)\mu + s\nu\big)\big)(\nu - \mu)(dq). \end{aligned} \tag{5.18}$$

Since

$$u(q,(1-s)\mu + s\nu) = u(q,\mu) + s \int_{\mathbb{M}} \frac{\delta u}{\delta \mu}(q,\mu)(x)(\nu - \mu)(dx) + o(s),$$

assuming enough differentiability properties on u, we conclude

$$\nabla_q u(q,(1-s)\mu+s\nu) = \nabla_q u(q,\mu) + s\int_{\mathbb{M}} \nabla_q\Big(\frac{\delta u}{\delta\mu}(q,\mu)(x)\Big)(\nu-\mu)(dx) + o(s).$$

This, together with (5.14) implies

$$\nabla_q u(q,(1-s)\mu+s\nu) = \nabla_q u(q,\mu) + s\int_{\mathbb{M}}\Big(\nabla_{\omega_2}u(x,\mu)(q)+\nabla_q u(q,\mu)\Big)(\nu-\mu)(dx) + o(s).$$

Since $\nabla_q u(q,\mu)$ is independent of x and $\nu-\mu$ is of null average, we obtain

$$\nabla_q u(q,(1-s)\mu+s\nu) = \nabla_q u(q,\mu) + s\int_{\mathbb{M}}\nabla_{\omega_2}u(x,\mu)(q)(\nu-\mu)(dx) + o(s). \tag{5.19}$$

We conclude

$$\begin{aligned} H\big(q,\nabla_q u(q,(1-s)\mu+s\nu)\big) =& H\big(q,\nabla_q u(q,\mu)\big) \\ &+ s\Big\langle D_pH\big(q,\nabla_q u(q,\mu)\big), \int_{\mathbb{M}}\nabla_{\omega_2}u(x,\mu)(q)(\nu-\mu)(dx)\Big\rangle + o(s) \end{aligned} \tag{5.20}$$

We combine (5.18) and (5.20) to obtain

$$\begin{aligned} \frac{\delta}{\delta\mu}\mathcal{H}\big(\mu,\nabla_{\omega_2}U(\mu)\big)(q)(\nu-\mu)(dq) =& \int_{\mathbb{M}}\Big\langle D_pH\big(q,\nabla_q u(q,\mu)\big), \int_{\mathbb{M}}\nabla_{\omega_2}u(x,\mu)(q)(\nu-\mu)(dx)\Big\rangle\mu(dq) \\ &+ \int_{\mathbb{M}} H\big(q,\nabla_q u(q,\mu)\big)(\nu-\mu)(dq) \\ =& \int_{\mathbb{M}}\int_{\mathbb{M}}\big\langle D_pH\big(x,\nabla_x u(x,\mu)\big),\nabla_{\omega_2}u(q,\mu)(x)\big\rangle\mu(dx)(\nu-\mu)(dq) \\ &+ \int_{\mathbb{M}} H\big(q,\nabla_q u(q,\mu)\big)(\nu-\mu)(dq). \end{aligned}$$

Thus, there is a constant $\lambda\equiv\lambda(\mu)$ such that

$$\frac{\delta}{\delta\mu}\mathcal{H}\big(\mu,\nabla_{\omega_2}U(\mu)\big)(q) = H\big(q,\nabla_q u(q,\mu)\big) + \int_{\mathbb{M}}\big\langle D_pH\big(x,\nabla_x u(x,\mu)\big),\nabla_{\omega_2}u(q,\mu)(x)\big\rangle\mu(dx) - \lambda(\mu)$$

We use that the weak Fréchet differential is of null μ–average to infer $\lambda(\mu)=\bar\lambda_0(u)(\mu)$.

(ii) We exploit (5.11) to obtain

$$\begin{aligned} O\Big(U((1-s)\mu+s\nu)\Big) =& \int_{\mathbb{M}}\mathrm{div}_q\Big(\nabla_q u(q,(1-s)\mu+s\nu))\Big)((1-s)\mu+s\nu))(dq) \\ =& \int_{\mathbb{M}}\mathrm{div}_q\Big(\nabla_q u(q,(1-s)\mu+s\nu))\Big)\mu(dq) \\ &+ s\int_{\mathbb{M}}\mathrm{div}_q\Big(\nabla_q u(q,(1-s)\mu+s\nu))\Big)(\nu-\mu)(dq). \end{aligned}$$

We use (5.19) to infer

$$\begin{aligned} O\Big(U((1-s)\mu + s\nu)\Big) = & \int_{\mathbb{M}} \operatorname{div}_q \Big(\nabla_q u(q,\mu)\Big)\mu(dq) \\ & + s\int_{\mathbb{M}}\int_{\mathbb{M}} \operatorname{div}_x\big(\nabla_{\omega_2} u(q,\mu)(x)\big)(\nu-\mu)(dq)\mu(dx) \\ & + s\int_{\mathbb{M}} \operatorname{div}_q\Big(\nabla_q u(q,(1-s)\mu+s\nu))\Big)(\nu-\mu)(dq) + o(s). \end{aligned} \tag{5.21}$$

Thus,

$$\begin{aligned} \int_{\mathbb{M}} \frac{\delta}{\delta\mu}\Big(O(U(\mu))\Big)(q)(\nu-\mu)(dq) = & \int_{\mathbb{M}}\int_{\mathbb{M}} \operatorname{div}_x\big(\nabla_{\omega_2} u(q,\mu)(x)\big)\mu(dx)(\nu-\mu)(dq) \\ & + \int_{\mathbb{M}} \operatorname{div}_q\Big(\nabla_q u(q,\mu)\Big)(\nu-\mu)(dq). \end{aligned}$$

This determines $\frac{\delta}{\delta\mu}\Big(O(U(\mu))\Big)$ up to a null constant which is uniquely identified by the fact that the weak Fréchet differential is of null μ–average . We verify (ii) . QED.

Exercise 5.5. Assume $U : \mathcal{P}_2(\mathbb{M}) \to \mathbb{R}$ is 4 times weakly Fréchet differentiable with continuous derivatives up to the boundary of $\mathbb{M}$. Show that there are functions

$$\phi_1, \phi_2, \phi_3 : \mathbb{M} \mapsto \mathbb{R}, \quad \phi_{12}, \phi_{13}, \phi_{23} : \mathbb{M}^2 \mapsto \mathbb{R}$$

such that

$$\frac{\delta^3 U}{\delta\mu^3}(q_1,q_2,q_3,\mu) - \frac{\delta^3 U}{\delta\mu^3}(q_3,q_2,q_1,\mu) = \sum_{i=1}^3 \phi_i(q_i) + \sum_{i<j}\phi_{ij}(q_i,q_j).$$

Proof: Let $\mu \in \mathcal{P}_2(\mathbb{M})$ and let ν_1,ν_2,ν_3 be signed Borel measure on $\mathbb{M}$ of finite total mass. Assume for s_1,s_2,s_3 small enough $\mu + s_1\nu_1 + s_2\nu_2 + s_3\nu_3$ remains nonnegative. The well–known calculus rule

$$\partial^3_{s_3 s_2 s_1}\Big(U(\mu + s_1\nu_1 + s_2\nu_2 + s_3\nu_3)\Big) = \partial^3_{s_1 s_2 s_3}\Big(U(\mu + s_1\nu_1 + s_2\nu_2 + s_3\nu_3)\Big),$$

means

$$\begin{aligned} & = \int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta\mu^3}(q_1,q_2,q_3,\mu + s_1\nu_1 + s_2\nu_2 + s_3\nu_3)\nu_1(dq_1)\nu_2(dq_2)\nu_3(dq_3) \\ & = \int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta\mu^3}(q_3,q_2,q_1,\mu + s_1\nu_1 + s_2\nu_2 + s_3\nu_3)\nu_1(dq_1)\nu_2(dq_2)\nu_3(dq_3) \end{aligned}$$

In particular when $(s_1,s_2,s_3) = (0,0,0)$, setting

$$w(q_1,q_2,q_3) := \frac{\delta^3 U}{\delta\mu^3}(q_3,q_2,q_1,\mu) - \frac{\delta^3 U}{\delta\mu^3}(q_1,q_2,q_3,\mu)$$

we reads off

$$\int_{\mathbb{M}^3} w(q_1,q_2,q_3)\nu_1(dq_1)\nu_2(dq_2)\nu_3(dq_3) = 0. \tag{5.22}$$

Let $\bar\nu_1, \bar\nu_2, \bar\nu_3$ be signed Borel measures of finite total mass and finite second moments. Let $\rho \in C_c(\mathbb{M})$ be a probability density. To alleviate the notation, we identify ρ with the measure $\rho\mathcal{L}^d$ and set

$$\nu_i := \bar\nu_i - \lambda_i\rho, \qquad \lambda_i := \bar\nu_i(\mathbb{M}).$$

By (5.22)

$$0 = \int_{\mathbb{M}^3} w(q_1,q_2,q_3)\big(\bar\nu_1(dq_1) - \lambda_1\rho(dq_1)\big)\big(\bar\nu_2(dq_2) - \lambda_2\rho(dq_2)\big)\big(\bar\nu_3(dq_3) - \lambda_3\rho(dq_3)\big)$$

Hence,

$$\begin{aligned}
0 = &\int_{\mathbb{M}^3} w(q_1,q_2,q_3)\Big(\bar\nu_1(dq_1)\bar\nu_2(dq_2)\bar\nu_3(dq_3) - \lambda_3\rho(dq_3)\bar\nu_1(dq_1)\bar\nu_2(dq_2)\Big)\\
&+ \int_{\mathbb{M}^3} w(q_1,q_2,q_3)\Big(-\bar\nu_1(dq_1)\lambda_2\rho(dq_2)\bar\nu_3(dq_3) + \bar\nu_1(dq_1)\lambda_2\rho(dq_2)\lambda_3\rho(dq_3)\Big)\\
&+ \int_{\mathbb{M}^3} w(q_1,q_2,q_3)\Big(-\lambda_1\rho(dq_1)\bar\nu_2(dq_2)\bar\nu_3(dq_3) + -\lambda_1\rho(dq_1)\bar\nu_2(dq_2)\lambda_3\rho(dq_3)\Big)\\
&+ \int_{\mathbb{M}^3} w(q_1,q_2,q_3)\Big(\lambda_1\rho(dq_1)\lambda_2\rho(dq_2)\bar\nu_3(dq_3) - \lambda_1\rho(dq_1)\lambda_2\rho(dq_2)\lambda_3\rho(dq_3)\Big).
\end{aligned}$$

Thus, setting $\bar\nu(dq) := \bar\nu_1(dq_1)\bar\nu_2(dq_2)\bar\nu_3(dq_3)$ we conclude

$$\begin{aligned}
0 = &\int_{\mathbb{M}^3} w(q_1,q_2,q_3)\bar\nu(dq) - \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}} w(q_1,q_2,q_3')\rho(dq_3')\Big)\bar\nu(dq)\\
&- \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}} w(q_1,q_2',q_3)\rho(dq_2')\Big)\bar\nu(dq) + \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}^2} w(q_1,q_2',q_3')\rho(dq_2')\rho(q_3')\Big)\bar\nu(dq)\\
&- \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}} w(q_1',q_2,q_3)\rho(dq_1')\Big)\bar\nu(dq) + \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}^2} w(q_1',q_2,q_3')\rho(dq_1')\rho(q_3')\Big)\bar\nu(dq)\\
&+ \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}^2} w(q_1',q_2',q_3)\rho(dq_1')\rho(q_2')\Big)\bar\nu(dq) - \int_{\mathbb{M}^3}\Big(\int_{\mathbb{M}^3} w(q_1',q_2',q_3')\rho(dq_1')\rho(q_2')\rho(dq_3')\Big)\bar\nu(dq).
\end{aligned}$$

Since $\bar\nu_1, \bar\nu_2, \bar\nu_3$ are arbitrary signed Borel measures we conclude that

$$\begin{aligned}
0 = {}& w(q_1,q_2,q_3) - \int_{\mathbb{M}} w(q_1,q_2,q_3')\rho(dq_3') - \int_{\mathbb{M}} w(q_1,q_2',q_3)\rho(dq_2') + \int_{\mathbb{M}^2} w(q_1,q_2',q_3')\rho(dq_2')\rho(q_3')\\
&- \int_{\mathbb{M}} w(q_1',q_2,q_3)\rho(dq_1') + \int_{\mathbb{M}^2} w(q_1',q_2,q_3')\rho(dq_1')\rho(q_3') + \int_{\mathbb{M}^2} w(q_1',q_2',q_3)\rho(dq_1')\rho(q_2')\\
&- \int_{\mathbb{M}^3} w(q_1',q_2',q_3')\rho(dq_1')\rho(q_2')\rho(dq_3').
\end{aligned}$$

This conclude the proof. QED.

Proposition 5.6. *The following identity hold:*

$$\frac{\delta}{\delta\mu}\big(B(U)\big) = 2\int_{\mathbb{M}} \mathrm{div}_q\Big(\nabla_{\omega_2} u(q,x,\mu)\Big)\mu(dx) + \int_{\mathbb{M}^2} \mathrm{Tr}\Big(\nabla^2_{\omega_2} u(q,x,y,\mu)\Big)\mu(dx)\mu(dy) - \lambda_2(u)(\mu)$$

Proof: We integrate by parts to write $B(U)$ in the form

$$B(U)(\mu) = \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dq), \nabla\mu(dx)\rangle$$

to conclude if $\mu_s := \mu + s(\nu-\mu)$ then

$$\begin{aligned}
B(U)(\mu+s(\nu-\mu)) &= \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu_s)\langle \nabla\mu_s(dq), \nabla\mu_s(dx)\rangle \\
&= \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu_s)\langle \nabla\mu(dq), \nabla\mu(dx)\rangle + o(s) \\
&+ s\int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu_s)\Big(\langle \nabla\mu(dq), \nabla(\nu-\mu)(dx)\rangle + \langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle\Big).
\end{aligned}$$

This implies

$$\begin{aligned}
B(U)(\mu+s(\nu-\mu)) &= \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dq), \nabla\mu(dx)\rangle \\
&+ s\int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta\mu^3}(q,x,y,\mu)\langle \nabla\mu(dq), \nabla\mu(dx)\rangle(\nu-\mu)(dy) \\
&+ s\int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dq), \nabla(\nu-\mu)(dx)\rangle \\
&+ s\int_{\mathbb{M}^2} \langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle + o(s).
\end{aligned} \tag{5.23}$$

We use first interchange the role of q and x and then (5.13) to conclude

$$\begin{aligned}
\int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dq), \nabla(\nu-\mu)(dx)\rangle &= \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(x,q,\mu)\langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle \\
&= \int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle \\
&= \int_{\mathbb{M}^2} \frac{\delta u}{\delta\mu}(q,x,\mu)\langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle.
\end{aligned}$$

We integrate by parts and then use the first identity in Remark 5.3 to conclude

$$\begin{aligned}
\int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta\mu^2}(q,x,\mu)\langle \nabla\mu(dq), \nabla(\nu-\mu)(dx)\rangle &= \int_{\mathbb{M}^2} \mathrm{div}_q\Big(\nabla_x \frac{\delta u}{\delta\mu}(q,x,\mu)\Big)\mu(dx)(\nu-\mu)(dq)\rangle \\
&= \int_{\mathbb{M}^2} \mathrm{div}_q\Big(\nabla_{\omega_2} u(q,x,\mu)\Big)\mu(dx)(\nu-\mu)(dq)\rangle
\end{aligned} \tag{5.24}$$

Similarly, integrating by parts and using the first identity in Remark 5.3 we have

$$\int_{\mathbb{M}^2} \frac{\delta^2 U}{\delta \mu^2}(q,x,\mu)\langle \nabla\mu(dx), \nabla(\nu-\mu)(dq)\rangle = \int_{\mathbb{M}^2} \mathrm{div}_q\Big(\nabla_{\omega_2} u(q,x,\mu)\Big)\mu(dx)(\nu-\mu)(dq)\rangle. \quad (5.25)$$

By Exercise 5.5 we may write

$$\frac{\delta^3 U}{\delta \mu^3}(q,x,y,\mu) = \frac{\delta^3 U}{\delta \mu^3}(y,x,q,\mu) + \phi_1(q) + \phi_2(x) + \phi_3(y) + \phi_{12}(q,x) + \phi_{13}(q,y) + \phi_{23}(x,y). \quad (5.26)$$

Note for any function f which is continuously differentiable, integrating by parts, we observe

$$0 = \int_{\mathbb{M}^3} f(q)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \quad (5.27)$$

$$0 = \int_{\mathbb{M}^3} f(x)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \quad (5.28)$$

$$0 = \int_{\mathbb{M}^3} f(y)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \quad (5.29)$$

Similarly, integrations by parts reveal

$$0 = \int_{\mathbb{M}^3} \phi_{13}(q,y)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \quad (5.30)$$

$$0 = \int_{\mathbb{M}^3} \phi_{23}(x,y)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \quad (5.31)$$

Note

$$\int_{\mathbb{M}^3} \phi_{12}(q,x)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) = \int_{\mathbb{M}}(\nu-\mu)(dy)\int_{\mathbb{M}^2} \phi_{12}(q,x)\langle \nabla\mu(dx), \nabla\mu(dq) = 0.$$

This, together with (5.26)- (5.31) implies

$$\begin{aligned}
&\int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta \mu^3}(q,x,y,\mu)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \\
&= \int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta \mu^3}(y,x,q,\mu)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) \\
&= \int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta \mu^3}(q,x,y,\mu)\langle \nabla\mu(dx), \nabla\mu(dy)\rangle(\nu-\mu)(dq) \\
&= \int_{\mathbb{M}^3} \mathrm{Tr}\Big(\nabla_{xy}\Big(\frac{\delta^3 U}{\delta \mu^3}(q,x,y,\mu)\Big)\Big)\mu(dx)\mu(dy)(\nu-\mu)(dq) \\
&= \int_{\mathbb{M}^3} \mathrm{Tr}\Big(\nabla_{xy}\Big(\frac{\delta^2 u}{\delta \mu^2}(q,x,y,\mu)\Big)\Big)\mu(dx)\mu(dy)(\nu-\mu)(dq)
\end{aligned} \quad (5.32)$$

We use the second identity in Remark 5.3 to conclude that

$$\int_{\mathbb{M}^3} \frac{\delta^3 U}{\delta \mu^3}(q,x,y,\mu)\langle \nabla\mu(dx), \nabla\mu(dq)\rangle(\nu-\mu)(dy) = \int_{\mathbb{M}^3} \mathrm{Tr}\Big(\nabla^2_{\omega_2} u\big(q,x,y,\mu\big)\Big)\mu(dx)\mu(dy)(\nu-\mu)(dq). \quad (5.33)$$

We combine (5.23), (5.24), (5.25) and (5.33) to conclude there exists a constant $\lambda \equiv \lambda(\mu)$ such that

$$\frac{\delta}{\delta\mu}\Big(B(U)\Big)(\mu) = 2\int_{\mathbb{M}^2} \mathrm{div}_q\Big(\nabla_{\omega_2} u(q,x,y,\mu)\Big)\mu(dx) + \int_{\mathbb{M}^3} \mathrm{Tr}\Big(\nabla^2_{\omega_2} u\big(q,x,y,\mu\big)\Big)\mu(dx)\mu(dy) - \lambda.$$

We use the fact that $\frac{\delta}{\delta\mu}\Big(B(U)\Big)(\mu)$ is of μ–null average to determine λ and verify the proof of the proposition. QED.

