SIAM J. MATH. ANAL.
Vol. 26, No. 2, pp. 280–304, March 1995

LOCAL INVERTIBILITY OF SOBOLEV FUNCTIONS*

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Abstract. A local inverse function theorem is established for mappings $v \in W^{1,N}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^N$ open set, such that $\det \nabla v(x) > 0$ almost everywhere in $x \in \Omega$. Regularity of the local inverse v^{-1} is obtained provided that $|\frac{\text{adj}(\nabla v)}{\det \nabla v}|^s \det \nabla v \in L^1(\Omega)$ for some $1 \leq s < +\infty$. The local invertibility property is used to study the weak lower semicontinuity of a functional involving variation of the domain.

Key words. local invertibility, topological degree, weak lower semicontinuity

AMS subject classification. 49

1. Introduction.

The aim of this paper is to give a simple proof of local invertibility of continuous functions $v \in W^{1,N}(\Omega, \mathbb{R}^N)$, where $\Omega \subset \mathbb{R}^N$ is an open set and $\det \nabla v(x) > 0$ almost everywhere in $x \in \Omega$ (Theorem 3.1). We show that the local inverse function w is $W^{1,1}$ and under suitable hypotheses we improve regularity of w to $W^{1,s}$ for some $s > 1$. Precisely, it is shown that v is locally invertible almost everywhere in the sense that for almost every $x \in \Omega$, there is an open neighborhood D of x and there is a function $w \in W^{1,1}(v(D), D)$ such that $v(D)$ is an open set,

$$v \circ w(y) = y \text{ a.e. } y \in v(D), \tag{1}$$

$$w \circ v(x) = x \text{ a.e. } x \in D, \tag{2}$$

and

$$\nabla w(y) = (\nabla v)^{-1}(w(y)) \text{ a.e. } y \in v(D), \tag{3}$$

where $(\nabla v)^{-1}(w(y))$ is the inverse matrix of $\nabla v(w(y))$. Moreover, if we assume that $|\frac{\text{adj}(\nabla v)}{\det \nabla v}|^s \det \nabla v \in L^1(\Omega)$ for some $1 \leq s < +\infty$, then, as in [Sv], we prove that $w \in W^{1,s}(v(D), D)$. One can then deduce easily that if $\det \nabla v(x) \geq \gamma > 0$ a.e. $x \in \Omega$, $v \in W^{1,q}(\Omega)^N$, and $q \geq N(N-1)$, then $v : D \to v(D)$ and $w : v(D) \to D$ are homeomorphisms, (1) holds for every $y \in v(D)$, (2) holds for every $x \in D$, $w \in W^{1,N}(v(D), D)$, and v is an open mapping on $\Omega \setminus L$ for a suitable $L \subset \mathbb{R}^N$ which has zero measure (see Corollary 3.3). In particular, we conclude that if $N = 2$, $v \in W^{1,2}(\Omega)^2$, and $\det \nabla v(x) \geq \gamma > 0$ a.e. $x \in \Omega$, then $w \in W^{1,2}(v(D), D)$ and there is a set of measure zero $L \subset \mathbb{R}^N$ such that v is an open mapping on $\Omega \setminus L$ and a weaker version of [IS] is obtained. Recently, we became aware of a result by Heinonen and Koskela [HK], where they show that if a mapping is in $W^{1,q}$ for some $q > N(N-1)$, its jacobian is positive almost everywhere and $N \geq 3$, then the mapping is open and discrete, and so $L = \emptyset$.

Conversely, if $v \in W^{1,q}(\Omega)^N$ for some $q > N$, $\det \nabla v(x) \neq 0$ a.e. $x \in \Omega$ and if for almost every $x_0 \in \Omega$ v is locally almost invertible in a neighborhood of x_0 in the sense of (1)–(3), then there are open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ and a set of measure zero $N \subset \mathbb{R}^N$

* Received by the editors May 5, 1993; accepted for publication (in revised form) October 1, 1993. This work was supported by the Army Research Office and the National Foundation through the Center for Nonlinear Analysis at Carnegie Mellon University. The research of the first author was partially supported by the National Science Foundation grant DMS-9201215.

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The paper is organized as follows: In the second section we fix notation and recall some definitions and well-known properties related to the Brouwer degree. In the third section we prove the local invertibility property of the mappings $v \in W^{1,q}(\Omega, \mathbb{R}^N)$, $q \geq N$, under the condition $\det \nabla v(x) > 0$ a.e. $x \in \Omega$. In view of our applications, in addition we prove that if $v_\epsilon \rightharpoonup v$ weakly in $W^{1,q}$ $q \geq N$, $\det \nabla v(x) > 0$ a.e. $x \in \Omega$ and $\det \nabla v_\epsilon(x) = 1$ a.e. $x \in \Omega$ then, up to a subsequence, v_ϵ and v are, respectively, locally invertible on open sets $D_\epsilon(x)$ and $D(x)$ for almost every $x \in \Omega$, where $D_\epsilon(x)$ and $D(x)$ are neighborhoods of x such that $v_\epsilon(D_\epsilon(x)) = v(D(x))$ does not depend on ϵ. The last section is devoted to the applications, where we obtain the weak lower semicontinuity for a class of functionals E on $B_{p,q}$.

2. Preliminaries. In the sequel we will use the following notation. For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $|x|$ stands for $(|x_1|^2 + \cdots + |x_N|^2)^{1/2}$ and $|x|_\infty := \max\{|x_1|, \ldots, |x_N|\}$. If $A \subset \mathbb{R}^N$ $|A|$ denotes the Lebesgue measure of A, A^c denotes its complement, $\operatorname{dist}(x, A)$ is defined by $\inf\{|x - y| : \ y \in A\}$, and $\rho(x, A)$ is given by $\inf\{|x - y|_\infty : \ y \in A\}$.

