Exam 1 - MATH 4305

Section, AU or AG:

Last name:

First Name:

Use the blue book provided. Indicate your name and section. Show all your work. All the problems are worth the same.

Problem 1 (4 points): Solve

$$-x_2 - x_3 + x_4 = 1$$
$$2x_1 - x_3 + 2x_4 = 2$$
$$2x_1 - x_2 - 2x_3 + 3x_4 = 3$$

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 & 2 \\ 2 & -1 & -2 & 3 & 3 \end{array}\right]$$

$$X = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1/2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{array}\right]$$

$x_1 = 1 + \frac{1}{2}t_1 - t_2$

$x_2 = -1 - t_1 + t_2$

$x_3 = t_1$

$x_4 = t_2$

Problem 2 (4 points): Give a basis of the kernel and a basis of the image of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

What is the dimension of the kernel? What is the dimension of the image? What is the rank of this matrix?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Basis of Image $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Basis of Kernel $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim \operatorname{Ker}(A) = 1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim \operatorname{Im} A = \operatorname{rank} A = 2$

Problem 3 (4 points): Let T be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$?

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) =$$

$$= T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Problem 4 (4 points): Compute the matrix associated with $\text{proj}_L(\mathbf{x})$, where $L = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$$\text{proj}_L(x) = \frac{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

$$= \frac{-x_1 + x_2}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x_1 - x_2}{2} \\ \frac{-x_1 + x_2}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

matrix

Problem 5 (4 points): Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right]$$

$= A^{-1}$