Exercise 5.7. Show $\lambda_0(u)(\mu) = \bar\lambda_0(u)(\mu)$.

Proof: By (5.14)

$$N_\mu\big[\nabla_{\omega_2} u(q,\mu)(\cdot), \nabla_x u(\cdot,\mu)\big] = \int_{\mathbb{M}} \Big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x\Big(\frac{\delta u}{\delta\mu}(x,\mu)(q)\Big) - \nabla_x u(x,\mu)\Big\rangle \mu(dx)$$

and so,

$$\begin{aligned}
\int_{\mathbb{M}} N_\mu\big[\nabla_{\omega_2} u(q,\mu)(\cdot), \nabla_x u(\cdot,\mu)\big]\mu(dq) &= \int_{\mathbb{M}\times\mathbb{M}} \big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x\Big(\frac{\delta u}{\delta\mu}(x,\mu)(q)\Big)\big\rangle \mu(dx)\mu(dq)\\
&\quad - \int_{\mathbb{M}} \big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x u(x,\mu)\big\rangle \mu(dx)\\
&= \int_{\mathbb{M}} \big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x \int_{\mathbb{M}} \Big(\frac{\delta u}{\delta\mu}(x,\mu)(q)\Big)\mu(dq)\big\rangle \mu(dx)\\
&\quad - \int_{\mathbb{M}} \big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x u(x,\mu)\big\rangle \mu(dx)\\
&= -\int_{\mathbb{M}} \big\langle D_p H\big(x, \nabla_x u(x,\mu)\big), \nabla_x u(x,\mu)\big\rangle \mu(dx) \qquad (5.34)
\end{aligned}$$

We use the identity

$$H(q,p) + L(q, D_p H(q,p)) = \langle p, D_p H(q,p)\rangle$$

in (5.34) to conclude that $\bar\lambda_0(u)(\mu) = \lambda_0(u)(\mu)$. QED.

5.4 The master equation via a transport equation on $\mathcal{P}_2(\mathbb{M})$

Assume

$$U : [0,T] \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

is a sufficiently smooth solution to the (local) Hamilton–Jacobi equation

$$\partial_t U(t,\mu) + \mathcal{H}(\mu, \nabla_{\omega_2} U(t,\mu)) + \mathcal{F}(\mu) = \epsilon_1 O(U)(t,\mu) + \epsilon_2 B(U)(t,q) \qquad (5.35)$$

in $(0,T) \times \mathcal{P}_2(\mathbb{M})$ along with the initial value condition

$$U(0,\mu) = U_*. \qquad (5.36)$$

In particular, U is a viscosity solution to (5.35-5.36) and so, it is uniquely determined (cf. [30]).

Set
$$w(t,q,\mu) := \frac{\delta U}{\delta \mu}(t,\mu)(q).$$
and
$$\lambda_{\epsilon_1}^{\epsilon_2}(w)(t,\mu) := \epsilon_1 \lambda_1(w)(t,\mu) + \epsilon_2 \lambda_2(w)(t,\mu) - \bar\lambda_0(w)(t,\mu).$$
Set
$$\begin{aligned}\Lambda_{\epsilon_1}^{\epsilon_2}(w,\beta)(t,\mu) := \epsilon_1 \int_{\mathbb{M}} &\mathrm{div}_x\Big(\nabla_{\omega_2}\beta(\mu)(x)\Big)\mu(dx)\\ &+ \epsilon_2 \int_{\mathbb{M}^2} \mathrm{Tr}\Big(\nabla^2_{\omega_2}\beta(x,y,\mu)\Big)\mu(dx)\mu(dy) - N_\mu\big[\nabla_{\omega_2}\beta(\mu), \nabla_q w(t,\cdot,\mu)\big].\end{aligned}$$

Theorem 5.8. *Suppose $\beta : [0,T]\times\mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ is sufficiently smooth and set $u \equiv w+\beta$. Then u satisfies the master equation* (5.3) *along with the initial value condition* (5.4) *if and only if β satisfies the differential equation*
$$\begin{aligned}\partial_t\beta(t,\mu) + N_\mu\big[\nabla_{\omega_2}\beta(t,\mu), \nabla_q w(t,\cdot,\mu)\big] = \lambda_{\epsilon_1}^{\epsilon_2}(w)(t,\mu) + \epsilon_1 \int_{\mathbb{M}} \mathrm{div}_x\Big(\nabla_{\omega_2}\beta(t,\mu)(x)\Big)\mu(dx)\\ + \epsilon_2 \int_{\mathbb{M}^2} \mathrm{Tr}\Big(\nabla^2_{\omega_2}\beta(t,x,y,\mu)\Big)\mu(dx)\mu(dy) \qquad (5.37)\end{aligned}$$
and
$$u(0,\cdot,\cdot) = w(0,\cdot,\cdot) + \beta(0,\cdot) \qquad (5.38)$$

Proof: We differentiate both sides of the expressions in (5.35) and apply Propositions 5.4 and 5.6 to conclude that
$$\begin{aligned}&\partial_t w(t,q,\mu) + H(q, \nabla_q w(t,q,\mu) + N_\mu\big[\nabla_{\omega_2} w(t,q,\mu), \nabla_x w(t,\cdot,\mu)\big] + F(q,\mu)\\ &= \mathcal{L}_{\epsilon_1}^{\epsilon_2} w(t,q,\mu) - \lambda_{\epsilon_1}^{\epsilon_2}(w)(t,q,\mu). \qquad (5.39)\end{aligned}$$
We use Corollary 5.2 to conclude that
$$\begin{aligned}&\partial_t u(t,q,\mu) - \partial_t\beta(t,\mu) + H(q, \nabla_q u(t,q,\mu) + N_\mu\big[\nabla_{\omega_2} u(t,q,\mu)(\cdot), \nabla_x u(t,\cdot,\mu)\big] + F(q,\mu)\\ =&\mathcal{L}_{\epsilon_1}^{\epsilon_2} u(t,q,\mu) - \lambda_{\epsilon_1}^{\epsilon_2}(w) - \Lambda_{\epsilon_1}^{\epsilon_2}(w,\beta)(t,\mu).\end{aligned}$$
The function β is determined by the initial condition (5.38). This concludes the proof of the theorem. QED.

Remark 5.9. Let l_0 be the Lipschitz constant of $D_{(q,p)}H$ and assume $\nabla_q u(t,\cdot,\cdot)$ ls l_1–Lipschitz and bounded. Then
$$\vec V(t,q,\mu) := D_p H\big(q, \nabla_q u(t,q,\mu)\big)$$
is Lipschitz $l_0(1+l_1)$–Lipschitz for each $t\in[0,T]$ and is uniformly bounded. Thanks to [26], given $\mu\in\mathcal{P}_2(\mathbb{M})$ and $t>0$ there exists a unique absolutely continuous path $\sigma[t,\mu] : [0,t] \mapsto \mathcal{P}_2(\mathbb{M})$ such that
$$\partial_\tau\Big(\sigma[t,\mu](\tau)\Big) + \mathrm{div}_q\Big(\sigma[t,\mu](\tau)\vec V(\tau,q,\sigma[t,\mu](\tau))\Big) = 0 \quad \text{in} \quad (0,t)\times\mathbb{M}, \qquad \sigma[t,\mu](0) = \mu.$$
in the sense of (4.1).

Remark 5.10. We continue using the notation of Theorem 5.8 .

(i) Note (5.37) is a first order infinite dimensional linear equation in β.

(ii) If there exists $r \geq 0$ such that $\epsilon_1 = \epsilon_2(1+r)$ then (5.37) is equivalent to

$$\partial_t \beta + N_\mu\big[\nabla_{\omega_2}\beta(\mu), \nabla_q w(t,\cdot,\mu)\big] = \lambda^{\epsilon_2}_{\epsilon+\epsilon_2}(w)(t,\mu) + \epsilon_2 \triangle_{\omega_2,r}\beta.$$

Lemma 5.11. *We continue using the notation of Theorem 5.8 and assume $\epsilon_1 = \epsilon_2 = 0$. Given a sufficiently smooth function $\beta_0 : \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$, (5.37) admits a solution starting at β_0.*

Proof: Let $(\sigma, \vec{V})$ be as in Remark 5.9. Note (5.37) is equivalent to

$$\frac{d}{d\tau}\Big(\beta\Big(\tau, \sigma[t,\mu](\tau)\Big)\Big) = \lambda^{\epsilon_2}_{\epsilon_1}(w)\Big(\tau, \sigma[\tau,\mu](\tau)\Big).$$

Thus,

$$\beta(t,\mu) = \beta_0\Big(\sigma[t,\mu](0)\Big) + \int_0^t \lambda^0_0(w)\Big(\tau, \sigma[\tau,\mu](\tau)\Big)d\tau.$$

QED.

Remark 5.12. One could wonder if we can choose $\beta \equiv 0$ in Theorem 5.8, meaning $u \equiv \frac{\delta U}{\delta \mu}$ is a solution to the master equation. We argue that such a statement is in general false. Indeed, assume for instance $\epsilon_1 = \epsilon_2 = 0$ and assume on the contrary that $u \equiv \frac{\delta U}{\delta \mu}$ is a solution to the master equation. By Lemma 5.11,

$$\int_0^t \lambda^0_0(w)\Big(\tau, \sigma[\tau,\mu](\tau)\Big)d\tau \equiv 0 \qquad \forall t \in [0,T].$$

By Exercise 5.7 $\lambda_0(u)(\mu) = \bar{\lambda}_0(u)(\mu)$ and so, we infer

$$0 = \int_{\mathbb{M}} L\Big(q, D_p H\big(q, \nabla_q w(t,q,\mu)\big)\Big)\mu(dq) \qquad \forall t \in [0,T].$$

Taking for instance $H(q,p) = |p|^2/2$, the above condition on w reads off

$$0 = \int_{\mathbb{M}} |\nabla_q w(t,q,\mu)|^2 \mu(dq) \qquad \forall t \in [0,T],$$

which means

$$w(t,q,\mu) \equiv w(t,\mu).$$

Since

$$0 = \int_{\mathbb{M}} \frac{\delta U}{\delta \mu}(t,q,\mu)\mu(dq) = \int_{\mathbb{M}} w(t,\mu)\mu(dq) = w(t,\mu),$$

we use that by definition $w \equiv \frac{\delta U}{\delta \mu}$ to conclude that

$$\frac{\delta U}{\delta \mu}(t,q,\mu) \equiv 0.$$

This proves that in general, we cannot expect that $u \equiv \frac{\delta U}{\delta \mu}$ unless we are dealing with trivial solutions.

Chapter 6

Stochastic calculus on $\mathcal{P}_2(\mathbb{R}^d)$

In this chapter, unless the contrary is explicitly stated, we assume

$$\mathbb{M} = \mathbb{R}^d.$$

6.1 Lecture 22: A crash course on stochastic analysis on $\mathbb{R}^d$ (Nov 13)

Let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space.

(i) The expectation of a Random variable $X : \Omega \mapsto \mathbb{R}^d$ is $\mathbb{E}(X) := \int_\Omega X(\omega)\mathbb{P}(d\omega)$.

(ii) We say that $A_1, \cdots, A_n \in \mathcal{B}_\Omega$ are independent if

$$\mathbb{P}\big(A_{i_1} \cap \cdots \cap A_{i_k}\big) = \mathbb{P}(A_{i_1}) \cdots P(A_{i_k})$$

for any $i_1, \cdots, i_k \in \{1, \cdots, n\}$ such that $i_1 < \cdots < i_k$.

(iii) If Λ is a non empty set and for each $\lambda \in \Lambda$, $\mathcal{A}_\lambda$ is a subset of $\mathcal{B}_\Omega$, we say that $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ is independent if for each natural number n and each distinct $\lambda_1, \cdots, \lambda_n \in \Lambda$ and each $A_1 \in \mathcal{A}_{\lambda_1}, \cdots, A_n \in \mathcal{A}_{\lambda_n}$ we have that $A_1, \cdots, A_n \in \mathcal{B}_\Omega$ are independent.

Assume for each $\lambda \in \Lambda$, X_λ is a d–dimensional Radon variable and

$$\mathcal{A}_\lambda = \{X_\lambda^{-1}(B) \mid B \in \mathcal{B}_{\mathbb{R}^d}\}.$$

We say that $\{X_\lambda \mid \lambda \in \Lambda\}$ are independent if $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ are independent.

Let $\dot{M} = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of $\mathbb{R}^d$, which is topologically identified with the unit sphere in $\mathbb{R}^{d+1}$. The set $\dot{M}$ is a compact Hausdorff space and so, by Tychonov theorem, if $I = [0, \infty)$ or is an open interval

$$\Omega := \prod_I \dot{M} \tag{6.1}$$

is a Hausdorff space for the product topology, which is the topology generated by the sets $\pi_i^{-1}(V_i)$ where V_i is an arbitrary open subset of $\dot{M}$ and $\pi_i : \Omega \mapsto \dot{M}$ denotes the ith projection on Ω. An element of Ω is any $\omega : I \to \dot{M}$.

Set

$$p_t(x,y) = \frac{e^{-\frac{|x-y|^2}{2t}}}{\sqrt{2\pi t}^d}, \quad x, y \in \mathbb{R}^d, t > 0$$

and set $p_0(dy) = \delta_x(dy)$.

Remark 6.1. The following are well–known fact of the theory of probability. There is a unique Radon measure $\mathbb{P}$ on Ω which satisfies the following (cf. e.g. Section 10.4 [46]):

(i) if for any natural number n, $\dot{M}_n$ denote the n cartesian product of $\dot{M}$ then

$$\int_{\dot{M}_n} F(x) p_{t_1}(0, x_1) p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx = \int_\Omega f(\omega) \mathbb{P}(d\omega).$$

Here

$$f(\omega) := F\big(\omega(t_1), \cdots, \omega(t_n)\big), \quad F \in C\big(\dot{M}_n\big), \quad x = (x_1, \cdots, x_n).$$

(ii) Any topology on Ω containing all the $\pi_i^{-1}(V_i)$ where $V_i \in \mathcal{B}_{\dot{M}}$ contains the smallest σ which contains the $\pi_i^{-1}(V_i)$'s. Therefore, the product topology generated contains the product σ–algebra generated by $(\pi_i)_{i \in I}$. For instance, $\Omega_c := C([0, \infty), \mathbb{R}^d)$ is a Borel subset of Ω such that $\mathbb{P}(\Omega_c) = 1$. However, Ω_c does not belong to the σ–algebra generated by $(\pi_i)_{i \in I}$ (cf. e.g Section 10.5 [46]).

(iii) It is apparent from (ii) that $\mathbb{P}$ is concentrated on the set of $\omega \in \Omega$ such that $\omega(0) = \vec{0}$.

