1a. Find the eigenvalues and corresponding eigenvectors to the matrix

\[ A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \]

b. Find \( A^{15} \) (Show all work).

c. Solve \( \vec{x}_n = A\vec{x}_{n-1}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

d. Suppose \( B \) is a \( 4 \times 4 \) matrix with three distinct eigenvalues. One eigenvalue has geometric multiplicity one and one has geometric multiplicity two. Is it possible that \( B \) is not diagonalizable?

2. Let \( L \) be the line in \( \mathbb{R}^3 \) given by

\[ L = \{ \vec{x} = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad -\infty < t < \infty \} \]

and \( \vec{y} = [2, 1, 1]^T \)

a. Find the orthogonal projection of \( \vec{y} \) onto \( L \).

b. Find the matrix representing the orthogonal projection onto \( L \).

c. Find the matrix that represents the reflection of a vector about \( L \).

3a. Two matrices \( A \) and \( B \) are similar if \( B = P^{-1}AP \) where \( P \) is an invertible matrix. Show that \( A \) and \( B \) have the same determinant.

3b. Show that \( A \) and \( B \) have the same characteristic polynomial.

3c. Find the characteristic polynomial of the matrix

\[ A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix} \]

4(a) Let \( A \) be an \( m \times n \) matrix. Show \( \text{Null} A \perp \text{Col}(A^T) \)

(b) Let,

\[ \vec{v}_1 = [1, 0, 1, 1]^T, \quad \vec{v}_2 = [1, 1, 0, 0]^T, \quad \vec{v}_3 = [1, 2, 1, 1]^T. \]

From the above vectors construct an orthonormal set of vectors.

(c) Let \( S \) be the space spanned by \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \). If \( \vec{x} = [1, 1, 1, 1]^T \) find the vector in \( S \) closest to \( \vec{x} \).

(d) Let \( A \) be the matrix whose columns are \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) and \( Q \) be the matrix whose columns are the orthonormal basis just constructed. Find \( R \) so that \( A = QR \).