If $\Omega \subset \mathbb{R}^N$ is an open set, $v \in W^{1,1}(\Omega)^N$, then ∇v is the $N \times N$ matrix of the distributional derivatives $\frac{\partial v_i}{\partial x_j}$. If, furthermore, $\nabla v \in L^N$, then $\det \nabla v$ is the determinant of ∇v.

We recall some properties of mappings.

LEMMA 2.1. *Let Ω be a bounded, open set in $\mathbb{R}^N$ and $v \in (W^{1,N}_{\mathrm{loc}}(\Omega))^N$ such that $\det \nabla v(x) > 0$ a.e. $x \in \Omega$. Then v is a continuous mapping on Ω. Futhermore, if K is a compact set and V is an open set such that $K \subset V \subset\subset \Omega$, then there is a constant C_N depending only on N, such that*

$$|v(x) - v(y)| \leq M^{\frac{1}{N}} C_N \theta(|x - y|)$$

for every $x, y \in K$ that verify $|x - y| \leq \delta$, where

$$M = \int_V |\nabla v(x)|^N dx,$$

$$\theta(t) = \left(\frac{2}{\log(\frac{2}{t})}\right)^{\frac{1}{N}},$$

and

$$\delta = \min\left\{2, \frac{1}{2}(\operatorname{dist}(K, \mathbb{R}^N \setminus V))^2\right\}.$$

Proof. This lemma is an immediate consequence of Theorem 3.5, p. 294, and Proposition 3.3, p. 292 in [GR] and Theorem 4.4, p. 339 in [Re] (see also [Man]). It can also be shown that, under the above hypotheses, v is a monotonic mapping (see the definition of monotonic mapping below).

DEFINITION 2.2 ([GR]). *Let Ω be a bounded, connected, open set in $\mathbb{R}^N$ and $v \in W^{1,N}(\Omega)^N$. We say that v is* monotonic *at the point $x \in \Omega$ if there is a number $0 < r(x) \leq d(x, \partial\Omega)$ such that for almost every $r \in (0, r(x))$ the pre-image of the intersection of the set $v(B(x,r))$ with the unbounded connected component of $\mathbb{R}^N \setminus v(\partial B(x,r))$ is of measure 0 in $B(x,r)$. We say that v is a monotonic mapping in Ω if v is monotonic at every point $x \in \Omega$.*

(iii) *If $U \subset\subset \Omega$ is an open set such that $|\partial U| = 0$ and $p \in \mathbb{R}^N \setminus v(\partial U)$, then*

$$d(v, U, p) = \int_U f(v(x)) \det \nabla v(x) dx \tag{13}$$

for any f nonnegative, continuous real-valued function that satisfies $\int_{\mathbb{R}^N} f(x)dx = 1$, with compact support in V, where V is the connected component of $\mathbb{R}^N \setminus v(\partial U)$ containing p.

Remark 2.5. A function $v : \Omega \to \mathbb{R}^N$ is said to satisfy the N property (Lusin's property) if

$$|v(E)| = 0$$

whenever $E \subset \Omega$ is a measurable set such that $|E| = 0$, and v is said to satisfy the N^{-1} property if

$$|v^{-1}(A)| = 0$$

whenever $A \subset \mathbb{R}^N$ is a measurable set such that $|A| = 0$.

(a) It is known that if $v \in W^{1,N}(\Omega)^N$, $\det \nabla v(x) > 0$ a.e. $x \in \Omega$, then v satisfies the N and the N^{-1} property. (See [GR], pp. 296–297.)

(b) Also, if $v \in W^{1,q}(\Omega)^N$ with $q > N$, then v satisfies the N-property. (For details we refer the reader to [MM].)

Proof of Lemma 2.4. We refer the reader to [GR], Theorem 1.8, p. 280, Theorem 2.6, p. 288, or also to [Sv] for the proof of (11) and (12) in the case where U is a domain.

First we prove that (12) is still valid even if U is not connected and (13) is a by-product of this fact. To achieve this, we remark that by Vitali's covering theorem there are $\{D_i\}$, a countable family of open balls mutually disjoint, and a set N of measure zero such that $(\cup_i D_i) \cap N = \emptyset$ and

$$(\cup_i D_i) \cup N = U.$$

Setting $B = \cup_i D_i$, we have $\cup_i \partial D_i \subset \partial B$. If $y \in \mathbb{R}^N \setminus (v(\partial B) \cup v(\partial U))$, then by the decomposition formula (10)

$$\sum_i \chi_{v(D_i)} d(v, D_i, y) = \sum_i d(v, D_i, y) = d(v, B, y). \tag{14}$$

Let $K = \bar{U} \setminus B$. As K is a compact set and $K \subset \partial U \cup N$, if $y \notin v(K)$ then, by the excision property of degree (9), we obtain

$$d(v, U, y) = d(v, U \setminus K, y) = d(v, B, y). \tag{15}$$

By using the fact that v has the N property (see Remark 2.5), $D_i \subset\subset \Omega$, $|\partial U| = |N| = |\partial D_i| = 0$, by (12), (14), and (15) we obtain

$$\begin{aligned}\int_U f \circ v(x) \det \nabla v(x) dx &= \int_B f \circ v(x) \det \nabla v(x) dx \\ &= \sum_i \int_{D_i} f \circ v(x) \det \nabla v(x) dx\end{aligned}$$

If, in addition, $|\frac{\operatorname{adj}(\nabla v)}{\det \nabla v}|^s \det \nabla v \in L^1(\Omega)$ *for some* $1 \le s < +\infty$ *then* $w \in W^{1,s}(B(y_0, r), D)$.