Theorem 6.2. *There exists a topological probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that there exists a d–dimensional stochastic process $\{W_t \mid t \geq 0\}$ satisfying the following properties:*
(i) $W_0 = 0$ $\mathbb{P}$–almost everywhere
(ii) For each $0 \leq s < t$, the law of $W_t - W_s$ is $N(0, t-s)$.
(iii) For any $0 < t_1 < \cdots < t_n$, $W_{t_1} - W_0, W_{t_2} - W_{t_1} \cdots W_{t_{n-1}} - W_{t_n}$ are independent.

Definition 6.3. Any d–dimensional process satisfying (i–iii) of Theorem 6.2 is called a d–dimensional Wiener measure.

Theorem 6.4. *Let $\{W_t \mid t \geq 0\}$ be a a d–dimensional Wiener measure and set $W_t := (W_t^1, \cdots, W_t^d)$. Let $t, s \geq 0$ and $k, l \in \{1, \cdots, d\}$. We have*

(i) $\mathbb{E}(W_t^k W_s^l) = \delta_{kl} \min\{t, s\}$.

(ii) $\mathbb{E}\Big((W_t^k - W_s^k)(W_t^l - W_s^l)\Big) = \delta_{kl}(t-s)$.

(iii) If $0 < t_1 < \cdots < t_n$ and $F \in C\big((\mathbb{R}^d)^n\big)$ then

$$\mathbb{E}\Big(F\big(W_{t_1}, \cdots, W_{t_n}\big)\Big) = \int_\Omega F\big(\omega(t_1), \cdots, \omega(t_n)\big) \mathbb{P}(d\omega).$$

Definition 6.5. Let $(W_t)_{t \geq 0}$ be a n–dimensional Brownian motion. Let $T > 0$, let $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $a : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d \times n}$ be measurable function such that there exist $C, D > 0$ such that

$$|b(t, x)| + |a(t, x)| \leq C(1 + |x|), \quad |b(t, x) - b(t, y)| + |a(t, x) - a(t, y)| \leq D|x - y|$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Let $Z : \Omega \mapsto \mathbb{R}^d$ be a Radon variable of finite second moment independent of the σ–algebra generated by $\{W_t \mid t \geq 0\}$. We say that $X : [0, T] \times \Omega \mapsto \mathbb{R}^d$ is a solution to the equation

$$dX_t = b(t, X_t)dt + a(t, X_t)dW_t, \quad X_0 = Z \tag{6.2}$$

if the following hold:

(i) X is adapted to the filtration $\mathcal{F}_t^Z$ generated by $\{W_s \mid s \in [0, t]\} \cup \{Z\}$ such that $t \to X_t(\omega)$ is continuous for $\mathbb{P}$–almost everywhere $\omega \in \Omega$,

(ii)

$$\mathbb{E}\Big(\int_0^T |X_t|^2 dt\Big) < \infty$$

(iii) for any $0 \leq t_1 < t_2 \leq T$ we have

$$X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} b(t, X_t)dt + \int_{t_1}^{t_2} a(t, X_t)dW_t.$$

Here, by X_0, we mean Z.

Theorem 6.6. *Under the assumptions of Definition 6.5, there exists a unique solution to* (6.2).

Theorem 6.7. *Impose the assumptions of Definition 6.5 and use the notation there. If f : $C^2([0, T] \times \mathbb{R}^d)$ a function of bounded second derivative then for any $0 \leq s < r \leq T$ then*

$$f(r, X_r) - f(s, X_s) = \int_s^r \Big(\partial_t f(t, X_t) + \big\langle \nabla_x f(t, X_t), b(t, X_t)dt + a(t, X_t)dW_t\big\rangle\Big)$$
$$\frac{1}{2}\int_s^r \mathrm{Tr}\Big(a^*(t, X_t)\nabla_x^2 f(t, X_t)a(t, X_t)\Big)dt.$$

Here, a^* is the transposed matrix of a.

6.2 Lecture 23: Stochastic motions on $\mathcal{P}_2(\mathbb{R}^d)$ (Nov 15)

In the sequel, we fix $(W_t)_{t\geq 0}$ is a standard d–dimensional Brownian motion starting at the origin. In what follows, we will define a stochastic process on the Wasserstein space as

$$t \mapsto \mathbb{B}_t^\mu := (Id + \sqrt{2}W_t)_\# \mu.$$

In other words, for any $\omega \in \Omega$

$$\mathbb{B}_t^\mu(\omega)(E) = \mu\big(E - \sqrt{2}W_t(\omega)\big) \qquad \forall E \subset \mathbb{M}$$

More generally, we define

$$\sigma_t^{\epsilon,\beta}[m] := (Id + \sqrt{2\beta}W_t)_\#(G_t^\epsilon * m).$$

Here, G_t^ϵ is the heat kernel for the heat equation define as

$$G_t^\epsilon(z) = p_{2\epsilon t}(z, 0) = \frac{1}{\sqrt{4\pi\epsilon t}^d} e^{-\frac{|z|^2}{4\epsilon t}}$$

so that

$$\partial_t\big(G_t^\epsilon * u_0\big) = \epsilon\triangle\big(G_t^\epsilon * u_0\big), \qquad \text{on} \quad (0,\infty)\times\mathbb{M}.$$

By definition, a path $t\to\sigma_t\in\mathcal{P}_2(\mathbb{M})$ solves the differentiable equation

$$d\sigma = \operatorname{div}\Big(a\nabla\sigma dt - \sqrt{2b}\sigma dW\Big) \text{ on } \quad (0,T)\times\mathcal{P}_2(\mathbb{M}), \quad \sigma_0=\mu \tag{6.3}$$

if for every $\phi\in C^1\big((0,T);C_c^2(\mathbb{M})\big)$ and every $0<s<r<T$ we have

$$\begin{aligned}&\int_{\mathbb{M}}\phi(r,y)\sigma_r(dy) - \int_{\mathbb{M}}\phi(s,x)\sigma_s(dx)\\ &=\int_s^r\Big(\int_{\mathbb{M}}\partial_t\phi(t,x)\sigma_t(dx)dt + \sqrt{2b}\langle\nabla\phi(t,x),\sigma_t(dx)dW\rangle + a\triangle\phi(t,x)\sigma_t dx)dt\Big)\end{aligned} \tag{6.4}$$

and

$$\lim_{t\to 0^+}\mathbb{E}\big(W_2(\sigma_t,\mu)\big)=0.$$

We would like to argue that $\mathbb{B}_t^\mu$ satisfies (6.3) and further satisfies the non–linear version in Theorem 6.8. Similarly, we have

$$d\sigma^{\epsilon,\beta} = \operatorname{div}\Big((\epsilon+\beta)\nabla\sigma^{\epsilon,\beta}dt - \sqrt{2\beta}\sigma^{\epsilon,\beta}dW\Big) \text{ on } \quad (0,T)\times\mathcal{P}_2(\mathbb{M}), \quad \sigma_0^{\epsilon,\beta}=\mu \tag{6.5}$$

Let $T>0$ and let $V:[0,T]\times\mathcal{P}_2(\mathbb{M})\to\mathbb{R}$ be a continuous map such that V is continuously differentiable on $(0,T)\times\mathcal{P}_2(\mathbb{M})$. We further assume that for each $t>0$, $V(t,\cdot)$ admits the Taylor expansion in (4.17) and $\partial_t V$, $\nabla_{\omega_2}V$, $\nabla(\nabla_{\omega_2}V)$ and $\nabla^2_{\omega_2}V$ are continuous on $[0,T]\times\mathcal{P}_2(\mathbb{M})$ in the sense that they have a continuous extension. Suppose that for any $\mu\in\mathcal{P}_2(\mathbb{M})$ there exists C_μ and a neighborhood $\mathcal{O}_\mu$ of μ such that

$$|\nabla_{\omega_2}V(t,\nu)(x)|\le C_\mu(1+|x|) \quad \forall t\in[0,T], \quad \forall x\in\mathbb{M}, \quad \forall\nu\in\mathcal{O}_\mu \tag{6.6}$$

and

$$\Big|\nabla_x\Big(\nabla_{\omega_2}V(t,\nu)(x)\Big)\Big|,\ \big|\nabla^2_{\omega_2}V(t,\nu)\big|\le C_\mu \quad \forall\nu\in\mathcal{O}_\mu. \tag{6.7}$$

Theorem 6.8. *Setting $\sigma_t := (Id+\sqrt{2\beta}W_t)_\# m$, and $\sigma_t^\epsilon := (Id+\sqrt{2\beta}W_t)_\#(G_t^\epsilon * m)$. Then for any $0<s<r<T$, we have*

(i)

$$V(r,\sigma_r)-V(s,\sigma_s) = \int_s^r\Big(\partial_t V(t,\sigma_t)dt+\sqrt{2\beta}\Big\langle\int_{\mathbb{M}}\nabla_{\omega_2}V(t,\sigma_t)(x)\sigma_t(dx);dW_t\Big\rangle+\beta\triangle_{\omega_2}V(t,\sigma_t)dt\Big).$$

(ii)

$$V(r,\sigma_r^\epsilon)-V(s,\sigma_s^\epsilon) = \int_s^r\Big(\partial_t V(t,\sigma_t^\epsilon)dt+\sqrt{2\beta}\Big\langle\int_{\mathbb{M}}\nabla_{\omega_2}V(t,\sigma_t^\epsilon)(x)\sigma_t^\epsilon(dx);dW_t\Big\rangle+\beta\triangle_{w,\frac{\epsilon}{\beta}}V(t,\sigma_t^\epsilon)dt\Big).$$

Proof: Given $(a,\mu) \in \mathbb{M} \times \mathcal{P}_2(\mathbb{M})$, we set

$$M(a,\mu) := (Id + \sqrt{2\beta}a)_{\#}\mu, \qquad \Lambda_1(t,a) := V\big(t, M(a,\mu)\big)$$

$$\Lambda_3(t,a) := \int_{\mathbb{M}} \nabla_{\omega_2} V\big(t, \big[M(a,\mu)\big](x)\big) M(a,\mu)(dx).$$

For $h \in \mathbb{R}^d$ we define the measure $\gamma_h \equiv \gamma_h^a$ on $\mathbb{M}^{2d}$ by

$$\int_{\mathbb{M}^{2d}} F(x,y)\gamma_h(dx,dy) = \int_{\mathbb{M}} F(x+\sqrt{2\beta}a, x+\sqrt{2\beta}(a+h))\mu(dx) \qquad \forall F \in C_c(\mathbb{R}^{2d}).$$

Note γ_h has $M(a,\mu)$ as its first marginal and $M(a+h,\mu)$ as its second marginal.

Claim 1. We claim that γ_h is an optimal coupling between $M(a,\mu)$ and $M(a+h,\mu)$.
Proof of Claim 1. If τ is a permutation of n letters and $(x_i)_{i=1}^n$ then setting $c=\sqrt{2\beta}a$ and $e=\sqrt{2\beta}(a+h)$ we have

$$\sum_{i=1}^n |(x_i+c)-(x_{\tau(i)}+e)|^2 = \sum_{i=1}^n |x_i-x_{\tau(i)}|^2+|c-e|^2+2\langle x_i-x_{\tau(i)}, c-e\rangle = \sum_{i=1}^n |x_i-x_{\tau(i)}|^2+n|c-e|^2.$$

This implies

$$\sum_{i=1}^n |(x_i+c)-(x_{\tau(i)}+e)|^2 \geq \sum_{i=1}^n |(x_i+c)-(x_i+e)|^2.$$

Hence, the support of γ_h is cyclically monotone and so, by Theorem 3.21, $\gamma_h \in \Gamma_0\big(M(a,\mu), M(a+h,\mu)\big)$. This proves Claim 1.

Claim 2. We have $\nabla\Lambda_1(t,a) = \sqrt{2\beta}\Lambda_3(t, M(a,\mu))$.
Proof of Claim 2. We have

$$\begin{aligned} &V\big(t, M(a+h,\mu)\big) - V\big(t, M(a,\mu)\big) \\ &= \int_{\mathbb{M}} \langle \nabla_{\omega_2} V\big(t, M(a,\mu)(x), y-x\rangle \gamma_h(dx,dy) + o\big(W_2 M(a,\mu), M(a+h,\mu)\big). \end{aligned}$$

This reads off

$$\Lambda_1(t,a+h) - \Lambda_1(t,a) = \sqrt{2\beta}\int_{\mathbb{M}} \langle \nabla_{\omega_2} V\big(t, M(a,\mu)\big)(x+\sqrt{2\beta}a), h\rangle \mu(dx) + o\big(|h|\big),$$

which means

$$\Lambda_1(t,a+h) - \Lambda_1(t,a) = \sqrt{2\beta}\int_{\mathbb{M}} \langle \nabla_{\omega_2} V\big(t, M(a,\mu)\big)(z), h\rangle M(a,\mu)(dz) + o\big(|h|\big),$$

This proves Claim 2.

Claim 3. We have $\triangle\Lambda_1(t,a) = 2\beta\triangle_{\omega_2} V\big(t, \big[M(a,\mu)\big]\big)$.

Proof of Claim 3. By Claim 2,

$$\nabla\Lambda_1(t,a+h) = \sqrt{2\beta}\int_{\mathbb{M}^{2d}} \nabla_{\omega_2} V\big(t, M(a+h,\mu)\big)(z)\gamma_h(dw,dz). \tag{6.8}$$

But

$$\begin{aligned}\nabla_{\omega_2} V\big(t, M(a+h,\mu)\big)(z) &= \nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w) + \nabla_x\nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w)(z-w)\\ &+ \int_{\mathbb{M}^{2d}} \nabla^2_{\omega_2} V\big(t, M(a,\mu)\big)(w,x)(y-x)\gamma_h(dx,dy)\\ &+ \Upsilon\Big(W_2(M(a,\mu), M(a+h,\mu)), |w-z|\Big)\end{aligned}$$

where

$$\Upsilon(s,t) = (s+t)\big(\rho(t)+\epsilon(s)\big),$$

ρ is a concave modulus and $\lim_{t\to 0}\epsilon(t) = 0$. Thus,

$$\begin{aligned}\nabla_{\omega_2} V\big(t, M(a+h,\mu)\big)(z) &= \nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w) + \nabla_x\nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w)(z-w)\\ &+ \sqrt{2\beta}\Big(\int_{\mathbb{M}^{d}} \nabla^2_{\omega_2} V\big(t, M(a,\mu)\big)(w,x)M(a,\mu)(dx)\Big)h\\ &+ \Upsilon\Big(W_2(M(a,\mu), M(a+h,\mu)), |w-z|\Big)\end{aligned}$$

We combine this together with (6.8) to conclude that

$$\begin{aligned}\nabla\Lambda_1(t,a+h) = \nabla\Lambda_1(t,a) &+ \sqrt{2\beta}\int_{\mathbb{M}^{2d}} \nabla_x\nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w)(z-w)\gamma_h(dw,dz)\\ &+ 2\beta\int_{\mathbb{M}^{2d}}\Big(\int_{\mathbb{M}^{d}} \nabla^2_{\omega_2} V\big(t, M(a,\mu)\big)(w,x)M(a,\mu)(dx)\Big)\gamma_h(dw,dz)h\\ &+ \sqrt{2\beta}\int_{\mathbb{M}^{2d}} \Upsilon\Big(W_2(M(a,\mu), M(a+h,\mu)), |w-z|\Big)\gamma_h(dw,dz)\end{aligned}$$

Thus,

$$\begin{aligned}\nabla\Lambda_1(t,a+h) = \nabla\Lambda_1(t,a) &+ 2\beta\Big(\int_{\mathbb{M}^{d}} \nabla_x\nabla_{\omega_2} V\big(t, M(a,\mu)\big)(w)M(a,\mu)(dw)\Big)h\\ &+ 2\beta\Big(\int_{\mathbb{M}^{2d}} \nabla^2_{\omega_2} V\big(t, M(a,\mu)\big)(w,x)M(a,\mu)(dx)M(a,\mu)(dw)\Big)h\\ &+ \sqrt{2\beta}\int_{\mathbb{M}^{2d}} \Upsilon\Big(|h|, |w-z|\Big)\gamma_h(dw,dz)\end{aligned} \tag{6.9}$$

But

$$\int_{\mathbb{M}^{2d}} \Upsilon\Big(|h|, |w-z|\Big)\gamma_h(dw,dz) = \Upsilon\Big(|h|, \sqrt{2\beta}|h|\Big) = o(|h|). \tag{6.10}$$

We combine (6.9) and (6.10) to conclude that

$$\nabla^2\Lambda_1(t,a) = 2\beta\Big(\int_{\mathbb{M}^d} \nabla_x\nabla_{\omega_2}V\big(t,M(a,\mu)\big)(w)M(a,\mu)(dw)\Big)$$
$$+\,2\beta\Big(\int_{\mathbb{M}^{2d}} \nabla^2_{\omega_2}V\big(t,M(a,\mu)\big)(w,x)M(a,\mu)(dx)M(a,\mu)(dw)\Big).$$

Hence,

$$\triangle\Lambda_1(t,a) = 2\beta\Big(\int_{\mathbb{M}^d} \mathrm{Tr}\Big(\nabla_x\nabla_{\omega_2}V\big(t,M(a,\mu)\big)(w)\Big)M(a,\mu)(dw)\Big)$$
$$+\,2\beta\Big(\int_{\mathbb{M}^{2d}} \mathrm{Tr}\Big(\nabla^2_{\omega_2}V\big(t,M(a,\mu)\big)(w,x)\Big)M(a,\mu)(dx)M(a,\mu)(dw)\Big)$$
$$= 2\beta\triangle_{\omega_2}V\big(t,M(a,\mu)\big).$$

Claim 4. By the standard finite dimensional Itô formula if $0\le s<r\le T$ then

$$\Lambda_1(r,W_r)-\Lambda_1(s,W_s) = \int_s^r\Big(\partial_t(t,W_t)dt + \langle\nabla\Lambda_1(t,W_t),dW_t\rangle + \frac{1}{2}\triangle\Lambda_1(t,W_t)dt\Big).$$

We use Claims 2 and 3 to conclude the proof of (i).

