Prob 1)

Yes it is a linear space

proof 1) If $f \& g \in V \Rightarrow$

$\deg(f) \leq 3 \; \& \; \deg(g) \leq 3 \Rightarrow \deg(f+g) \leq 3$.

$f(1) = g(1) = 0 \Rightarrow (f+g)(1) = 0 \Rightarrow$

$f+g \in V$

2) If $f \in V \; \& \; \lambda \in \mathbb{R} \Rightarrow \deg(f) \leq 3$

$\Rightarrow \deg(\lambda f) \leq 3$.

$f(1) = 0 \Rightarrow (\lambda f)(1) = 0 \Rightarrow \lambda f \in V$

Finding basis.

$f \in V \Rightarrow f = ax^3 + bx^2 + cx + d$ &

$f(1) = 0 \Rightarrow a+b+c+d = 0$

then $a = -t_1 - t_2 - t_3$ $\quad d = t_3$

$b = t_1$

$c = t_2$

then $f = (-t_1 - t_2 - t_3)x^3 + t_1 x^2 + t_2 x + t_3$

$= t_1(x^2 - x^3) + t_2(x - x^3) + t_3(1 - x^3)$

then basis $= \{x^2 - x^3,\ x - x^3,\ 1 - x^3\}$

$\dim V = 3$

Prob 2 a) yes

1) $T(f+g) = \begin{bmatrix} (f+g)(-1) \\ (f+g)'(0) \\ (f+g)(0) \\ (f+g)'(1) \end{bmatrix} = \begin{bmatrix} f(1) \\ f'(0) \\ f(0) \\ f'(1) \end{bmatrix} + \begin{bmatrix} g(1) \\ g'(0) \\ g(0) \\ g'(1) \end{bmatrix} =$

$= T(f) + T(g)$

2) $T(\lambda f) = \begin{bmatrix} (\lambda f)(-1) \\ (\lambda f)'(0) \\ (\lambda f)(0) \\ (\lambda f)'(1) \end{bmatrix} = \lambda \begin{bmatrix} f(-1) \\ f'(0) \\ f(0) \\ f'(1) \end{bmatrix} = \lambda T(f)$

b) $T(f)=0 \Rightarrow$
$$\begin{aligned} f(-1)&=0\\ f'(0)&=0\\ f(0)&=0\\ f'(1)&=0 \end{aligned}$$

$$f = ax^2+bx+c$$

$$\left.\begin{aligned} f(-1)=0 &\Rightarrow & a-b+c&=0\\ f'(0)=0 &\Rightarrow & b&=0\\ f(0)=0 &\Rightarrow & c&=0 \end{aligned}\right\} \Rightarrow a=0 \Rightarrow f=0$$

$\operatorname{Ker}(T) = \{0\}$ bais $= \emptyset$

c) Always ~~Span~~ $\operatorname{Im}(T) = \operatorname{Span}\{T(x^2), T(x), T(1)\} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$ because

$\operatorname{Span}\{1, x, x^2\} = V$ $\dim(\operatorname{Im}(T)) + \underbrace{\dim(\operatorname{Ker} T)}_{=0} = \underbrace{\dim V}_{=3}$

then $\dim(\mathrm{Im}(T)) = 3 \Longrightarrow$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis of $\mathrm{Im}(T)$

Prob 3

$$u_1 = \frac{\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} \right\|} = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \underbrace{\left(\begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} \right)}_{=10} \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix}1\\8\\1\\6\end{bmatrix} - \begin{bmatrix}1/10\\7/10\\1/10\\7/10\end{bmatrix} \cdot \begin{bmatrix}1\\8\\1\\6\end{bmatrix} \begin{bmatrix}1/10\\7/10\\1/10\\7/10\end{bmatrix} - \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix} \cdot \begin{bmatrix}1\\8\\1\\6\end{bmatrix} \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\8\\1\\6\end{bmatrix} - \begin{bmatrix}1\\7\\1\\7\end{bmatrix} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$u_3 = \begin{bmatrix}0\\1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$$

the 3 vectors in circle

Prob 4 $\quad A^T A x = A^T b$

$$A^T A = \begin{bmatrix}3 & 5 & 4\\2 & 3 & 5\end{bmatrix} \begin{bmatrix}3 & 2\\5 & 3\\4 & 5\end{bmatrix} = \begin{bmatrix}50 & 41\\41 & 38\end{bmatrix}$$

$$A^T b = \begin{bmatrix}3 & 5 & 4\\2 & 3 & 5\end{bmatrix} \begin{bmatrix}5\\9\\2\end{bmatrix} = \begin{bmatrix}68\\47\end{bmatrix}$$

$$\left[\begin{array}{cc|c} 50 & 41 & 68 \\ 41 & 38 & 47 \end{array}\right]$$

$$\begin{array}{ccc} 1 & 41/50 & 68/50 \\ 0 & 38 - \frac{41^2}{50} & 47 - \frac{(41)\,68}{50} \end{array}$$

$$\begin{array}{cc} \cancel{249} & \cancel{-4} \end{array}$$

$$\left[\begin{array}{ccc} 1 & 41/50 & 68/50 \\ 0 & 1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$