Before proving Theorem 3.1, we list some of its consequences.

COROLLARY 3.2. *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded, open set,* $q \ge N$, *and* $v \in W^{1,q}(\Omega)^N$ *be a function such that* $\det \nabla v(x) \ne 0$ a.e. $x \in \Omega$.

(a) *Assume that* $\Omega_1, \Omega_2 \subset \mathbf{R}^N$ *are two open sets and* $N \subset \mathbf{R}^N$ *is a set of measure zero such that* $\Omega = \Omega_1 \cup \Omega_1 \cup N$, $\det \nabla v(x) > 0$ *a.e.* $x \in \Omega_1$, *and* $\det \nabla v(x) < 0$ *a.e.* $x \in \Omega_2$. *Then for almost every* $x_0 \in \Omega$ v *is locally almost invertible in a neighborhood of* x_0 *in the sense above.*

(b) *Conversely, if* $q > N$, $v \in W^{1,q}(\Omega)^N$ *and if for almost every* $x_0 \in \Omega$ v *is locally almost invertible in a neighborhood of* x_0, *then there are open sets* $\Omega_1, \Omega_2 \subset \mathbf{R}^N$ *and a null set* $N \subset \mathbf{R}^N$ *such that* $\Omega = \Omega_1 \cup \Omega_2 \cup N$, $\det \nabla v(x) > 0$ *a.e.* $x \in \Omega_1$, *and* $\det \nabla v(x) < 0$ *a.e.* $x \in \Omega_2$.

COROLLARY 3.3. *Let* $q \ge N$, *let* $\Omega \subset \mathbf{R}^N$ *be a bounded, open set and let* $v \in W^{1,q}(\Omega)^N$ *be a function such that* $\det \nabla v(x) = 1$ *a.e.* $x \in \Omega$. *Then the inverse function* w *of Theorem* 3.1 *is such that*

$$w \in W^{1,\frac{q}{N-1}}(v(D))^N.$$

If, in addition, $q \ge N(N-1)$ *then* $w \circ v(x) = x$ *for every* $x \in D$, $v \circ w(y) = y$ *for every* $y \in B(y_0, r)$, v *is a local homeomorphism and* v *is an open mapping on* $\Omega \setminus L$ *for some set* $L \subset \Omega$ *of zero measure. In particular, if* $N = 2$ *then* $N(N-1) = N = 2$ *and* v *is a local homeomorphism at* x_0.

We make some remarks and state some lemmas needed for the proofs of Corollaries 3.2 and 3.3, which will appear at the end of this section.

Remark 3.4.

1. As mentioned in the introduction, it has been proven recently by Heinonen and Koskela [HK, Cor. 1.10] that if a mapping is in $W^{1,q}$ for some $q > N(N-1)$ and if its jacobian is positive and $N \ge 3$, then the mapping is open and discrete and so $L = \emptyset$.

2. Recall that $v \in W^{1,N}(\Omega)^N$ is said to be a *mapping of bounded distortion* (or usually a *quasi-regular mapping*) if $|\nabla v(x)|^N \le K(\det \nabla v(x))$ for almost every $x \in \Omega$ and for some constant K. It is well known that every mapping of bounded distortion $v \in W^{1,N}(\Omega)^N$ is locally a homeomorphism at almost every point $x_0 \in \Omega$. (See [Re, Thm. 6.6, p. 187].) Moreover, mappings of bounded distortion are open mappings or constant in Ω. (See [Re, Thm. 6.4, p. 184].)

3. Note that even if $v \in C^1(\bar{\Omega})^N$ is such that $\det \nabla v(x) \ge \gamma > 0$ $\forall x \in \Omega$, we cannot expect a global invertibility of v without any regularity assumptions on the trace of v (see [Ba]).

4. Under the assumptions of Theorem 3.1, we cannot expect v to be locally invertible everywhere (see [Ba]).

5. An example of a mapping $v \in W^{1,\infty}(\Omega)^2$, $(\Omega \subset \mathbf{R}^2)$, is exhibited in [Ba], with $\det \nabla v(x) = 1$ a.e. $x \in \Omega$, for which there is no sequence $v_r \in C^1(\bar{\Omega})^2$ such that $v_r \to v$ uniformly and $J_{v_r}(x) > 0$ a.e. $x \in \Omega$. Therefore, to prove Theorem 3.1, one cannot approximate the function v by a sequence of smooth functions v_r, expecting the functions v_r to be locally invertible.

6. Note that for every bounded, open set $\Omega \subset \mathbf{R}^N$, there exists a measurable set $E \subset \Omega$ of nonzero measure and a homeomorphism $v \in W^{1,\infty}(\Omega)^N$ such that $\det \nabla v(x) = 0$ for every $x \in E$. (See [MZ, Remarks 3.7].)

Since $y \in v(B)$, there is $x \in B$ such that $y = v(x)$. By (6) we have

$$\lim_{\epsilon \to 0} \int_B \rho_\epsilon(v(z) - y) \det \nabla v(z) dz = d(v, B, y) \tag{22}$$

and by using the continuity of v at x, we deduce that for every $\epsilon > 0$ there is $\delta > 0$ such that $|v(z) - y| \leq \frac{\epsilon}{2}$ for every $z \in B(x, \delta)$. By recalling that $\det \nabla v(z) > 0$ a.e. $z \in B(x, \delta)$, by (21) and (22) we obtain

$$d(v, B, y) > 0. \tag{23}$$

Finally, since the degree $d(v, \cdot, y)$ is a nondecreasing function of the set, by using (19) and the fact that $B \subset v^{-1}(C_R) \cap B(x_0, R)$, we obtain

$$d(v, B, y) \leq d(v, B(x_0, R), y) = 1, \tag{24}$$

which, together with (23) and the fact that the degree is an integer number, yields (20). □