The proof of (ii) follows the same lines of arguments.

Remark 6.9. Theorem 6.8 is in fact an extension of Itô formula to the set of probability measures. An extension based on probabilistic arguments (hence different from ours) was proposed in books [14] [15]. For very general processes, [12] gave a proof which uses a heavy machinery. We have opted to offer the above proof since our setting is different from that of the above cited prior works.

Consider the following σ–algebra.

$$\Sigma(t) := \sigma\{W_s \mid 0\le s\le t\} = \mathcal{W}(t), \quad \mathcal{W}^+(t) := \sigma\{W_s - W_t \mid t\le s\}. \tag{6.11}$$

We have that

$$\Sigma(s)\subset\Sigma(t), \quad \mathcal{W}(t)\subset\Sigma(t), \qquad \Sigma(t) \text{ is independent of } \mathcal{W}^+(t)$$

for any $0\le s\le t$. In other words $\Sigma(\cdot)$ is a non anticipating filtration with respect to $\mathcal{W}(\cdot)$.

Recall $\mathbb{L}^2_d(0,T)$ is the set of $G:(0,T)\times\Omega\mapsto\mathbb{R}^d$ such that $G(t)$ is $\Sigma(t)$–measurable (G is called progressively measurable) and

$$\mathbb{E}\Big(\int_0^T |G|^2dt\Big)<\infty.$$

Under this condition, we recall

$$\mathbb{E}\Big(\int_0^T\langle G_t,dW_t\rangle\Big)=0. \tag{6.12}$$

Exercise 6.10. Let μ, V and σ be as in Theorem 6.8. Show that for any $0\le s<r\le T$ we have

$$\mathbb{E}\Big(\int_s^r\Big\langle\int_{\mathbb{M}}\nabla_{\omega_2}V(t,\sigma_t)(x)\sigma_t(dx),dW_t\Big\rangle\Big)=0$$

Proof: Set

$$G(t,\cdot) := \int_{\mathbb{M}} \nabla_{\omega_2} V(t,\sigma_t)(x)\sigma_t(dx)$$

It suffices to show that $G \in \mathbb{L}^2_d(0,T)$ and use (6.12) to conclude the proof.

Let Λ_3 be the continuous function defined in the proof of Theorem 6.8. As

$$G(t,\cdot) = \Lambda_3(t, W_t),$$

we conclude G is progressively measurable with respect to the filtration $\Sigma(\cdot)$ in (6.11). We use that $\nabla^2_{\omega_2} V(t,\cdot)$ is uniformly bounded by a constant C to conclude

$$\Big|\nabla_{\omega_2} V(t,\sigma_t)(x) - \nabla_{\omega_2} V(t,\mu)(x)\Big| \le C W_2(\sigma_t,\mu) = C\sqrt{2\beta}|W_t|.$$

This, together with (6.6) yields

$$\Big|\nabla_{\omega_2} V(t,\sigma_t)(x)\Big| \le C_\mu(1+|x|) + C\sqrt{2\beta}|W_t|.$$

Since

$$\int_{\mathbb{M}} |x|^2 \sigma_t(dx) = \int_{\mathbb{M}} |x+\sqrt{2\beta}W_t|^2 \mu(dx) \le 2\|Id\|^2_\mu + 2(\sqrt{2\beta}|W_t|)^2$$

we conclude there is a constant $\bar{C}_\mu$ such that

$$\Big\|\nabla_{\omega_2} V(t,\sigma_t)(x)\Big\|^2_{\sigma_t} \le \bar{C}_\mu\Big(1+|W_t|^2\Big).$$

This implies

$$\mathbb{E}\Big(\int_0^T \Big\|\nabla_{\omega_2} V(t,\sigma_t)(x)\Big\|^2_{\sigma_t} dt\Big) < \infty,$$

which is enough to verify the desired result. QED.

Theorem 6.11. *Assume $U_0 : \mathcal{P}_2(\mathbb{M}) \to \mathbb{R}$ is twice continuously differentiable (cf. Definition 4.30). We assume for any $\mu \in \mathcal{P}_2(\mathbb{M})$, $\nabla\big(\nabla_{\omega_2} U_0[\mu]\big) : \mathbb{M} \to \mathbb{R}^{d\times d}$ and $\nabla^2_{\omega_2} U_0[\mu] : \mathbb{M}\times\mathbb{M} \to \mathbb{R}^{d\times d}$ are uniformly bounded and have a concave modulus of continuity independent of μ. We fix $\epsilon, \beta > 0$ and define*

$$U(t,\mu) := \mathbb{E}\Big(U_0\big(\sigma_t^{\epsilon,\beta}[\mu]\big)\Big) \qquad \forall (t,\mu) \in (0,\infty)\times\mathcal{P}_2(\mathbb{M}).$$

Then

(i) U is continuously differentiable in $(0,\infty)\times\mathcal{P}_2(\mathbb{M})$ and for any $t > 0$, $U(t,\cdot)$ is twice continuously differentiable on $\mathcal{P}_2(\mathbb{M})$.

(ii) U satisfies the heat equation

$$\partial_t U = \beta \triangle_{\omega_2,\frac{\epsilon}{\beta}} U \quad in \quad (0,\infty)\times\mathcal{P}_2(\mathbb{M}), \quad U(0,\cdot) = U_0.$$

Proof:

Part I: Proof in a particular case. While in general the proof of the theorem is based on Theorem 6.8, we shall give here a more direct proof when $\epsilon = 0$ and

$$U_0(\mu) = \frac{1}{k}\int_{\mathbb{M}^k} \exp\Big(-2\pi i \sum_{j=1}^k \langle x_i, \xi_i\rangle\Big)\mu(dx_1)\cdots\mu(dx_k).$$

In this case

$$U_0(\sigma_t) = \exp\Big(-2\pi i \Big\langle \sqrt{2\beta} W_t, \sum_{j=1}^k \xi_j\Big\rangle\Big) U_0(\mu) \tag{6.13}$$

But, changing variable, we have

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-2\pi i \sum_{j=1}^k \langle \xi_j, \sqrt{2\beta}W_t\rangle\Big)\Big) &= \int_{\mathbb{M}} \exp\Big(-2\pi i \sum_{j=1}^k \langle \xi_j, \sqrt{2\beta} z\rangle\Big) G_t^{0.5}(z)dz \\ &= \int_{\mathbb{M}} \frac{1}{\sqrt{2\beta}^d} \exp\Big(-2\pi i \Big\langle \sum_{j=1}^k \xi_j, w\Big\rangle\Big) G_t^{0.5}\Big(\frac{w}{\sqrt{2\beta}}\Big)dw \\ &= \int_{\mathbb{M}} \exp\Big(-2\pi i \Big\langle \sum_{j=1}^k \xi_j, w\Big\rangle\Big) G_t^{\beta}(w)dw. \end{aligned} \tag{6.14}$$

We then face the computation of the Fourier transform of G_t^β which by the Fourier transform table is

$$\widehat{G}_t^\beta(\sum_{j=1}^k \xi_j) = \exp\Big(-4\pi^2 \Big|\sum_{j=1}^k \xi_j\Big|^2 \beta t\Big)$$

This, together with (6.14) yields

$$\mathbb{E}\Big(\exp\Big(-2\pi i \sum_{j=1}^k \langle \xi_j, \sqrt{2\beta}W_t\rangle\Big)\Big) = \exp\Big(-4\pi^2 \Big|\sum_{j=1}^k \xi_j\Big|^2 \beta t\Big). \tag{6.15}$$

We combine (6.13) and (6.15) to obtain

$$U(t,\mu) = \exp\Big(-\beta t \lambda_k^2(\xi)\Big) U_0(\mu).$$

Since $\triangle_{\omega_2} U_0 = -\lambda_k^2(\xi) U_0$, we conclude the proof of the theorem in a particular case.

Part II: Proof in the general case. Set $V(t,) := U_0$ so that using the notation Theorem 6.8 we have

$$\Lambda_1(t,a) := U_0\big(M(a,\mu)\big).$$

We have

$$U_0(\sigma_r^\epsilon) - U_0(\sigma_s^\epsilon) = \int_s^r \Big(\sqrt{2\beta}\Big\langle \int_{\mathbb{M}} \nabla_{\omega_2} U_0(\sigma_t^\epsilon)(x)\sigma_t^\epsilon(dx); dW_t\Big\rangle + \beta \triangle_{w,\frac{\epsilon}{\beta}} U_0(\sigma_t^\epsilon)dt\Big).$$

We apply Exercise 6.10 to obtain

$$\frac{U(s+h,\mu)-U(s,\mu)}{h}=\mathbb{E}\left(\frac{1}{h}\int_s^{s+h}\beta\triangle_{w,\frac{\epsilon}{\beta}}U_0(\sigma_t^\epsilon)dt\right).$$

Letting $h\to 0$ we deduce the desired result. QED.

Chapter 7

Existence of solutions to the master equation

Throughout this chapter $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is endowed with the quotient metric: if $x, y \in \mathbb{R}^d$,

$$|x - y|_{\mathbb{T}^d} = \min_{k \in \mathbb{Z}^d} |x - y - k|.$$

The goal of this section is to state some existence results for the master equation.

We denote as $\mathcal{P}(\mathbb{T}^d)$ the quotient of $\mathcal{P}_2(\mathbb{R}^d)$ by the following equivalence relation (cf. [29]): $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ are equivalent if

$$\int_{\mathbb{R}^d} \varphi(q)\mu_0(dq) = \int_{\mathbb{R}^d} \varphi(q)\mu_1(dq) \qquad \forall \varphi \in C(\mathbb{T}^d).$$

We denote as $[\mu_0]$ the class of equivalence of μ_0. The Wasserstein distance $\mathcal{P}(\mathbb{T}^d)$ which can be realized as a quotient distance is

$$\mathcal{W}(\mu_0, \bar{\mu}_0) = \mathcal{W}([\mu_0], [\bar{\mu}_0]) = \min_{\mu_1, \bar{\mu}_1} \Big\{ W_2(\mu_1, \bar{\mu}_1) \mid \mu_1 \in [\mu_0], \bar{\mu}_1 \in [\bar{\mu}_0] \Big\}.$$

We assume $H \in C^3(\mathbb{R}^{2d})$ is such that $H(\cdot, p)$ is $\mathbb{Z}^d$–periodic for every $p \in \mathbb{R}^d$. We write

$$H \in C^3(\mathbb{T}^d \times \mathbb{R}^d).$$

We assume there exists $\kappa > 0$ and κ–Lipchitz functions $F, u_* : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \mapsto \mathbb{R}$ such that for each $\mu \in \mathcal{P}(\mathbb{T}^d)$, $F(\cdot, \mu)$, $u_*(\cdot, \mu)$ is three times continuously differentiable with

$$|\nabla_q F|, \;\; |\nabla_{qq} F|, \;\; |\nabla_{qqq} F|, \;\; |\nabla_q u_*|, \;\; |\nabla_{qq} u_*|, \;\; |\nabla_{qqq} u_*| \leq \kappa.$$

As done in (1.1) for $w : \mathbb{M} \times \mathbb{R}^d \mapsto \mathbb{R}$ sufficiently smooth, we define

$$\mathcal{L}_{\epsilon_1}^{\epsilon_2}(w) := \mathcal{L}_{\epsilon_1}(w) + \mathcal{L}^{\epsilon_2}(w).$$

We fix $T > 0$ and would like to find

$$u : [0, T] \times \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$$

such that

$$\partial_t u(t,q,\mu) + H(q, \nabla_q u(t,q,\mu)) + N_\mu[\nabla_{\omega_2} u(t,q,\mu), \nabla_q u(t,\cdot,\mu)] + F(q,\mu) = \mathcal{L}_{\epsilon_1}^{\epsilon_2}(u)(t,q,\mu), \quad (7.1)$$

subject to the initial value condition

$$u(0,\cdot,\cdot) = u_*. \quad (7.2)$$

7.1 Lectures 24, 25: Potential games (Nov 20, 27)

Given $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $s > 0$, the Mean Field Games system we consider consists in finding $\tilde{U} \in \text{Lip}(0, s] \times \mathbb{T}^d)$ and $\sigma \in AC_2(0, s; \mathcal{P}(\mathbb{T}^d))$ such that

$$\begin{cases} \partial_t \tilde{U}(t,q) + H\big(q, \nabla_q \tilde{U}(t,q)\big) + F(q, \tilde{\sigma}_t) & = 0 \quad \text{in} \quad (0,s) \times \mathbb{T}^d \\ \partial_t \tilde{\sigma} + \text{div}\Big(\tilde{\sigma} D_p H(\cdot, \nabla_q \tilde{U}(t,\cdot))\Big) & = 0 \quad \text{in} \quad \mathcal{D}'((0,s) \times \mathbb{T}^d) \\ \tilde{U}(0,\cdot) & = u_*(\cdot, \sigma_0) \quad \text{in} \quad \mathbb{T}^d \\ \tilde{\sigma}_s & = \mu. \end{cases} \quad (7.3)$$

Theorem 7.1. *There exists $T > 0$ such that the following hold:*

(i) The system of differential equations

$$\begin{cases} \dot{\Sigma}^1 & = D_p H(\Sigma^1, \Sigma^2) \quad in \quad (0,T) \times \mathbb{T}^d \\ \dot{\Sigma}^2 & = -D_q H(\Sigma^1, \Sigma^2) - \nabla_q F(\Sigma^1, \Sigma^1_\sharp \mu) \quad in \quad (0,T) \times \mathbb{T}^d \\ \Sigma^1(s,\cdot) & = \mathbf{id}, \\ \Sigma^2(0,\cdot) & = \nabla_q u_*\big(\Sigma^1(0,\cdot), \Sigma^1(0,\cdot)_\sharp \mu\big) \end{cases} \quad (7.4)$$

has a solution $\Sigma = (\Sigma^1, \Sigma^2)$, for any $(s, \mu) \in (0,T) \times \mathcal{P}(\mathbb{T}^d)$.

(ii) There exist $A_1, A_2, B, E > 0$ such that the solution to (7.4) is unique in the set of $\Sigma \in W^{2,\infty}\big((0,T) \times \mathbb{T}^d; \mathbb{T}^{2d}\big)$ satisfying

$$|\partial_t \Sigma^1|, \; |\nabla_q \Sigma^1|, \; |\nabla_{qq} \Sigma^1| \le A_1, \quad |\partial_t \Sigma^2|, \; |\nabla_q \Sigma^2|, \; |\nabla_{qq} \Sigma^2| \le A_2$$

and

$$|\nabla_q \Sigma^1(0,\cdot)|, \; |\nabla_{qq} \Sigma^1(0,\cdot)| \le E, \quad |\Sigma^2| \le B$$

(iii) Under the conditions imposed in (ii) which ensure uniquness of solution, we may write $\Sigma \equiv \Sigma[s,\mu](t,q)$. There is a constant $D_1 > 0$ (depending only $T, A_1, A_2, B, E > 0$) such that $\Sigma[\cdot,\mu](t,q)$ is Lipschitz and $\text{Lip}(\Sigma[\cdot,\mu](t,q)) \le D_1$.

(iv) Under the conditions imposed in (iii), $(t,s,q,\mu)\to\mathcal{M}(t,s,q,\mu):=(t,s,q,\Sigma^1[s,\mu)(t,q))$ *is continuous and* $\mathcal{M}(\cdot,\cdot,\cdot,\mu)$ *is a* C^1 *diffeomeorphism.*

Proof: The proof is based on a standard fixed point argument and is due to [41]. QED.

In the sequel
$$X[s,\mu](t,\cdot)=\Big(\Sigma^1[s,\mu](t,\cdot)\Big)^{-1},$$
and we define
$$v[s,\mu](t,\cdot)=\partial_t\Sigma^1[s,\mu](t,\cdot)\circ X[s,\mu](t,\cdot)=D_pH\Big(\cdot,\mathcal{V}[s,\mu](t,\cdot)\Big). \tag{7.5}$$
where
$$\mathcal{V}[s,\mu](t,\cdot)=\Sigma^2[s,\mu](t,\cdot)\circ X[s,\mu](t,\cdot). \tag{7.6}$$

Exercise 7.2. Show that if we set $\tilde\sigma_t:=\Sigma^1_{t\,\sharp}\mu$ then
$$\begin{cases}\partial_t\tilde\sigma+\mathrm{div}\Big(v[s,\mu](t,\cdot)\tilde\sigma\Big) &=0 \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{T}^d)\\ \qquad\qquad\qquad\qquad\tilde\sigma_s &=\mu.\end{cases}$$

Proof: Despite the fact that the proof is standard, we still repeat it here. Let $\varphi\in C^1(0,T;C^1_c(\mathbb{T}^d))$. Setting
$$a(\varphi):=\int_0^T dt\int_{\mathbb{T}^d}\Big(\partial_t\varphi(t,q)+\langle\nabla\varphi(t,\cdot),v[s,\mu](t,\cdot)\rangle\Big)\tilde\sigma_t(dq),$$
we have
$$\begin{aligned}a(\varphi)&=\int_0^T dt\int_{\mathbb{T}^d}\Big(\partial_t\varphi\big(t,\Sigma^1(t,\cdot)\big)+\big\langle\nabla\varphi\big(t,\Sigma^1(t,\cdot)\big),v\big(t,\Sigma^1(t,\cdot)\big)\big\rangle\Big)\mu(dq)\\&=\int_0^T dt\int_{\mathbb{T}^d}\Big(\partial_t\varphi(t,\Sigma^1(t,\cdot))+\big\langle\nabla\varphi(t,\Sigma^1(t,\cdot)),\dot\Sigma^1(t,\cdot))\big\rangle\Big)\mu(dq)\\&=\int_0^T dt\int_{\mathbb{T}^d}\frac{d}{dt}\Big(\varphi(t,\Sigma^1(t,\cdot))\Big)\mu(dq)\\&=\int_{\mathbb{T}^d}\Big(\varphi(T,\Sigma^1(T,\cdot))-\varphi(0,\Sigma^1(0,\cdot))\Big)\mu(dq)\\&=\int_{\mathbb{T}^d}\varphi(T,\cdot)\tilde\sigma_T(dq)-\int_{\mathbb{T}^d}\varphi(0,\cdot)\tilde\sigma_0(dq)\end{aligned}$$
This concludes the proof. QED.

Exercise 7.3. Show that if T is sufficiently small and $(\mu_n)_n\subset\mathcal{P}(\mathbb{T}^d)$ converges narrowly to μ then $(v[\cdot,\mu_n])_n$ converges uniformly to $v[\cdot,\mu]$ on $(0,T)^2\times\mathbb{T}^d$.

Hint: Show we can choose T sufficiently small such that
$$\frac{1}{2}<\det\nabla_q\Sigma^1[s,\mu](t,q).$$
Conclude
$$\nabla_q X^1[s,\mu](t,q)\le 4(1+\sqrt{d})^{d-1}.$$

Exercise 7.4. For T sufficiently small define $z : [0,T] \times \mathbb{T}^d \mapsto \mathbb{R}$ as

$$z(0,q) = u_*\Big(\Sigma^1[s,\mu](0,q), \Sigma^1[s,\mu](0,\cdot)_\sharp\mu\Big)$$

and

$$\partial_t z_t = \Big\langle \Sigma^2[s,\sigma]_t, D_pH\big(\Sigma[s,\sigma]_t\big)\Big\rangle - H\big(\Sigma[s,\sigma]_t\big) - F\Big(\Sigma^1[s,\mu]_t, \Sigma^1[s,\mu]_{t\,\sharp}\mu\Big) \tag{7.7}$$

Set

$$\tilde U(t,\cdot) := z(t, X(t,\cdot)), \qquad \tilde\sigma_t := \Sigma^1_{t\,\sharp}\mu.$$

Show that $\nabla_q \tilde U_t \circ \Sigma^1_t = \Sigma^2_t$ and $(\tilde U, \tilde \sigma)$ satisfies (7.3).

Proof: By its definition, $\partial_t z$ is continuous in t and continuously differentiable in q. That z is continuously differentiable on $(0,T) \times \mathbb{T}^d$. We have $\tilde U(t, \Sigma^1(t,\cdot)) = z(t,\cdot)$ and so,

$$\begin{aligned}\partial_t z(t,\cdot) &= \partial_t \tilde U\big((t,\Sigma^1(t,\cdot)\big) + \langle \nabla_q \tilde U\big((t,\Sigma^1(t,\cdot)\big), \dot\Sigma^1(t,\cdot)\rangle \\ &= \partial_t \tilde U\big((t,\Sigma^1(t,\cdot)\big) + \Big\langle \nabla_q \tilde U\big((t,\Sigma^1(t,\cdot)\big), D_pH\big(\Sigma(t,\cdot)\big)\Big\rangle \end{aligned} \tag{7.8}$$

Set

$$r_t := \nabla_q z_t - (\nabla_q \Sigma^1_t)^T \Sigma^2_t.$$

Note

$$r_0 = (\nabla_q \Sigma^1_0)^T \nabla u_*\big(\Sigma^1_0, \Sigma^1_{0\,\sharp}\mu\big) - (\nabla_q\Sigma^1_0)^T\Sigma^2_0 \equiv 0. \tag{7.9}$$

Since

$$\partial_t z_t = \Big\langle \Sigma^2_t, \partial_t\Sigma^1_t)\Big\rangle - H\big(\Sigma_t\big) - F\Big(\Sigma^1_t, \Sigma^1_\sharp\mu\Big),$$

using the fact that because $\partial_t z$ is differentiable in q we have $\partial_t\nabla_q z = \nabla_q\partial_t z$, we obtain

$$\begin{aligned}\partial_t\nabla_q z = \nabla_q\partial_t z &= (\nabla_q\Sigma^2)^T\partial_t\Sigma^1 + \big(\nabla_q\partial_t\Sigma^1\big)^T\Sigma^2 \\ &\quad - (\nabla_q\Sigma^1)^T D_qH(\Sigma) - (\nabla_q\Sigma^2)^T D_pH(\Sigma) - (\nabla_q\Sigma^1)^T D_qF(\Sigma, \Sigma_\sharp\mu)\end{aligned} \tag{7.10}$$

We have

$$\begin{aligned}\partial_t\Big(\nabla_q\Sigma^1_t)^T\Sigma^2_t\Big) &= (\nabla_q\partial_t\Sigma^1)^T\Sigma^2 + (\nabla_q\Sigma^1)^T\partial_t\Sigma^2 \\ &= \Big(\nabla_q\partial_t\Sigma^1\Big)^T\Sigma^2 - (\nabla_q\Sigma^1)^T\Big(D_qH(\Sigma) + \nabla_qF(\Sigma,\Sigma_\sharp\mu)\Big).\end{aligned} \tag{7.11}$$

We combine (7.10) and (7.11) to conclude that

$$\partial_t r = (\nabla_q\Sigma^2)^T\partial_t\Sigma^1 - (\nabla_q\Sigma^2)^T D_pH(\Sigma) = (\nabla_q\Sigma^2)^T\Big(\partial_t\Sigma^1 - D_pH(\Sigma)\Big) = 0.$$

This, together with (7.9) implies

$$\nabla_q z_t \equiv (\nabla_q\Sigma^1_t)^T\Sigma^2_t. \tag{7.12}$$

Direct calculations show

$$\nabla_q\tilde U_t = (\nabla_q X_t)^T \nabla_q z_t \circ X_t,$$

which combined with (7.12) implies

$$\nabla_q \tilde{U}_t = (\nabla_q X_t)^T (\nabla_q \Sigma_t^1 \circ X_t)^T \Sigma_t^2 \circ X_t = \Sigma_t^2 \circ X_t.$$

We combine (7.7) and (7.8) and replace Σ_t^2 by $\nabla_q \tilde{U}_t \circ \Sigma_t^1$ to conclude that

$$\partial_t \tilde{U}\big(t, \Sigma_t^1\big) + \Big\langle \nabla_q \tilde{U}\big(t, \Sigma_t^1\big), D_p H\big(\Sigma_t\big) \Big\rangle = \Big\langle \nabla_q \tilde{U}\big(t, \Sigma_t^1\big), D_p H\big(\Sigma_t\big) \Big\rangle - H\big(\Sigma_t\big) - F\Big(\Sigma_t^1, \Sigma^1_{t\,\sharp}\mu\Big).$$

We simplify the previous identity and replace Σ_t by $(\Sigma_t^1, \nabla_q \tilde{U}_t \circ \Sigma_t^1)$ to conclude that

$$\partial_t \tilde{U}\big(t, \Sigma_t^1\big) = -H\big(\Sigma_t^1, \nabla_q \tilde{U}_t \circ \Sigma_t^1\big) - F\Big(\Sigma_t^1, \Sigma^1_{t\,\sharp}\mu\Big).$$

This verifies the first identity in (7.3). We use (7.5) and Exercise 7.2 to obtain the second identity in (7.3). The definition of $\tilde{U}$ ensures the third identity in (7.3) holds, while the last identity is due to the fact that $\Sigma^1[s, \mu]_0 = Id$. QED.

Theorem 7.5. *There exists $T > 0$ such that the following holds for any $(s, \mu) \in (0, T) \times \mathcal{P}(\mathbb{T}^d)$.*

(i) The system (7.3) admits at least one solution $(\tilde{U}, \sigma)$ such that $\tilde{U} \in W^{2,\infty}\big((0,T) \times \mathbb{T}^d\big)$ and $\sigma \in AC_2(0, T; \mathcal{P}(\mathbb{T}^d))$.

(ii) The system (7.3) admits at most one solution $(\tilde{U}, \tilde{\sigma})$ such that $\tilde{\sigma} \in AC_2(0, T; \mathcal{P}(\mathbb{T}^d))$, $\tilde{U} \in W^{2,\infty}\big((0,T) \times \mathbb{T}^d\big)$ and $\tilde{W}$ is $W^{3,\infty}$ in the q–variables.

Proof: The verification of (i) can be found in Exercise 7.4.

(ii) Assume $(\tilde{U}, \tilde{\sigma})$ is the solution constructed earlier and $(\bar{U}, \bar{\sigma})$ is another solution. Set

$$\bar{v}(t, q) = D_p H\big(q, \nabla_q \bar{U}(t, q)\big). \tag{7.13}$$

Since $\mathbb{T}^d$ is a compact set and $\bar{v}$ is continuous and Lipschitz in q, the following system of ODEs admits a unique solution:

$$\bar{v}(t, \bar{\Sigma}^1(t, q)) = \partial_t \bar{\Sigma}^1(t, q), \quad (t, q) \in [0, T] \times \mathbb{T}^d \qquad \bar{\Sigma}^1(s, \cdot) = Id. \tag{7.14}$$

We conclude by the uniqueness theory of continuity equations on $\mathcal{P}(\mathbb{T}^d) \subset \mathcal{P}_2(\mathbb{R}^d)$ in [2] that

$$\bar{\sigma}_t = \bar{\Sigma}^1_{t\,\sharp}\mu.$$

Set

$$\bar{\mathcal{V}}(t, q) = \nabla_q \bar{U}(t, q), \quad \bar{\Sigma}^2(t, q) = \bar{\mathcal{V}}\big(t, \bar{\Sigma}^1(t, q)\big). \tag{7.15}$$

By (7.13), (7.14) and (7.15)

$$\partial_t \bar{\Sigma}^1 = D_p H\big(\bar{\Sigma}^1, \bar{\Sigma}^2\big). \tag{7.16}$$

Since $\bar{U}$ satisfies the Hamilton–Jacobi equation

$$\partial_t \bar{U}_t + H(q, \nabla_q \bar{U}_t) + F(q, \bar{\sigma}_t) = 0$$

differentiating with respect to q and replacing q by $\bar{\Sigma}^1$, we have

$$\partial_t\big(\nabla_q\bar{U}\big)(t,\bar{\Sigma}^1)+D_qH\big(\bar{\Sigma}^1,\nabla_q\bar{U}(t,\bar{\Sigma}^1)\big)+\nabla_{qq}\bar{U}(t,\bar{\Sigma}^1)D_qH\big(\bar{\Sigma}^1,\nabla_q\bar{U}_t(t,\bar{\Sigma}^1)\big)+\nabla_qF(\bar{\Sigma}^1,\bar{\sigma}_t)=0.$$

We use (7.15) to obtain

$$\partial_t\big(\nabla_q\bar{U}\big)(t,\bar{\Sigma}^1)+D_qH\big(\bar{\Sigma}^1,\nabla_q\bar{U}(t,\bar{\Sigma}^1)\big)+\nabla_{qq}\bar{U}(t,\bar{\Sigma}^1)\partial_t\bar{\Sigma}^1+\nabla_qF(\bar{\Sigma}^1,\bar{\sigma}_t)=0.$$

This means

$$\partial_t\Big(\nabla_q\bar{U}(t,\bar{\Sigma}^1))\Big)+D_qH\big(\bar{\Sigma}^1,\nabla_q\bar{U}(t,\bar{\Sigma}^1)\big)+\nabla_qF(\bar{\Sigma}^1,\bar{\sigma}_t)=0.$$

or equivalently,

$$\partial_t\bar{\Sigma}^2+D_qH\big(\bar{\Sigma}^1,\bar{\Sigma}^2\big)+\nabla_qF(\bar{\Sigma}^1,\bar{\sigma}_t)=0. \tag{7.17}$$

We have

$$\bar{\Sigma}^2_0=\nabla_q\bar{U}\big(0,\bar{\Sigma}^1_0\big)=\nabla_qu_*(\bar{\Sigma}^1_0,\bar{\sigma}_0). \tag{7.18}$$

Since $\bar{U}(t,\cdot)$ is $W^{3,\infty}$, we use (7.15) to conclude that $\bar{\Sigma}^2(t,\cdot)$ is $W^{2,\infty}$.

We combine (7.15), (7.17) with (7.18) and the second identity in (7.14) to conclude that $\bar{\Sigma}$ is a solution to (7.4). By Theorem 7.1, the latter system admits a unique solution for $T>0$ small enough provided that the bounds in (ii) of that Theorem are verified. This is the task remaining to complete to conclude the proof of the Theorem. QED.

Theorem 7.6. *Assume T is small enough to that Σ is uniquely determined and $\Sigma^1[s,\mu](t,\cdot)$ is invertible for any $s,t\in[0,T]$ and any $\mu\in\mathcal{P}(\mathbb{T}^d)$. Define*

$$Q^t_s[\mu]:=\Sigma^1[s,\mu](t,\cdot),\;\;P^t_s[\mu]=\Sigma^2[s,\mu](t,\cdot),\;\;\sigma_t:=\Sigma^1[s,\mu](t,\cdot)_\sharp\mu.$$

Then the following properties for any $t_0\in[0,T]$:

(i)

$$Q^t_{t_0}[\sigma_{t_0}]\circ Q^{t_0}_s[\mu]=Q^t_s[\mu],\quad P^t_{t_0}[\sigma_{t_0}]\circ Q^{t_0}_s[\mu]=P^t_s[\mu],\quad X^{t_0}_t[\mu]=Q^t_{t_0}\Big[Q^{t_0}_t[\mu]_\sharp\mu\Big].$$

(ii)

$$\mathcal{V}^t_s[\mu]=\mathcal{V}^t_{t_0}[\sigma_{t_0}],\quad v^t_s[\mu]=v^t_{t_0}[\sigma_{t_0}].$$

(iii)

$$\partial_sQ^t_s[\mu]=-\nabla_qQ^t_s[\mu]v^s_t[\sigma_t]$$

Proof: (i) Note that $t\to Q^t_s[\mu]\circ X^{t_0}_s[\mu]$ satisfies the same odes as $t\to Q^t_s[\mu]$ at the point $X^{t_0}_s[\mu]$. We make the analogous observation for $P^t_s[\mu]\circ X^{t_0}_s[\mu]$ to conclude that

$$\Big(Q^t_s[\mu]\circ X^{t_0}_s[\mu],P^t_s[\mu]\circ X^{t_0}_s[\mu]\Big)=\Sigma_t[t_0,\sigma_{t_0}].$$

This proves the first and second identities in (i). In particular

$$Q^t_{t_0}\Big[Q^{t_0}_t[\mu]_\sharp\mu\Big]Q^{t_0}_t[\mu]=Q^t_t[\mu]=Id,$$

which verifies the last identity in (i).

(ii) Note that the first identity (ii) is equivalent to showing that

$$P_s^t[\mu] \circ X_s^t[\mu] = P_{t_0}^t\big[\sigma_{t_0}\big] \circ X_{t_0}^t\big[\sigma_{t_0}\big]. \tag{7.19}$$

Using the second identity in (i) this is equivalent to showing that

$$P_{t_0}^t[\sigma_{t_0}] \circ Q_s^{t_0}[\mu] \circ X_s^t[\mu] = P_{t_0}^t\big[\sigma_{t_0}\big] \circ X_{t_0}^t\big[\sigma_{t_0}\big].$$

It suffices to show $Q_s^{t_0}[\mu] \circ X_s^t[\mu] = X_{t_0}^t\big[\sigma_{t_0}\big]$. But the first identity in (i) implies

$$Q_{t_0}^t\big[\sigma_{t_0}\big] \circ Q_s^{t_0}[\mu] \circ X_s^t[\mu] = Q_s^t[\mu] \circ X_s^t[\mu] = Id$$

which implies that $Q_s^{t_0}[\mu] \circ X_s^t[\mu] = X_{t_0}^t\big[\sigma_{t_0}\big]$. Since $v_s^t[\mu] = D_pH(\cdot, \mathcal{V}_s^t[\mu])$, we verify the second identity in (ii).