Lemma 3.8. *Let Ω, v, R_0 and x_0 be as in Lemma 3.7, (18), and (19). Let C_{R_0} be the connected component of $\mathbb{R}^N \setminus v(\partial B(x_0, R_0))$ containing $y_0 := v(x_0)$. Then for every $r > 0$ such that $B(y_0, r) \subset\subset C_{R_0}$, if $O := v^{-1}(B(y_0, r)) \cap B(x_0, R_0) \subset\subset B(x_0, R_0)$ then*

$$v(O) = B(y_0, r), \quad v(\partial O) \subset \partial v(O) = \partial B(y_0, r). \tag{25}$$

Proof. It is clear that $v(O) \subset B(y_0, r)$. Conversely, if $y \in B(y_0, r)$, by (19) $d(v, B(x_0, R_0), y) = 1$ and so by (5) there exists $x \in B(x_0, R_0)$ such that $y = v(x)$, which implies $y \in v(O)$. Let $x \in \partial O$ and let $\{a_n\} \subset O$, $\{b_n\} \subset B(x_0, R_0) \setminus O$ be such that

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = x.$$

We have $v(a_n) \in v(O) = v(v^{-1}(B(y_0, r))) = B(y_0, r)$ and $v(b_n) \notin v(O) = B(y_0, r)$. By using the continuity of v at x, we have

$$v(x) = \lim_{n \to +\infty} v(a_n) = \lim_{n \to +\infty} v(b_n),$$

which gives $x \in \partial v(O)$. □

Lemma 3.9. *Let $v \in W^{1,N}(\Omega)^N$, $\det \nabla v(x) > 0$ a.e. $x \in \Omega$ and let $x_0 \in D$ be such that $v(x) \neq v(x_0)$ for every $x \in \bar{B}(x_0, R_0) \setminus \{x_0\}$. Let $0 < R < R_0$ and let C be an open set containing $y_0 = v(x_0)$. Then there is $r > 0$ such that $v^{-1}(B(y_0, r)) \cap B(x_0, R) \subset\subset B(x_0, R)$.*

Proof. Define

$$d(\delta) = \sup\{|x - x_0| : x \in \bar{B}(x_0, R), |v(x) - v(x_0)| \leq \delta\}.$$

Since $v(x) \neq v(x_0)$ for every $x \in \bar{B}(x_0, R) \setminus \{x_0\}$ and v is uniformly continuous on $\bar{B}(x_0, R)$, we have

$$\lim_{\delta \to 0} d(\delta) = 0.$$

Take now $r > 0$ such that $d(r) < \frac{R}{2}$. We have

$$v^{-1}(B(y_0, r)) \cap B(x_0, R) \subset B\left(x_0, \frac{R}{2}\right) \subset\subset B(x_0, R). \quad □$$

By using the fact that for every $n \in \mathbb{N}$, $\{x \in \bar{B}(x_0, R_0) : \ \alpha + n \le x_i \le \alpha + n + 1\}$ is a compact set, v is a continuous function, and $v(D) \setminus N$ is measurable, we obtain that A_1 is measurable and we conclude that $w \in L^\infty(B(y_0, r))^N$.

Claim 2.

$$v \circ w(y) = y \ \text{ for every } \ y \in v(D) \equiv B(y_0, r), \tag{34}$$

$$w \circ v(x) = x \ \text{ for every } \ x \in D \setminus v^{-1}(N). \tag{35}$$

This follows immediately from (32) and (33). One notices that, due to (30) and Remark 2.5, $|v^{-1}(N)| = 0$.

Claim 3. $f \circ w$ is measurable for every $f : D \to \mathbb{R}$ measurable.

We know that every Lebesgue measurable set is a union of a Borel measurable set and a set of measure zero. To show that $f \circ w$ is measurable, by Claim 1 it suffices to show that $w^{-1}(R)$ is measurable for every $R \subset D$ such that $|R| = 0$. Let R be a subset of D such that $|R| = 0$. We have by (34) that

$$w^{-1}(R) \subset v(R),$$

and since $|R| = 0$, by the N property of v, we obtain that $|w^{-1}(R)| = 0$. Thus $w^{-1}(R)$ is measurable.

Let $g : v(D) = B(y_0, r) \to \mathbb{R}$ be defined by

$$g(y) = \frac{|\text{adj}\nabla v(w(y))|}{\det \nabla v(w(y))}.$$

Claim 4. $g \in L^1(v(D))$.

By Claim 3, g is measurable. By Lemma 2.4 and (11), where we set $f = \chi_{v(D)}$ the indicator of the set $v(D)$, and by Claim 2 and (31) we obtain

$$\int_{v(D)} |g(y)|dy \ = \ \int_D |g \circ v(x)| \det \nabla v(x) dx = \int_D |\text{adj}\nabla v(x)| dx.$$

Therefore $g \in L^1(v(D))$.

Claim 5. $w \in W^{1,1}(v(D))^N$ and $\nabla w(y) = (\frac{\text{adj}\nabla v(w(y))}{\det \nabla v(w(y))})^T$.