(iv) By (i) $Q_s^t[\mu] \circ Q_t^s[\sigma_t] = Id$ and so, since the differentiability property of the map $\mathcal{M}$ in Theorem 7.1 (iv) implies

$$0 = \partial_s\Big(Q_s^t[\mu]\big(Q_t^s[\sigma_t]\big)\Big) = \partial_s Q_s^t[\mu]\Big(Q_t^s[\sigma_t]\Big) + \nabla_q Q_t^s[\sigma_t]\Big(Q_t^s[\sigma_t]\Big)\partial_s Q_t^s[\sigma_t].$$

We use the fact that

$$\partial_s Q_t^s[\sigma_t] = D_pH\big(Q_t^s[\sigma_t], P_t^s[\sigma_t]\big) = v_t^s[\sigma_t]\big(Q_t^s[\sigma_t]\big)$$

to verify (iv). QED.

Theorem 7.7. *Assume $\epsilon_1 = \epsilon_2 = 0$, Σ is as in Theorem 7.1 and $\mathcal{V}$ is as in (7.6). Then there exists $T > 0$ such that the function $u : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \mapsto \mathbb{R}$ defined as*

$$u(s,q,\mu) = u_*\big(q, \Sigma^1[s,\mu](0,\cdot)_\sharp\mu\big) - \int_0^s \Big(H\big(q, \mathcal{V}[s,\mu](\tau,q)\big) + F\big(q, \Sigma^1[s,\mu](\tau,\cdot)_\sharp\mu\big)\Big)d\tau,$$

is a solution of class C^1 to (7.1)-(7.2).

Proof: Here, we skip the proof on the smoothness property of u and let the reader use the above results (including Theorem 7.6) to prove it.

We write

$$Q_s^t[\mu] = \Sigma^1[s,\mu](t,\cdot), \quad X_s^t[\mu] = X[s,\mu](t,\cdot), \quad \mathcal{V}_s^t[\mu] = \mathcal{V}[t,\mu](t,\cdot), \quad \text{etc...}$$

We fix $(s,q,\mu) \in (0,T) \times Td \times \mathcal{P}(\mathbb{T}^d)$ and set

$$\sigma_t := Q_s^t[\mu]_\sharp\mu \tag{7.20}$$

so that $\mu = \sigma_s$.

1. By the definition

$$u(t,q,\sigma_t) = u_*\big(q, Q_t^0[\sigma_t]_\sharp\sigma_t\big) - \int_0^t \Big(H\big(q, \mathcal{V}_t^\tau[\sigma_t](q)\big) + F\big(q, Q_t^\tau[\sigma_t]_\sharp\sigma_t\big)\Big)d\tau, \tag{7.21}$$

We use the first identity in Theorem 7.6 (i) to conclude

$$\sigma_\tau = Q_s^\tau[\mu]_\sharp\mu = Q_t^\tau[\sigma_t] \circ Q_s^t[\mu]_\sharp\mu = Q_t^\tau[\sigma_t]_\sharp\sigma_t \qquad \forall t, \tau \in (0, T). \tag{7.22}$$

We use Theorem 7.6 (ii) to conclude

$$\mathcal{V}[s, \mu](\tau, q) = \mathcal{V}[t, \sigma_t](\tau, q) \qquad \forall t, \tau \in (0, T).$$

This, together with (7.21) and (7.22) implies

$$u(t, q, \sigma_t) = u_*(q, \sigma_0) - \int_0^t \Big(H\big(q, \mathcal{V}_\tau^\tau[\sigma_\tau](q)\big) + F\big(q, \sigma_\tau\big)\Big) d\tau.$$

Hence,

$$\frac{d}{dt}\Big(u(t, q, \sigma_t)\Big)\Big|_{t=s} = -H\big(q, \mathcal{V}_s^s[\sigma_s](q)\big) - F\big(q, \sigma_s\big) \tag{7.23}$$

Recall

$$\partial_t Q_s^t[\mu] = D_p H\Big(Q_s^t[\mu], P_s^t[\mu]\Big) = v_s^t[\mu] \circ Q_s^t[\mu].$$

and so, thanks to (7.20), we conclude $v_s^t[\mu]$ is a velocity driving σ. Thus

$$\frac{d}{dt}\Big(u(t, q, \sigma_t)\Big)\Big|_{t=s} = \partial_s u(s, q, \sigma_s) + \big\langle \nabla_{\omega_2} u(s, q, \sigma_s), v_s^s[\mu]\big\rangle.$$

Since $v_s^t[\mu] = D_p H\big(\cdot, \mathcal{V}_s^t[\mu]\big)$, this, together with (7.23) implies

$$\partial_s u(s, q, \sigma_s) + N_\mu\Big[\nabla_{\omega_2} u(s, q, \mu), \mathcal{V}_s^s[\mu]\Big] + H\big(q, \mathcal{V}_s^s[\mu](q)\big) + F(q, \mu) = 0. \tag{7.24}$$

2. To alleviate the notation, let us write Σ^2 in place of $\Sigma^2[s, \mu]$. By definition

$$\Sigma^2(t, q) = \mathcal{V}\big(t, \Sigma^1(t, q)\big)$$

and so,

$$\partial_t \Sigma^2(t, q) = \partial_t \mathcal{V}\big(t, \Sigma^1(t, q)\big) + \nabla_q \mathcal{V}\big(t, \Sigma^1(t, q)\big)\partial_t \Sigma^1(t, q).$$

We use (7.4) to conclude that

$$-D_q H\big(\Sigma(t, q)\big) - \nabla_q F\big(\Sigma^1(t, q), \sigma_t\big) = \partial_t \mathcal{V}\big(t, \Sigma^1(t, q)\big) + \nabla_q \mathcal{V}\big(t, \Sigma^1(t, q)\big) D_q H\big(\Sigma(t, q)\big).$$

Thus,

$$- D_q H\big(\cdot, \mathcal{V}(t, \cdot)\big) - \nabla_q F\big(\cdot, \sigma_t\big) = \partial_t \mathcal{V}\big(t, \cdot\big) + \nabla_q \mathcal{V}\big(t, \cdot\big) D_q H\big(\cdot, \mathcal{V}(t, \cdot)\big). \tag{7.25}$$

Differentiating u with respect to q we obtain

$$\nabla_q u(s, q, \mu) = \nabla_q u_*(q, \sigma_0) - \int_0^s \Big(D_q H\big(q, \mathcal{V}(t, q)\big) + (\nabla_q \mathcal{V}(t, q))^T D_q H\big(q, \mathcal{V}(t, q)\big) + \nabla_q F\big(q, \sigma_t\big)\Big) dt.$$

By Exercise 7.4, there exists a function $\tilde{U}$ such that $\mathcal{V} = \nabla_q \tilde{U}$ and so, $\nabla_q \mathcal{V} = (\nabla_q \mathcal{V})^T$. We conclude

$$\nabla_q u(s, q, \mu) = \nabla_q u_*(q, \sigma_0) - \int_0^s \Big(D_q H\big(q, \mathcal{V}(t, q)\big) + \nabla_q \mathcal{V}\big(t, q\big) D_q H\big(q, \mathcal{V}(t, q)\big) + \nabla_q F\big(q, \sigma_t\big)\Big) dt.$$

This, together with (7.25) implies

$$\nabla_q u(s,q,\mu) = \nabla_q u_*(q,\sigma_0) + \int_0^s \partial_t \mathcal{V}(t,q)dt = \nabla_q u_*(q,\sigma_0) + \mathcal{V}(s,q) - \mathcal{V}(0,q).$$

Thus,

$$\nabla_q u(s,q,\mu) = \mathcal{V}(s,q) \equiv \mathcal{V}_s^s[\mu](q) \tag{7.26}$$

We use this in (7.24) to conclude that u satisfies (7.1). The identity (7.2) is straightforward to check. QED.

Remark 7.8. The first proof of the theorem was based on a variational approach and was due to [27] when $H(q,p) = |p|^2/2$ and u_* and F are interaction potential functions. Under the same assumptions [5] proposed an alternative proof. A major improvement which goes beyond variational approaches, has recently been achieved in [41] for more general Hamiltonians.

Exercise 7.9. Show the following Lagrangian representation of the function u of Theorem 7.7:

$$u(s,q,\mu) = u_*\Big(Q_s^0[\mu](q), Q_s^0[\mu]_\sharp\mu\Big) + \int_0^s \Big(L\Big(Q_s^t[\mu](q), \partial_t Q_s^t[\mu](q)\Big) - F\Big(Q_s^t[\mu](q), Q_s^t[\mu]_\sharp\mu\Big)\Big)dt$$

Proof: Fix $(s,q,\mu) \in (0,T) \times Td \times \mathcal{P}(\mathbb{T}^d)$ and let σ_t be as in (7.20). Set

$$E(t) := u\big(t, Q_s^t[\mu](q), \sigma_t\big) - u_*\big(Q_s^0[\mu](q), \sigma_0\big) - \int_0^t \Big(L\Big(Q_s^\tau[\mu](q), \partial_\tau Q_s^\tau[\mu](q)\Big) - F\Big(Q_s^\tau[\mu](q), \sigma_\tau\Big)\Big)d\tau.$$

Since (7.2) is satisfied, we have $E(0) = u\big(0, Q_s^0[\mu](q), \sigma_0\big) - u_*\big(Q_s^0[\mu](q), \sigma_0\big)$ and so,

$$E(0) = 0. \tag{7.27}$$

Furthermore,

$$\dot{E}(t) = \frac{d}{dt}\Big(u\big(t, Q_s^t[\mu](q), \sigma_t\big)\Big) - L\Big(Q_s^t[\mu](q), \partial_t Q_s^t[\mu](q)\Big) + F\Big(Q_s^t[\mu](q), \sigma_t\Big). \tag{7.28}$$

We use the fact that $D_p H(Q_s^t[\mu](q), \cdot)$ and $D_v L(Q_s^t[\mu](q), \cdot)$ are inverse of each other and use the first equation in (7.4) to conclude that

$$L\Big(Q_s^t[\mu](q), \partial_\tau Q_s^t[\mu](q)\Big) = -H\Big(Q_s^t[\mu](q), P_s^t[\mu](q)\Big) + \Big\langle D_p H\Big(Q_s^t[\mu](q), P_s^t[\mu](q)\Big), P_s^t[\mu](q)\Big\rangle. \tag{7.29}$$

But since by (7.20) $v_s^t[\mu]$ is a velocity for σ_t, we have

$$\begin{aligned}\frac{d}{dt}\Big(u\big(t, Q_s^t[\mu](q), \sigma_t\big)\Big) &= \partial_t u\big(t, Q_s^t[\mu](q), \sigma_t\big) + \langle \nabla_q u\big(t, Q_s^t[\mu](q), \sigma_t\big), \partial_t Q_s^t[\mu](q)\rangle \\ &\quad + \big\langle \nabla_{\omega_2} u\big(t, Q_s^t[\mu](q), \sigma_t\big), v_s^t[\mu]\big\rangle_{\sigma_t} \\ &= \partial_t u\big(t, Q_s^t[\mu](q), \sigma_t\big) + \langle \nabla_q u\big(t, Q_s^t[\mu](q), \sigma_t\big), D_p H(Q_s^t[\mu](q), P_s^t[\mu](q))\rangle \\ &\quad + \big\langle \nabla_{\omega_2} u\big(t, Q_s^t[\mu](q), \sigma_t\big), D_p H(\cdot, \mathcal{V}_s^t[\mu])\big\rangle_{\sigma_t}.\end{aligned} \tag{7.30}$$

We use Theorem 7.6 (ii) and (7.26) to obtain

$$\mathcal{V}_s^t[\mu] = \mathcal{V}_s^t[\sigma_t] = \mathcal{V}_t^t[\sigma_t] = \nabla_q u(t, \cdot, \sigma_t) \tag{7.31}$$

Thus,

$$\mathcal{V}_s^t[\mu] \circ Q_s^t[\mu] = \nabla_q u(t, Q_s^t[\mu], \sigma_t)$$

Using (7.6) we conclude

$$\nabla_q u(t, Q_s^t[\mu], \sigma_t) = P_s^t[\mu] \tag{7.32}$$

Since u is a solution to the master equation (7.1), (7.30)–(7.32) imply

$$\begin{aligned}
\frac{d}{dt}\Big(u\big(t, Q_s^t[\mu](q), \sigma_t\big)\Big) &= \partial_t u\big(t, Q_s^t[\mu](q), \sigma_t\big) + \langle P_s^t[\mu], D_p H(Q_s^t[\mu](q), P_s^t[\mu](q))\rangle \\
&+ N_{\sigma_t}\Big[\nabla_{\omega_2} u\big(t, Q_s^t[\mu](q), \sigma_t\big), \nabla_q u(t, \cdot, \sigma_t)\Big] \\
&= -F\big(Q_s^t[\mu](q), \sigma_t\big) - H\Big(Q_s^t[\mu](q), \nabla_q u\big(t, Q_s^t[\mu](q), \sigma_t\big)\Big) \\
&+ \langle P_s^t[\mu], D_p H(Q_s^t[\mu](q), P_s^t[\mu](q))\rangle \\
&= -F\big(Q_s^t[\mu](q), \sigma_t\big) - H\Big(Q_s^t[\mu](q), P_s^t[\mu]\Big) \\
&+ \langle P_s^t[\mu], D_p H(Q_s^t[\mu](q), P_s^t[\mu](q))\rangle.
\end{aligned}$$

This, together with (7.29) implies

$$\frac{d}{dt}\Big(u\big(t, Q_s^t[\mu](q), \sigma_t\big)\Big) = -F\big(Q_s^t[\mu](q), \sigma_t\big) + L\Big(Q_s^t[\mu](q), \partial_t Q_s^t[\mu](q)\Big).$$

We ise (7.28) to conclude $\dot{E}(t) \equiv 0$, which together with (7.27) implies

$$E(t) = E(0) = 0 \qquad \forall t \in (0, T).$$

In particular, when $t = s$, we read off the desired identity. QED.

7.2 Lectures 26, 27: Convexity and displacement convexity (Nov 29, Dec 04)

Let $F : \mathbb{M} \times \mathcal{P}_2(\mathbb{M}) \mapsto \mathbb{R}$ be such that there exists $C > 0$ such that

$$|F(q, \mu)| \leq C(1 + |q|^2) \qquad \forall (q, \mu) \in \mathbb{M} \times \mathcal{P}_2(\mathbb{M}).$$

Definition 7.10. We say that F is monotone if

$$I[\mu_0, \mu_1] =: \int_{\mathbb{M}} \big(F(q, \mu_1) - F(q, \mu_0)\big)(\mu_1 - \mu_0)(dq) \geq 0 \qquad \forall \mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{M}).$$

In the remainder of this section, we assume

$$\mathbb{M} = \mathbb{R}^d$$

and assume we are given a weakly Fréchet continuously differentiable function $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ such that

$$F(q, \mu) := \frac{\delta \mathcal{F}}{\delta \mu}(q, \mu), \qquad \forall (q, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).$$

The goal of this section is to shade some light on the notion of the monotonicity property of functions. We show that F is monotone if and only of $\mathcal{F}$ is convex along the traditional interpolation paths $(1-t)\mu_0 + t\mu_1$. We refer to this convexity property as traditional convexity. The property solely depends on the trace of the Wasserstein second derivative matrix $\nabla^2_{\omega_2} \mathcal{F}$ and not on the full Wasserstein hessian

$$\text{Hess}\mathcal{F} = \Big(\nabla_q \big(\nabla_{\omega_2} \mathcal{F} \big), \nabla^2_{\omega_2} \mathcal{F} \Big).$$

The other notion of convexity, the so-called displacement convexity of $\mathcal{F}$, expresses convexity of $\mathcal{F}$ along geodesics in $\mathcal{P}_2(\mathbb{R}^d)$. To see if these two notions compare to another, let us consider functions of the form

$$\mathcal{F}(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} \Phi * \mu(q) \mu(dq)$$

where $\Phi \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ grows at most quadratically at ∞. On the one hand $\mathcal{F}$ is convex in the traditional sense if and only if the Fourier transform $\hat{\Phi}$ is nonnegative (cf. Lemma 7.15), which means $\widehat{\nabla^2 \Phi} \leq 0$. On the other hand the displacement convexity of $\mathcal{F}$ is equivalent to $\nabla^2 \Phi \geq 0$ (cf. Remark 7.16).

Consider the particular of the function

$$M_2(\mu) := \frac{1}{2} \int_{\mathbb{R}^{2d}} |x - y|^2 \mu(dx) \mu(dy).$$

While M_2 is displacement convex, it is $-M_2$ which is convex in the traditional sense (cf. Exercise 7.17). This means it is $-\frac{\delta M_2}{\delta \mu}$ which is monotone.

Remark 7.11. Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and let X_0, (resp. X_1) be the laws of μ_0 (resp. μ_1).

(i) Note that by Lemma 7.12 (i) F monotone implies $t \mapsto a(t) := \mathcal{F}\big((1-t)\mu_0 + t\mu_1\big)$ is convex. Conversely, assume $\mathcal{F}$ is convex. Then

$$0 \leq a'(1) - a'(0) = \int_{\mathbb{R}^d} \big(F(q, \mu_1) - F(q, \mu_0) \big)(\mu_1 - \mu_0)(dq).$$

Hence, F is monotone.