To prove Claim 5, we fix $\phi \in C_0^\infty(v(D))$ and set $K = \text{supp}\phi$. We show that

$$\int_{v(D)} w_\alpha(y) \frac{\partial \phi}{\partial y_j}(y) dy = -\int_{v(D)} \frac{(\text{adj}\, \nabla v(w(y)))_\alpha^j}{\det \nabla v(w(y))} \phi(y) dy.$$

Set $\delta = \text{dist}(K, \partial v(D)) > 0$. By using the uniform continuity of v on $\bar{D} \subset B(x_0, R_0)$, we choose $\epsilon > 0$ such that

$$|v(x) - v(x')| \le \frac{\delta}{4} \ \text{ for every } \ x, x' \in \bar{D}, \ |x - x'| \le \epsilon. \tag{36}$$

Let $\{v_n\} \subset C^\infty(\bar{D})^N$ be such that

$$v_n \to v \ \text{ in } \ C^0(\bar{D})^N \tag{37}$$

and

$$v_n \rightharpoonup v \ \text{ in } \ W^{1,N}(D)^N.$$

To see this, we recall that by Lemma 2.7 (iii) v is approximatively differentiable a.e. in Ω and by adapting the proof of Lemma 3.5 accordingly, it is possible to show that

$$d(v, B(x_0, r), v(x_0)) = 1$$

for some $r > 0$. Let C_0 be the connected component of $\mathbb{R}^N \setminus v(\partial B(x_0, r))$ which contains $v(x_0)$. Then

$$d(v, B(x_0, r), y) = 1 \tag{39}$$

for every $y \in C_0$, so if we choose $0 < r' < r$ such that

$$B(x_0, r') \subset B(x_0, r) \cap v^{-1}(C_0),$$

then by (39) (and since $\det \nabla v > 0$ a.e.) we have

$$d(v, B(x_0, r'), y) \leq 1$$

for every $y \in \mathbb{R}^N \setminus v(\partial B(x_0, r'))$. It suffices now to use the results in [TQ], (1.3)–(1.5), (2.26), and Theorem 3.7 (i). Note, however, that in [TQ], it is assumed that $\operatorname{adj} \nabla v \in L^r$, $r \geq \frac{q}{q-1}$ and if $N - 1 < q < N$, then $\frac{q}{q-1} > \frac{N}{N-1}$.

As it turns out, [TQ]'s results still hold for $r = \frac{N}{N-1}$ as remarked by [MTY] (see Theorem 5.3 in [MTY]).

Proof of Corollary 3.2.

Proof of (a). We have

$$v \in W^{1,N}(\Omega_1)^N, \quad \det \nabla v(x) > 0 \text{ a.e. } x \in \Omega_1$$

and

$$v \in W^{1,N}(\Omega_2)^N, \quad \det \nabla v(x) < 0 \text{ a.e. } x \in \Omega_2.$$

It suffices to apply Theorem 3.1 to v and to $R_0 v$ in Ω_2, where R_0 is a constant rotation with $\det R_0 = -1$.

Proof of (b). We now assume that $v \in W^{1,q}(\Omega)^N$, $q > N$, $\det \nabla v(x) \neq 0$ a.e. $x \in \Omega$, and for almost every $x_0 \in \Omega$, v is locally almost injective in a neighborhood of x_0 in the sense that there is an open set $D \equiv D(x_0) \subset\subset \Omega$ and there is a function $w : v(D) \to D$ such that

$$w \circ v(x) = x \text{ a.e. } x \in D. \tag{40}$$

By Vitali's covering theorem there is a countable family of nonempty, open, mutually disjoint balls $\{B_i,\ i \in \mathbb{N}\}$ and there is a sequence of functions $w_i : v(\bar{B}_i) \to \Omega$ such that $\bar{B}_i \subset \Omega$ and

$$\begin{aligned} &|\Omega \setminus \cup_{i=1}^{+\infty} B_i| = 0, \\ &w \circ v(x) = x \text{ a.e. } x \in B_i. \end{aligned} \tag{41}$$

The task ahead will be to partition B_i into three subsets B_i^1, B_i^2, and N_i such that B_i^1, B_i^2 are two open sets. N_i is a set of measure zero. and

$$\begin{aligned} &\det \nabla v(x) > 0 \text{ a.e. } x \in B_i^1. \\ &\det \nabla v(x) < 0 \text{ a.e. } x \in B_i^2. \end{aligned}$$

4. Semicontinuity involving variation of the domain. The variational treatment of crystals with defects leads to the study of functionals of the type

$$E(u,v) = \int_\Omega W(\nabla u(x)(\nabla v(x))^{-1})dx,$$

where $\Omega \subset \mathbb{R}^N$ is a reference domain, W is the strain energy density, u is the elastic deformation and v represents the slip (rearrangement) or plastic deformation with $\det(\nabla v(x)) = 1$ a.e. $x \in \Omega$. The underlying kinematical mode for slightly defective crystals was introduced by Davini [Dav] and later developed by Davini and Parry [DP]. As it turns out, matrices of the form

$$\nabla u(x)(\nabla v(x))^{-1}$$

represent lattice matrices of defect-preserving deformations (neutral deformations) and by taking the viewpoint that equilibria correspond to a variational principle, Fonseca and Parry [FP] studied the structure of some kind of generalized minimizers (the Young measure solutions) for the energy $E(\cdot,\cdot)$. (Related variational problems were also investigated in [DP].)

Using the div-curl lemma, it follows that if $u_n \rightharpoonup u$ in $W^{1,\infty}$ $w*$ and $v_n \rightharpoonup v$ in $W^{1,\infty}$ $w*$, then

$$\nabla u_n(\nabla v_n)^{-1} \rightharpoonup \nabla u(\nabla v)^{-1} \text{ in } L^\infty\ w*.$$

Lower semicontinuity and relaxation properties of $E(\cdot,\cdot)$ were addressed only under additional material symmetry assumptions on W. Existence and regularity properties for minimizers of $E(\cdot,\cdot)$ were obtained in [DF]. Following this work, we stress the fact that the direct methods of the calculus of variations fail to apply to this problem, as sequential weak lower semicontinuity of $E(\cdot,\cdot)$ is not sufficient to guarantee the existence of minimizers. Indeed, with $W(F) = |F|^r$, it is shown in [DF] that there are no minimizers in $\{(u,v) \in W^{1,\infty} \times W^{1,\infty} : \ u(x) = x \text{ on } \partial\Omega, \det(\nabla v(x)) = 1 \text{ a.e.}\}$ if $0 < r < N = 2$, while for $r > N$ existence is obtained for smooth (u,v) (see Theorem 2.3 in [DF]).

It is clear that if $\{(u_n, v_n)\}$ is a minimizing sequence and if $|\nabla u_n(\nabla v_n)^{-1}|^r$ is bounded in L^1, then

$$\nabla u_n(\nabla v_n)^{-1} \rightharpoonup L \text{ in } L^r, u_n|_{\partial\Omega} = u_0, \det(\nabla v_n) = 1 \text{ a.e.}$$

and so if some type of lower semicontinuity prevails, then

$$\int_\Omega W(L)dx \le \liminf \int_\Omega W(\nabla u_n(\nabla v_n)^{-1})dx. \tag{44}$$

It would remain to show that L would still have the same structure, precisely

$$L = \nabla u(\nabla v)^{-1},$$

where $u|_{\partial\Omega} = u_0$, $\det(\nabla v) = 1$ a.e. Note that (44) is always satisfied if W is a convex function. On the other hand, formally, as $\det(\nabla v) = 1$ a.e. and setting $w = u(v^{-1})$, the energy becomes

$$\int_{v(\Omega)} W(\nabla w(y))dy,$$

4. We may ask if these results can be extended to the case $\frac{N^2}{N+1} < q < N$, since, due to Müller's result ([Mu]), if we assume that $\text{Det}\nabla v = 1$ a.e. then $\text{Det}\nabla v = \det \nabla v$ a.e. in Ω.

5. Since lower semicontinuity of the energy is obtained in Theorem 4.1, the question now amounts to showing that one can find a minimizing sequence $\{\nabla u_n(\nabla v_n)^{-1}\}$ where $\{u_n\}$ is bounded in $W^{1,p}$ and $\{v_n\}$ is bounded in $W^{1,q}$. Actually, one only needs to show that there exists a sequence $\{f_n\} \subset W^{1,\infty}(\Omega,\Omega)$ such that $v_n \circ f_n$ is bounded in $W^{1,q}$ and

$$\begin{cases} \det \nabla f_n(x) &= 1 \text{ a.e. } x \in \Omega, \\ f_n(x) &= x \;\; x \in \partial\Omega. \end{cases}$$

Due to the examples provided in [DF], we know that this may not be possible since the infimum of E may be zero, which may prevent the existence of minimizing sequences bounded in $W^{1,p} \times W^{1,q}$.

As usual in variational problems for which existence of minimizers is not guaranteed (such as variational problems for material that change phase and, here, for slightly defective materials), we focus on the properties of the minimizing sequences rather than study the macroscopic limit of $\nabla u_n(\nabla v_n)^{-1}$.

What follows may help to understand better why boundedness of $\{\nabla u_n(\nabla v_n)^{-1}\}$ may not entail the boundedness of $\{\nabla u_n\}$ and $\{\nabla v_n\}$. Using Theorem 4.1, we show that we may construct a minimizing sequence $\{\nabla u_\epsilon(\nabla v_\epsilon)^{-1}\}$ with $|\nabla u_\epsilon|_p = 0(\frac{1}{\epsilon^\alpha})$, $|\nabla v_\epsilon|_q = 0(\frac{1}{\epsilon^\beta})$, for any $\alpha, \beta > 0$.

Consider the "perturbed" family of variational problems

$$E_\epsilon(u,v) = \int_\Omega W(\nabla u(\nabla v)^{-1})dx + \epsilon^{\alpha p}|\nabla u_\epsilon|_p^p + \epsilon^{\beta q}|\nabla v_\epsilon|_q^q,$$

where $u|_{\partial\Omega} = u_0$, $\det \nabla v = 1$ a.e., $\frac{1}{|\Omega|}\int_\Omega v(x)dx = 0$. Using the direct method of the calculus of variations, Poincaré's inequality, and Theorem 4.1, it follows immediately that there exists $(u_\epsilon, v_\epsilon) \in W^{1,p} \times W^{1,q}$ such that

$$E_\epsilon(u_\epsilon, v_\epsilon) = \inf\{E_\epsilon(u,v) : \;(u,v) \in W^{1,p} \times W^{1,q}, \;\det \nabla v = 1 \text{ a.e.}\}.$$

Then, given an admissible pair (u,v)

$$\begin{aligned} E(u,v) &= \lim_{\epsilon \to 0+} E_\epsilon(u,v) \\ &\geq \limsup_{\epsilon \to 0+} E_\epsilon(u_\epsilon, v_\epsilon) \\ &\geq \limsup_{\epsilon \to 0+} E(u_\epsilon, v_\epsilon), \\ &\geq \inf E. \end{aligned}$$

Doing the same with $\liminf_{\epsilon \to 0+} E(u_\epsilon, v_\epsilon)$ and taking the infimum in (u,v), we conclude that

$$\inf E = \lim_{\epsilon \to 0+} E(u_\epsilon, v_\epsilon)$$

and $|\nabla u_\epsilon|_p = 0(\frac{1}{\epsilon^\alpha})$, $|\nabla v_\epsilon|_q = 0(\frac{1}{\epsilon^\beta})$.

The following two lemmas will be useful to prove Theorem 4.1.