(ii) We don't have $\mu_t := (1-t)\mu_0 + t\mu_1 = \big((1-t)X_0 + tX_1\big)_\sharp \mathcal{L}^d_{(0,1)^d} =: \bar{\mu}_t$. Take for instance μ_0 and μ_1 be two distinct dirac masses. Then the support of μ_t has two elements while the support of $\bar{\mu}_t$ has only one element.

(iii) We shall show in Exercise 7.17 that we may choose $\mathcal{F}$ to be displacement concave while F is monotone.

Lemma 7.12. *Assume $\mathcal{F}$ is twice weakly Fréchet continuously differentiable and $\frac{\delta \mathcal{F}}{\delta \mu}$ is monotone.*

(i) If $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\frac{\delta^2 \mathcal{F}}{\delta \mu^2}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d \times \{(1-t)\mu_0 + t\mu_1 \mid t \in [0,1]\}$ then

$$\int_{\mathbb{R}^{2d}} \left(\frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, \mu_0) \right) (\mu_1 - \mu_0)(dq)(\mu_1 - \mu_0)(dx) \geq 0.$$

(ii) Let $\sigma_t \in AC_2(0, T, \mathcal{P}_2(\mathbb{R}^d))$ and let v be an admissible velocity for σ. Assume $\frac{\delta^2 \mathcal{F}}{\delta \mu^2}(\cdot, \cdot, \mu)$ is twice continuously at every $\mu \in \sigma[0,1]$ and $\frac{\delta^2 \mathcal{F}}{\delta \mu^2}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d \times \sigma[0,1]$. If t is a point of continuity of $v_t \sigma_t$ then

$$\int_{\mathbb{R}^{2d}} \Big\langle \nabla^2_{\omega_2} \mathcal{F}(q, x, \sigma_t), v(t, x) \otimes v(t, q) \Big\rangle \sigma_t(dq) \sigma_t(dx \geq 0.$$

Note $\nabla^2_{\omega_2} \mathcal{F}(q, x, \mu) = \nabla_{xq} \big(\frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, \mu) \big)$.

Proof: (i) Let $\bar{\sigma}_t := (1-t)\mu_0 + t\mu_1$. We have

$$\begin{aligned} 0 \leq I[\mu_0, \mu_1] &= \int_{\mathbb{R}^d} \left(\frac{\delta \mathcal{F}}{\delta \mu}(q, \mu_1) - \frac{\delta \mathcal{F}}{\delta \mu}(q, \mu_0) \right) (\mu_1 - \mu_0)(dq) \\ &= \int_0^1 dt \int_{\mathbb{R}^{2d}} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, \bar{\sigma}_t)(\mu_1 - \mu_0)(dq)(\mu_1 - \mu_0)(dx) \end{aligned}$$

Replacing μ_1 by $\bar{\sigma}_t$, we obtain

$$\begin{aligned} 0 \leq I[\mu_0, \bar{\sigma}_t] &= \int_0^1 ds \int_{\mathbb{R}^{2d}} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, (1-s)\mu_0 + s\bar{\sigma}_t)(\bar{\sigma}_t - \mu_0)(dq)(\bar{\sigma}_t - \mu_0)(dx) \\ &= t^2 \int_0^1 ds \int_{\mathbb{R}^{2d}} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, (1-s)\mu_0 + s\bar{\sigma}_t)(\mu_1 - \mu_0)(dq)(\mu_1 - \mu_0)(dx) \end{aligned}$$

Thus,

$$0 \leq \liminf_{t \to 0^+} \frac{I[\mu_0, \bar{\sigma}_t]}{t^2} = \int_{\mathbb{R}^{2d}} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q, x, \mu_0)(\mu_1 - \mu_0)(dq)(\mu_1 - \mu_0)(dx)$$

This proves (i).

(ii) Let t be a Lebesgue point for v_t, σ_t. By Definition 7.10

$$0 \leq I[\sigma_t, \sigma_{t+h}] = \int_0^1 ds \int_{\mathbb{R}^{2d}} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}\Big(q, x, (1-s)\sigma_t + s\sigma_{t+h}\Big)(\sigma_{t+h} - \sigma_t)(dq)(\sigma_{t+h} - \sigma_t)(dx) \quad (7.33)$$

Set

$$A(h, q, x) := \int_0^1 \frac{\delta^2 \mathcal{F}}{\delta \mu^2}\Big(q, x, (1-s)\sigma_t + s\sigma_{t+h}\Big) ds$$

and

$$E(h, x) := \int_t^{t+h} ds \int_{\mathbb{R}^d} \langle \nabla_q A(h, q, x), v(s, q) \rangle \sigma_s(dq)$$

Then

$$\int_{\mathbb{R}^d} A(h,q,x)(\sigma_{t+h}-\sigma_t)(dq) = \int_t^{t+h} dl \int_{\mathbb{R}^d} \langle \nabla_q A(h,q,x), v(l,q)\rangle \sigma_l(dq)$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} E(h,x)(\sigma_{t+h}-\sigma_t)(dx) &= \int_t^{t+h} d\tau \int_{\mathbb{R}^d} \langle \nabla_x E(h,x), v(\tau,x)\rangle \sigma_\tau(dx)\\ &= \int_t^{t+h} d\tau \int_t^{t+h} dl \int_{\mathbb{R}^{2d}} \langle \nabla_{xq} A(h,q,x), v(\tau,x)\otimes v(l,q)\rangle \sigma_l(dq)\sigma_\tau(dx)\end{aligned}$$

This, together with (7.33) yields

$$0 \le \liminf_{h\to 0^+} \frac{I[\sigma_t,\sigma_{t+h}]}{h^2} = \int_{\mathbb{R}^{2d}} \Big\langle \nabla_{xq}\Big(\frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\sigma_t)\Big), v(t,x)\otimes v(t,q)\Big\rangle \sigma_t(dq)\sigma_t(dx)$$

QED.

Remark 7.13. If we had instead $v(t,q,x)\otimes v(t,q,x)\sigma_t(dq,dx)\sigma_t(dq,dx)$ in the above identity, we could conclude that the symmetric part of $\nabla^2_{\omega_2}\mathcal{F}$ is nonnegative.

Remark 7.14. Let $\mathcal{F}$ be as in Lemma 7.12. In particular, we assume for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\frac{\delta^2\mathcal{F}}{\delta\mu^2}(\cdot,\cdot,\mu)$ admits a continuous extension on $\mathbb{R}^{2d}$ and so $\frac{\delta^2\mathcal{F}}{\delta\mu^2}$ is continuous. By the first identity in the Lemma, if we approximate μ_0 by $\mu_0^n = \varrho_0^n\mathcal{L}^d \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\varrho_0^n \in C(\mathbb{R}^d)$ and $\varrho_0^n > 0$, we have

$$\int_{\mathbb{R}^{2d}} \frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\mu_0^n)(\mu_1-\mu_0^n)(dq)(\mu_1-\mu_0^n)(dx) \ge 0 \qquad \forall \mu_1 \in \mathcal{P}_2(\mathbb{R}^d).$$

Let $f \in C_c(\mathbb{R}^d)$ be of null average and let B be a ball containing the support of f. For $r \in \mathbb{R}$ such that $|r|$ is small enough, we have that $\varrho_0^n + rf \ge 0$ and so, $\mu_1^n := (\varrho_0^n + rf)\mathcal{L}^d \in \mathcal{P}_2(\mathbb{R}^d)$. Thus,

$$0 \le \int_{\mathbb{R}^{2d}} \frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\mu_0^n)(\mu_1^n-\mu_0^n)(dq)(\mu_1^n-\mu_0^n)(dx) = r^2 \int_{\mathbb{R}^{2d}} \frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\mu_0^n) f(q)f(x)dqdx.$$

Dividing the latter expresssion by r^2 and letting n tend to ∞, we conclude

$$0 \le \int_{\mathbb{R}^{2d}} \frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\mu_0) f(q)f(x)dqdx. \tag{7.34}$$

It suffices to have that $\frac{\delta^2\mathcal{F}}{\delta\mu^2}(q,x,\cdot)$ is uniformly integrable on bounded subsets of $\mathbb{R}^{2d}$ to conclude that (7.34) holds for every function $f \in C_b(\mathbb{R}^d)$ of null average.

Lemma 7.15. *Let $\Phi \in C^2(\mathbb{R}^d)$ be an even function which growth at most quadratically at ∞. Set*

$$\mathcal{F}(\mu) := \frac{1}{2}\int_{\mathbb{R}^{2d}} \Phi(x-y)\mu(dx)\mu(dy), \quad F(q,\mu) = \frac{\delta\mathcal{F}}{\delta\mu}(q,\mu) \qquad \forall (q,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).$$

The following hold:

(i)
$$\frac{\delta \mathcal{F}}{\delta \mu}(q,\mu) = \int_{\mathbb{R}^d} \Phi(q-w)\mu(dw) - 2\mathcal{F}(\mu)$$

(ii)
$$\frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q,y,\mu) = \Phi(q-y) - \int_{\mathbb{R}^d} \Phi(q-w)\mu(dw) - 2\int_{\mathbb{R}^d} \Phi(y-w)\mu(dw) + 4\mathcal{F}(\mu)$$

(iii)
$$\nabla^2_{\omega_2}\mathcal{F}(q,y,\mu) = -\nabla^2\Phi(x-y), \qquad \nabla_q\Big(\nabla_{\omega_2}\mathcal{F}(q,\mu)\Big) = \int_{\mathbb{R}^d} \nabla^2\Phi(x-w)\mu(dw).$$

(iv) Further assume $\Phi \in L^1(\mathbb{R}^d)$. Then F is monotone if and only if its Fourier transform $\hat{\Phi}$ is nonnegative.

Proof: (i) Direct computations show there exists a constant $\lambda \equiv \lambda(\mu)$ such that

$$\frac{\delta \mathcal{F}}{\delta \mu}(q,\mu) = \int_{\mathbb{R}^d} \Phi(q-w)\mu(dw) - \lambda(\mu).$$

Thus

$$0 = \int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta \mu}(q,\mu)\mu(dq) = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \Phi(q-w)\mu(dw) - \lambda(\mu)\Big)\mu(dq) = 2\mathcal{F}(\mu) - \lambda(\mu).$$

This proves (i).

(ii) Differentiating (i) and using the fact that we know how to differentiate $\mathcal{F}(\mu)$, we obtain a constant $\bar{\lambda} \equiv \bar{\lambda}(\mu)$ such that

$$\frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q,y,\mu) = \Phi(q-y) - \bar{\lambda}(\mu) - 2\int_{\mathbb{R}^d} \Phi(y-w)\mu(dw) + 4\mathcal{F}(\mu).$$

We determine $\bar{\lambda}(\mu)$ thanks to the identity

$$\int_{\mathbb{R}^d} \frac{\delta^2 \mathcal{F}}{\delta \mu^2}(q,y,\mu)\mu(dy) = 0.$$

(iii) We use Lemma 4.23 of Chapter 5 to obtain the first identity in (iii) Similarly, using Remark 4.25 of Chapter 5 yields the second identity in (iii).

(iv) Further assume $\Phi \in L^1(\mathbb{R}^d)$. If $f \in L^2(\mathbb{R}^d)$ then $\Phi * f \in L^2$ and so $f\Phi * f \in L^1$. Note $\hat{\Phi} \in C(\mathbb{R}^d)$ and so, to prove that proving that $\hat{\Phi} \geq 0$ on $\mathbb{R}^d$, it suffices to show that $\hat{\Phi} \geq 0$ on $\mathbb{R}^d \setminus \{0\}$.

Using the expression of $\frac{\delta^2 \mathcal{F}}{\delta \mu^2}$ in (ii), by Remark 7.11, thanks to Plancherel theorem, we obtain that F is monotone if and only if for any $f \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(q)dq = 0$ we have we have

$$0 \leq \int_{\mathbb{R}^d} \Phi * f(q)f(q)dq = \int_{\mathbb{R}^d} \widehat{\Phi * f}(\xi)\hat{f}^*(\xi)d\xi = \int_{\mathbb{R}^d} \hat{\Phi}(\xi)\hat{f}(\xi)\hat{f}^*(\xi)d\xi = \int_{\mathbb{R}^d} \hat{\Phi}(\xi)|\hat{f}(\xi)|^2 d\xi. \tag{7.35}$$

1. Claim. We claim that $\hat\Phi$ is even and assumes only real values.

Proof of Claim 1. Note first that as Φ is even, $x \to \sin(2\pi\langle \xi, x\rangle)$ is odd and so,

$$\mathrm{Im}(\hat\Phi(\xi)) = \int_{\mathbb{R}^d} \sin(2\pi\langle \xi, x\rangle)\Phi(x)dx = 0. \tag{7.36}$$

We have

$$\mathrm{Re}(\hat\Phi(-\xi)) = \int_{\mathbb{R}^d} \cos(-2\pi\langle \xi, x\rangle)\Phi(x)dx = \int_{\mathbb{R}^d} \cos(2\pi\langle \xi, x\rangle)\Phi(x)dx = \mathrm{Re}(\hat\Phi(\xi)) \tag{7.37}$$

and so, $\mathrm{Re}(\hat\Phi)$ is even. This verifies the claim.

2. Claim. By (7.35), if $\hat\Phi(\xi) \geq 0$ on $\mathbb{R}^d \setminus \{0\}$ then F is monotone.

3. Claim. Conversely, we claim that if F is monotone then $\hat\Phi(\xi) \geq 0$ on $\mathbb{R}^d \setminus \{0\}$.

Proof of Claim 3. Assume F is monotone and assume on the contrary there exists $\xi_0 \neq 0$ such that $\hat\Phi(\xi_0) < 0$. Since $\hat\Phi$ is continuous, we may find $r > 0$ such that $\hat\Phi(\xi) < 0$ for any ξ in the ball of radius $2r$, centered at ξ_0. We assume r is small enough so that $B(\xi_0, 2r)$ does not contain the origin. Let $\varphi \in C(B(\xi_0, r))$ be a nonnegative function such that $\varphi(\xi_0) > 0$ on $B(\xi_0, r/2)$. Set

$$g(\xi) := \varphi(\xi) + \varphi(-\xi), \quad f := \hat g.$$

As $g \in L^2(\mathbb{R}^d)$, we have $g = \hat f$. Since g is of compact support, we have $f \in L^1(\mathbb{R}^d; \mathbb{C})$ and

$$\int_{\mathbb{R}^d} f(x)dx = \hat f(0) = g(0) = 2\varphi(0) = 0.$$

We are yet need to show the range of f is contained in $\mathbb{R}$. But since g is even and $\xi \to \sin(2\pi\langle x, \xi\rangle)$ is odd, we conclude

$$\mathrm{Im}\Big(f(x)\Big) = \int_{\mathbb{R}^d} \sin(2\pi\langle x, \xi\rangle) g(\xi)d\xi = 0.$$

This proves $f \in C(\mathbb{R}^d)$.

By Claim 1 $\hat\Phi$ is even and so,

$$\int_{\mathbb{R}^d} \hat\Phi(\xi)|\hat f(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \hat\Phi(\xi)|\varphi(\xi)|^2 d\xi + \int_{\mathbb{R}^d} \hat\Phi(\xi)|\varphi(-\xi)|^2 d\xi = 2\int_{B(\xi_0,r)} \hat\Phi(\xi)|\varphi(\xi)|^2 d\xi < 0. \tag{7.38}$$

As $\int_{\mathbb{R}^d} f(x)dx = 0$, (7.38) is at variance with (7.35). QED.

Remark 7.16. Let $\mathcal{F}$ be as in Lemma 7.15 and further assume $\nabla^2\Phi \in C(\mathbb{R}^d)$ is bounded. Let $\sigma \in AC_2(0, 1; \mathcal{P}_2(\mathbb{R}^d))$ be a geodesic of constant length given as $\sigma_t = (Id + t\xi)_\sharp \mu_0$. We have

$$\mathcal{F}(\sigma_t) = \frac{1}{2}\int_{\mathbb{R}^{2d}} \Phi\Big((x-y) + t(\xi(x) - \xi(y))\Big)\mu(dx)\mu(dy).$$

Hence,

$$\frac{d^2}{dt^2}\mathcal{F}(\sigma_t) = \int_{\mathbb{R}^{2d}} \Big\langle \nabla^2\Phi\Big((x-y) + t(\xi(x) - \xi(y))\Big)(\xi(x) - \xi(y)), \xi(x) - \xi(y)\Big\rangle \mu(dx)\mu(dy).$$

It becomes clear that if $\nabla^2\Phi \geq 0$ then $\mathcal{F}$ is convex along σ which means $\mathcal{F}$ is displacement convex.