LEMMA 4.3. *Let Ω', Ω be two open sets of $\mathbb{R}^N$ such that $\Omega' \subset\subset \Omega$; let $q \geq N$ and $v, v_n \in W^{1,q}(\Omega)^N$ be such that* $\det \nabla v(x) = \det \nabla v_n(x) = 1$ *a.e. $x \in \Omega$. Assume*

Set $D = v^{-1}(B(y_0, r_0)) \cap B(x_0, R_0) \subset\subset \Omega'$ and $D_n = v_n^{-1}(B(y_0, r_0)) \cap B(x_0, R_0) \subset\subset \Omega'$. By using (45)–(47) and arguments similar to the ones of the proof of Theorem 3.1, together with Corollary 3.3, we deduce that for $n \geq n_0$ there is $w_n : \bar{B}(y_0, r_0) \to \bar{D}_n$, there is $w : \bar{B}(y_0, r_0) \to \bar{D}$ such that

$$\begin{aligned}
&w_n, w \in W^{1,\frac{q}{N-1}}(B(y_0, r_0))^N,\\
&w_n \circ v_n(x) = x \ \text{ a.e. } \ x \in \bar{D}_n,\\
&v_n \circ w_n(y) = y \ \text{ a.e. } \ y \in \bar{B}(y_0, r_0),\\
&w \circ v(x) = x \ \text{ a.e. } \ x \in \bar{D} \ \text{ and } \ v(x_0) \neq v(x) \ \text{ for } \ x \in \bar{D}, x \neq x_0,\\
&v \circ w(y) = y \ \text{ a.e. } \ y \in \bar{B}(y_0, r_0).
\end{aligned}$$

Finally by Lemma 3.8, $v_n(D_n) = v(D) = B(y_0, r_0)$.

Remark 4.4.

1. It follows from the proof above that if the conclusion of Lemma 4.3 holds for $r \equiv r(x_0) > 0$ then it holds also for $0 < r' < r$. Thus, as v is continuous on $\bar{D}$, $v(x) \neq v(x_0)$ for $x \in D$ and $x \neq x_0$, we deduce that

$$\lim_{r \to 0} \max\{|x - x_0| : \ x \in \bar{D}, v(x) \in B(y_0, r_0)\} = 0.$$

2. It is possible to show that $\lim_{n \to +\infty} |D \Delta D_n| = 0$. We divide the proof into two cases.

Claim 1. $\lim_{n \to +\infty} |D \setminus D_n| = 0$.

Let $F_\epsilon = B(y_0, r_0 - \epsilon)$ and $O_\epsilon = v^{-1}(F_\epsilon) \cap D$. We prove first that for each ϵ fixed there exists $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $n \geq n_0$ implies $O_\epsilon \subset D_n$. Indeed, since $\{v_n\}$ converges to v uniformly, there exists $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $|v - v_n|_\infty \leq \frac{\epsilon}{2}$ for every $n \geq n_0$. If $x \in O_\epsilon$, we obtain

$$|v_n(x) - y_0| \leq |v(x) - y_0| + |v(x) - v_n(x)| < r_0$$

and so $x \in D_n$. As $\cup_\epsilon O_\epsilon = D$ and the sequence (O_ϵ) is nonincreasing, we have

$$\lim_{\epsilon \to 0} |D \setminus O_\epsilon| = 0$$

which, together with the fact that $|D \setminus D_n| \leq |D \setminus O_\epsilon|$ for $n \geq n_0$, yields Claim 1.

Claim 2. $\lim_{n \to +\infty} |D_n \setminus D| = 0$.

For $\epsilon > 0$, take $n_0 \equiv n_0(\epsilon) \in \mathbb{N}$ such that $|v - v_n|_\infty \leq \frac{\epsilon}{2}$ for every $n \geq n_0$. For $n \geq n_0$, we have

$$\left\{x \in B(x_0, R_0) : r - \frac{\epsilon}{2} \leq |v_n(x) - y_0| < r\right\} \subset \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}$$

and since v has the N^{-1} property (see Remark 2.5) we obtain

$$|\cap_\epsilon \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}| = |\{x \in B(x_0, R_0) : |v(x) - y_0| = r\}| = 0.$$

To conclude the proof of Claim 2, it suffices to remark that for $n \geq n_0$ we obtain

$$D_n \setminus D \subset \{x \in B(x_0, R_0) : r - \epsilon \leq |v(x) - y_0| < r + \epsilon\}.$$

LEMMA 4.5. *Let $p \geq 1$, $q \geq N$, $r \geq 1$ be such that $\frac{1}{p} + \frac{N-1}{q} = \frac{1}{r}$. Assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $u_n, u \in W^{1,q}(\Omega)^N$, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)^N$, $v_n, v \in W^{1,q}\Omega)^N$, $\det \nabla v_n = \det \nabla v = 1$ a.e. in Ω and $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$. Let $x_0 \in \Omega$, and w_n, w be, respectively, the local inverse function of v_n, v, in the open neighborhoods D_n, D of x_0, let $y_0 = v(x_0)$ and $B(y_0, r_0)$ be as in Lemma 4.3 and Remark 4.4. Then the following conditions hold:*

Also

$$\int_{B(y_0,r_0)} |\nabla u_n \circ w_n(y)|^r dy = \int_D |\nabla u_n(x)(\nabla w_n(x))^{-1}|^r dx$$
$$\leq C'[\int_\Omega |\nabla u_n(x)|^p dx]^{\frac{r}{p}} [\int_\Omega |\nabla v_n(x)|^{\frac{q}{N-1}} dx]^{\frac{r(N-1)}{p}} \leq C$$

for some constant C which does not depend on y_0, r, and n. Thus $\{u_n \circ w_n\}$ is bounded in $W^{1,r}(B(y_0,r_0))^N$.

Third step. We prove that, up to a subsequence, $u_n \circ w_n$ converges strongly in $L^1(B(y_0,r_0))$ to $u \circ w$. Let $f \in C(\bar{B}(y_0,r_0))$. By Remark 4.4, $\lim_{n\to+\infty} |D \Delta D_n| = 0$ and so

$$\chi_{D_n}(x) \to \chi_D(x) \ \text{ a.e. } \ x \in \Omega.$$

By using the fact that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)^N$, $v_n \rightharpoonup v$ in $W^{1,q}(\Omega)^N$ and assuming, without loss of generality, that $u_n \to u$ a.e., $v_n \to v$ a.e., we obtain by (11) and the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty} \int_{B(y_0,r_0)} u_n \circ w_n(y) f(y) dy = \lim_{n\to+\infty} \int_{D_n} u_n(x) f(v_n(x)) dx$$
$$= \int_D u(x) f(v(x)) dx$$
$$= \int_{B(y_0,r_0)} u \circ w(y) f(y) dy.$$

Therefore $u_n \circ w_n$ converges strongly to $u \circ w$ in measure and by applying the Sobolev imbedding theorem to the bounded sequence $\{u_n \circ w_n\}$ in $W^{1,r}(\Omega)$, we conclude that, up to a subsequence, $u_n \circ w_n$ converges strongly in $L^1(B(y_0,r_0))$ to $u \circ w$.

Fourth step. Using the second and the third step we conclude that $\{\nabla u_n \circ w_n\}$ is bounded in $W^{1,r}(\Omega)^N$,

$$u_n \circ w_n \rightharpoonup u \circ w \ \text{ in } \ W^{1,r}(\Omega)^N \ \text{ if } r > 1,$$

and

$$u_n \circ w_n \to u \circ w \ \text{ in } \ L^1(\Omega)^N \ \text{ if } r = 1. \qquad \square$$

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1. Without loss of generality (and, if necessary, after extracting a subsequence of $\{(u_n, v_n)\}$), we assume that

$$\liminf_{n\to+\infty} \int_\Omega W(\nabla u_n(x)(\nabla v_n(x))^{-1}) dx = \lim_{n\to+\infty} \int_\Omega W(\nabla u_n(x)(\nabla v_n(x))^{-1}) dx < +\infty.$$

Fix $\epsilon > 0$ and let $\Omega_\epsilon \subset\subset \Omega$ be an open set such that $|\Omega \setminus \Omega_\epsilon| < \epsilon$. By Lemma 2.1 and the Ascoli–Arzela theorem, without loss of generality we assume that v_n converges to v uniformly in $\bar{\Omega}_\epsilon$. Set

$$C = \{x \in \Omega_\epsilon \ : v \text{ is differentiable and almost invertible at } \ x\},$$

$$A = \{D(x) : x \in C, D(x) \text{ is an open set of } \Omega_\epsilon, \ v(D(x)) \text{ is an open ball}\},$$

We divide the rest of the proof of Theorem 4.1 into two cases.

First case. We assume that $1 = r = \frac{1}{p} + \frac{N-1}{q}$ and that there is a constant C such that $0 \leq W(F) \leq C(1 + |F|)$ for every $F \in M^{N\times N}$. Since $W \geq 0$ and $\{D^j(\eta)\}, \{D^j_n(\eta)\}$ are mutually disjoint for every $n \in \mathbf{N}$, we have by [FM]

$$\begin{aligned}
\int_{\cup_{j=1}^k D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx &= \sum_{j=1}^k \int_{D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx \\
&= \sum_{j=1}^k \int_{B(y^j, r^j-\eta)} W((\nabla u \circ w^j)(y))dy \\
&\leq \sum_{j=1}^k \liminf_{n\to+\infty} \int_{B(y^j,r^j-\eta)} W((\nabla u_n \circ w^j_n)(y))dy \qquad (52)\\
&= \sum_{j=1}^k \liminf_{n\to+\infty} \int_{D^j_n(\eta)} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\leq \liminf_{n\to+\infty} \sum_{j=1}^k \int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\leq \liminf_{n\to+\infty} \int_\Omega W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx.
\end{aligned}$$

By letting η go to zero, k go to infinity, and ϵ go to zero, we have

$$E(u,v) \leq \liminf_{n\to+\infty} E(u_n, v_n).$$

Second case. We assume that $1 < r = \frac{1}{p} + \frac{N-1}{q}$ and that there are some constants $C_1, C_2 > 0, 1 \leq s \leq r$ such that $-C_1(1+|F|^s) \leq W(F) \leq C_2(1+|F|^r)$ for every $F \in M^{N\times N}$. The proof follows as in the first case, where on step (52) we use the lower semicontinuity results of [Da] instead of [FM]. Since $\{\nabla u_n(x)(\nabla v_n)^{-1}(x)\}$ is weakly relatively compact in Ω, we have

$$\begin{aligned}
\int_{\cup_{j=1}^k D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx &= \sum_{j=1}^k \int_{D^j(\eta)} W(\nabla u(x)(\nabla v)^{-1}(x))dx \\
&= \sum_{j=1}^k \int_{B(y^j, r^j-\eta)} W((\nabla u \circ w^j)(y))dy \\
&\leq \sum_{j=1}^k \liminf_{n\to+\infty} \int_{B(y^j,r^j-\eta)} W((\nabla u_n \circ w^j_n)(y))dy \\
&= \sum_{j=1}^k \liminf_{n\to+\infty} \int_{D^j_n(\eta)} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&= \sum_{j=1}^k \liminf_{n\to+\infty} [\int_{D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx \\
&\quad + \int_{D^j_n(\eta)\setminus D^j} W(\nabla u_n(x)(\nabla v_n)^{-1}(x))dx
\end{aligned}$$

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