Conversely, assume $\mathcal{F}$ is displacement convex. Let $a, c \in \mathbb{R}^d \setminus \{0\}$ and set

$$\sigma_t := \frac{1}{2}(\delta_{a+tc} + \delta_0), \qquad \gamma_t = \frac{1}{2}\big(\delta_{(0,0)} + \delta_{(a,a+tc)}\big).$$

If $(x_1, y_1), (x_2, y_2)$ belongs to the support of γ_t then for instance

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = (a, a + tc)$$

or vice versa. In any case

$$\langle y_2 - y_1, x_2 - x_1\rangle = \langle a + tc, a\rangle = |a|^2 + t\langle c, a\rangle \geq |a|(|a| - t|c|) \geq 0, \qquad \forall t \in [0, T]$$

if $|a| \geq T|c|$. Thus, the support of $\gamma_T \in \Gamma(\sigma_0, \sigma_T)$ is cyclically monotone and so,

$$\sigma : [0, T] \mapsto \mathcal{P}_2(\mathbb{R}^d)$$

is a geodesic of constant speed if $|a| \geq T|c|$. We have

$$\mathcal{F}(\sigma_t) = \frac{1}{8}\Big(2\Phi(0) + 2\Phi(a + tc)\Big)$$

and so, since $\mathcal{F}$ is displacement convex,

$$0 \leq \frac{d^2}{dt^2}\mathcal{F}(\sigma_t)|_{t=0} = \frac{1}{4}\big\langle\nabla^2\Phi(a)c, c\big\rangle.$$

Consequently, $\nabla^2\Phi(a) \geq 0$ when $a \neq 0$. Since $\nabla^2\Phi$ is continuous, we conclude $\nabla^2\Phi(0) \geq 0$.

Exercise 7.17. Set

$$\mathcal{F}(\mu) := \frac{1}{2}\int_{\mathbb{R}^{2d}} |x - y|^2\mu(dx)\mu(dy), \quad F(q, \mu) = \frac{\delta\mathcal{F}}{\delta\mu}(q, \mu) \qquad \forall (q, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).$$

Show that $-F$ is monotone and F fails to be monotone.

Proof: By Lemma 7.15

$$\frac{\delta\mathcal{F}}{\delta\mu}(q, \mu) = \int_{\mathbb{R}^d} |q - w|^2\mu(dw) - 2\mathcal{F}(\mu) = |q|^2 - 2\int_{\mathbb{R}^d}\langle q, w\rangle\mu(dw) + \int_{\mathbb{R}^d} |w|^2\mu(dw) - 2\mathcal{F}(\mu).$$

Hence,

$$\frac{\delta\mathcal{F}}{\delta\mu}(q, \mu_1) - \frac{\delta\mathcal{F}}{\delta\mu}(q, \mu_0) = 2\int_{\mathbb{R}^d}\langle q, w\rangle(\mu_0 - \mu_1)(dw) + \int_{\mathbb{R}^d} |w|^2\mu_1(dw) - \int_{\mathbb{R}^d} |w|^2\mu_0(dw).$$

We conclude that

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\frac{\delta\mathcal{F}}{\delta\mu}(q, \mu_1) - \frac{\delta\mathcal{F}}{\delta\mu}(q, \mu_0)\right)(\mu_1 - \mu_0)(dq) &= 2\int_{\mathbb{R}^{2d}}\langle q, w\rangle(\mu_0 - \mu_1)(dw)(\mu_1 - \mu_0)(dq) \\ &= -2\left|\int_{\mathbb{R}^{2d}} w(\mu_0 - \mu_1)(dw)\right|^2 \leq 0\end{aligned}$$

This conclude the proof. QED.

7.3 Lecture 28: More general master equations (Dec 06)

The results in this section are due to [12]. We will use the 1–Wasserstein metric between $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{T}^d)$ defines as

$$\mathcal{W}_1(\mu_0, \mu_1) = \sup_{\phi \in \mathrm{Lip}(\mathbb{T}^d)} \left\{ \int_{\mathbb{T}^d} \phi(q)\big(\mu_1 - \mu_0\big)(dq) \mid \mathrm{Lip}(\phi) \leq 1 \right\}.$$

Given $l \in (\mathbb{N} \cup \{0\})^d$, we set $|l| = l_1 + \cdots + l_d$ and we write

$$\nabla^l = \frac{\partial^{|l|}}{\partial x_1^{l_1} \cdot \partial x_d^{l_d}}.$$

We use the following notations for

$$\phi \in C(\mathbb{T}^d), \quad \psi \in C(\mathbb{T}^{2d}), \quad \varphi \in C\big([0,T] \times \mathbb{T}^d\big)$$

$n, m \in \mathbb{N}$ and $\alpha \in (0,1)$:

$$\big\|\phi\big\|_{(n+\alpha)} := \big\|\phi\big\|_{C^{n,\alpha}(\mathbb{T}^d)}$$

$$\big\|\psi\big\|_{(m,n)} := \sum_{|l| \leq m, |l'| \leq n} \|\nabla^{(l,l')}\psi\|_\infty,$$

$$\big\|\psi\big\|_{(m+\alpha,n+\alpha)} := \big\|\psi\big\|_{(m,n)} + \sum_{|l|=m, |l'|=n} \big\|\nabla^{(l,l')}\psi\big\|_{C^{0,\alpha}(\mathbb{T}^d)}$$

and

$$\big\|\varphi\big\|_{(\frac{n}{2}+\frac{\alpha}{2}, n+\alpha)} := \sum_{|l|+2j \leq n} \|\nabla^l \partial_t^j \varphi\|_\infty + \sum_{|l|+2j=n} \sup_{t \in [0,T]} \|\nabla^l \partial_t^j \varphi(t,\cdot)\|_{C^{0,\alpha}(\mathbb{T}^d)} + \sup_{x \in \mathbb{T}^d} \|\nabla^l \partial_t^j \varphi(\cdot, x)\|_{C^{0,\alpha}([0,T])}$$

Throughout the section, we assume

$$H \in \mathrm{Lip}(\mathbb{T}^d \times \mathbb{R}^d)$$

is sufficiently smooth and there exists a constant $C > 0$ such that

$$0 < D_p^2 H(q,p) \leq C I_d \qquad \forall (q,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

We assume (cf. Definition 7.10) to be given monotone maps

$$F, u_* \in \mathrm{Lip}\big(\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)\big)$$

We set

$$\mathrm{Lip}_{(n,\alpha)}\Big(\frac{\delta F}{\delta \mu}\Big) := \sup_{\mu \neq \nu} \frac{1}{\mathcal{W}_1(\mu,\nu)} \Big\| \frac{\delta F}{\delta \mu}(\cdot,\cdot,\mu) - \frac{\delta F}{\delta \mu}(\cdot,\cdot,\nu) \Big\|_{(n+\alpha, n+\alpha)}.$$

We assume

$$\mathrm{Lip}_{(2,\alpha)}\Big(\frac{\delta F}{\delta\mu}\Big) + \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \|F(\cdot,\mu)\|_{2+\alpha} + \Big\|\frac{\delta F}{\delta\mu}(\cdot,\cdot,\mu)\Big\|_{(2+\alpha,2+\alpha)} < \infty \tag{7.39}$$

and

$$\mathrm{Lip}_{(3,\alpha)}\Big(\frac{\delta u_*}{\delta\mu}\Big) + \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \|u_*(\cdot,\mu)\|_{3+\alpha} + \Big\|\frac{\delta u_*}{\delta\mu}(\cdot,\cdot,\mu)\Big\|_{(3+\alpha,3+\alpha)} < \infty \tag{7.40}$$

Theorem 7.18. *Let H be as above and assume $\epsilon_2 = 0$. Assume F satisfies (7.39) and u_* satisfies (7.40). There exists a unique $u : [0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\mapsto\mathbb{R}$ of class C^1 which satisfies (7.1) - (7.2) and such that $\frac{\delta u}{\delta\mu}$ is continuous, $u(t,\cdot,\mu)$ is bounded in $C^{3,\alpha}$, $\frac{\delta u}{\delta\mu}(t,\cdot,\cdot,\mu)$ is bounded in $C^{3,\alpha}\times C^{2,\alpha}$ and*

$$\sup_{t\in[0,T]}\sup_{\mu\neq\nu}\frac{1}{\mathcal{W}_1(\mu,\nu)}\Big\|\frac{\delta u}{\delta\mu}(t,\cdot,\cdot,\mu) - \frac{\delta u}{\delta\mu}(t,\cdot,\cdot,\nu)\Big\|_{(2+\alpha,1+\alpha)} < \infty.$$

Proof: A more general version of the theorem was first proven [12]. QED.

Remark 7.19. Let H be as above and assume $\epsilon_1 > \epsilon_2 \geq 0$.

(i) Note that for any $(\xi_1,\cdots,\xi_k)\in(\mathbb{Z}^d)^k$, the function Ψ^k_ξ in (4.50) is periodic. Since

$$\epsilon_1 O + \epsilon_2 B = (\epsilon_1-\epsilon_2)O + \epsilon_2\triangle_{\omega_2},$$

the eigenvalue of $\epsilon_1 O + \epsilon_2 B$ associated to the eigenfunction

$$\mu \to \int_{(\mathbb{T}^d)^k}\Psi^k_\xi(x)\mu(dx_1)\cdots\mu(dx_k)$$

is

$$-\Big(\lambda^{\epsilon_2}_{\epsilon_1}(\xi)\Big)^2 = -4\pi^2\Big((\epsilon_1-\epsilon_2)\sum_{j=1}^k|\xi_j|^2 + \epsilon_2\Big|\sum_{j=1}^k\xi_j\Big|^2\Big)$$

(ii) When we further assume $(\xi_1,\cdots,\xi_k)\neq(0,\cdots,0)$ then

$$\lambda^{\epsilon_2}_{\epsilon_1}(\xi) > \max\Big\{\lambda^{\epsilon_2}_{\epsilon_2}(\xi), \lambda^0_{\frac{\epsilon_1-\epsilon_2}{2}}(\xi)\Big\} \geq 0.$$

In some sense, $\epsilon_1 O + \epsilon_2 B$ is more elliptic than both $\epsilon_2\triangle_{\omega_2}$ and $\frac{\epsilon_1-\epsilon_2}{2}O$ which means in some sense $\mathcal{L}^{\epsilon_2}_{\epsilon_1}$ is more elliptic that $\mathcal{L}^{\epsilon_2}_{\epsilon_2}$ and $\mathcal{L}^0_{\frac{\epsilon_1-\epsilon_2}{2}}$.

(iii) Despite the observation made in (ii) when $\epsilon_2 > 0$, in order to ensure existence of a smooth enough solution in (7.1) - (7.2), with $\mathcal{L}^{\epsilon_2}_{\epsilon_1}$ [12] required stronger regularity assumptions on F and u_* than they did in the case $\mathcal{L}^0_{\frac{\epsilon_1-\epsilon_2}{2}}$.

Appendix A

Appendix

A.1 Master equation as a system of conservation laws

Example A.1. Let $f : \mathbb{M} \mapsto \mathbb{R}$ and $\varphi : [0, T] \times \mathbb{M} \mapsto \mathbb{R}$ be smooth and such that

$$\partial_t \varphi(t, q) + H(q, \nabla \varphi(t, q)) + f(q) = 0 \tag{A.1}$$

We define

$$\mathcal{H}(\mu, \zeta) := \int_{\mathbb{M}} H(q, \zeta(q))\mu(dq).$$

We lift the above functions up to $\mathcal{P}_2(\mathbb{M})$ by setting for $\mu \in \mathcal{P}_2(\mathbb{M})$ and $\bar{\mu} := \int_{\mathbb{M}} x\mu(dx)$,

$$\mathcal{U}(t, \mu) := \mathcal{U}(t, \delta_{\bar{\mu}}) = \varphi(t, \bar{\mu}), \qquad \mathcal{F}(\mu) = \mathcal{F}(\delta_{\bar{\mu}}) = f(\bar{\mu}).$$

We have the property $\mathcal{U}(t, \delta_q) = \varphi(t, q)$ and so,

$$\nabla_{\omega_2} \mathcal{U}(t, \delta_q)(q) = \nabla_q \varphi(t, q). \tag{A.2}$$

Note

$$\mathcal{H}(\delta_{\bar{\mu}}, \zeta) = H(\bar{\mu}, \zeta(\bar{\mu})) \tag{A.3}$$

If $\zeta(t, \cdot) := \nabla_{\omega_2} \mathcal{U}(t, \mu)$ then when $\mu = \delta_q$, $\zeta(t, \cdot)$ is only unequivocally defined on $\{q\} = \{\bar{\mu}\}$. We use (A.2) to obtain

$$\zeta(t, \bar{\mu}) = \nabla_{\omega_2} \mathcal{U}(t, \delta_{\bar{\mu}})(\bar{\mu}) = \nabla_q \varphi(t, \bar{\mu}).$$

This, together with (A.3) implies

$$\mathcal{H}(\delta_{\bar{\mu}}, \zeta) = \mathcal{H}(\bar{\mu}, \nabla \varphi(t, \bar{\mu})).$$

Set

$$\bar{\mathcal{H}}(t, \mu) := \mathcal{H}(\bar{\mu}, \nabla \varphi(t, \bar{\mu})), .$$

For any $\nu \in \mathcal{P}_2(\mathbb{M})$

$$\lim_{s \to 0^+} \frac{\partial_t \mathcal{U}(t, \mu + s(\nu - \mu)) - \partial_t \mathcal{U}(t, \mu)}{s} = \lim_{s \to 0^+} \frac{\partial_t \varphi(t, \bar{\mu} + s(\bar{\nu} - \bar{\mu})) - \partial_t \varphi(t, \bar{\mu})}{s} = \langle \nabla \partial_t \varphi(t, \bar{\mu}), \bar{\nu} - \bar{\mu} \rangle.$$

Hence,

$$\frac{\delta}{\delta\mu}\big(\partial_t\mathcal{U}(t,\mu)\big)(q) = \langle \partial_t\nabla\varphi(t,\bar\mu), q\rangle - \int_{\mathbb{M}} \langle \nabla\varphi(t,\bar\mu), x\rangle \mu(dx) = \langle \partial_t\nabla\varphi(t,\bar\mu), q-\bar\mu\rangle. \tag{A.4}$$

Similarly,

$$\frac{\delta}{\delta\mu}\big(\mathcal{F}(\mu)\big)(q) = \langle \nabla f(\bar\mu), q-\bar\mu\rangle. \tag{A.5}$$

We have

$$\begin{aligned}\bar{\mathcal{H}}(t,\mu+s(\nu-\mu)) &= H\Big(\bar\mu + s(\bar\nu-\bar\mu), \nabla\varphi(t,\bar\mu+s(\bar\nu-\bar\mu))\Big)\\ &= H\Big(\bar\mu + s(\bar\nu-\bar\mu), \nabla\varphi(t,\bar\mu) + s\nabla^2\varphi(t,\bar\mu)(\bar\nu-\bar\mu)+o(s)\Big)\\ &= H\big(\bar\mu, \nabla\varphi(t,\bar\mu)\big)\\ &+ s\Big\langle D_qH(\bar\mu,\nabla\varphi(t,\bar\mu)) + \nabla^2\varphi(t,\bar\mu)D_pH\big(\bar\mu,\nabla\varphi(t,\bar\mu)\big), \bar\nu-\bar\mu\Big\rangle + o(s).\end{aligned}$$

Consequently,

$$\frac{\delta}{\delta\mu}\big(\bar{\mathcal{H}}(t,\mu)\big)(q) = \Big\langle D_qH(\bar\mu,\nabla\varphi(t,\bar\mu)) + \nabla^2\varphi(t,\bar\mu)D_pH\big(\bar\mu,\nabla\varphi(t,\bar\mu)\big), q-\bar\mu\Big\rangle. \tag{A.6}$$

By (A.1)

$$\partial_t\mathcal{U}(t,\mu) + \bar{\mathcal{H}}(t,\mu) + \mathcal{F}(\mu) = 0 \tag{A.7}$$

and so,

$$\frac{\delta}{\delta\mu}\Big(\partial_t\mathcal{U}(t,\mu) + \bar{\mathcal{H}}(t,\mu) + \mathcal{F}(\mu)\Big) = 0.$$

Plugging the expressions in (A.4) –(A.6) in (A.7) we obtain

$$\Big\langle \partial_t\nabla\varphi(t,\bar\mu) + D_qH(\bar\mu,\nabla\varphi(t,\bar\mu)) + \nabla^2\varphi(t,\bar\mu)D_pH\big(\bar\mu,\nabla\varphi(t,\bar\mu)\big) + \nabla f(\bar\mu), q-\bar\mu\Big\rangle = 0.$$

Since q is arbitrary, we conclude

$$\partial_t\nabla\varphi(t,\bar\mu) + D_qH(\bar\mu,\nabla\varphi(t,\bar\mu)) + \nabla^2\varphi(t,\bar\mu)D_pH\big(\bar\mu,\nabla\varphi(t,\bar\mu)\big) + \nabla f(\bar\mu) = 0.$$

In particular for $\mu=\delta_x$ we recover the conservation laws

$$\partial_t\nabla\varphi(t,x) + D_qH(x,\nabla\varphi(t,x)) + \nabla^2\varphi(t,x)D_pH\big(x,\nabla\varphi(t,x)\big) + \nabla f(x) = 0. \tag{A.8}$$

A.2 Acknowledgments.

These notes were written up when the author was in residence at UC Berkeley and MSRI, holding the 2018-2019 Chancellor Professor. This occurred during the Fall 2018 Thematic Program on “Hamiltonian systems, from topology to applications through analysis”. The author acknowledges generous supports provided by MSRI and UC Berkeley. The research of WG was supported by NSF under grant DMS–17 00 202 and by the Air Force under grant AFOSR–20181771. The author also acknowledges support of the PGMO project VarPDEMFG: ANR-11-LABX-0056-LMH, LabEx LMH.

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